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Fractional arboricity, strength and eigenvalues of graphs with fixed girth or clique number

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ABSTRACT

Let $c(G)$, $g(G)$, $\omega(G)$ and $\mu_{n-1}(G)$ denote the number of components, the girth, the clique number and the second smallest Laplacian eigenvalue of the graph G, respectively. The strength $\eta(G)$ and the fractional arboricity $\gamma(G)$ are defined by

$$\eta(G) = \min_{F \subseteq E(G)} \frac{|F|}{c(G-F) - c(G)} \quad \text{and}$$
$$\gamma(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)| - 1},$$

where the optima are taken over all edge subsets F and all subgraphs H whenever the denominator is non-zero, respectively. Nash-Williams and Tutte proved that G has k edge-disjoint spanning trees if and only if $\eta(G) \geq k$; and Nash-Williams showed that G can be covered by at most k forests if and only if $\gamma(G) \leq k$. In this paper, for integers $r \geq 2$, s and t, and any simple graph G of order n with minimum degree $\delta \geq \frac{2s}{t}$ and either clique number $\omega(G) \leq r$ or girth $g \geq 3$, we

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prove that if $\mu_{n-1}(G) > \frac{2s-1}{t\varphi(\delta,r)}$ or $\mu_{n-1}(G) > \frac{2s-1}{tN(\delta,g)}$, then $\eta(G) \geq \frac{s}{t}$, where $\varphi(\delta, r) = \max\{\delta+1, \lfloor \frac{r\delta}{r-1} \rfloor\}$ and $N(\delta, g)$ is the Moore bound on the smallest possible number of vertices such that there exists a δ-regular simple graph with girth g. As corollaries, sufficient conditions on $\mu_{n-1}(G)$ such that G has k edge-disjoint spanning trees are obtained. Analogous result involving $\mu_{n-1}(G)$ to characterize fractional arboricity of graphs with given clique number is also presented. Former results in Liu et al. (2014) [17] and Hong et al. (2016) [11] are extended, and the result in Liu et al. (2019) [18] is improved.

1. Introduction

We only consider finite undirected simple graphs in this paper. Undefined notation and terminologies will follow Bondy and Murty [1]. Throughout the paper, k, r, s, t are positive integers. Let $G = (V, E)$ be a graph of order n. We use $\delta(G)$ and $c(G)$ to denote the minimum degree and the number of components of a graph G, respectively. The clique number $\omega(G)$ of a graph G is the maximum cardinality of a complete subgraph of G. For a vertex subset $X \subseteq V(G)$, $G[X]$ is the subgraph of G induced by X. For any subset $X, Y \subseteq V(G)$ with $X \cap Y = \emptyset$, $e(X, Y)$ denotes the number of edges between X and Y, and $d_G(X)$ (or simply $d(X)$) is the number of edges between X and $V(G) \setminus X$, that is $d(X) = e(X, V(G) \setminus X)$.

For a simple graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, the adjacency matrix of G is defined to be a $(0,1)$-matrix $A(G) = (a_{ij})_{n\times n}$, where $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. As G is simple and undirected, $A(G)$ is a symmetric $(0,1)$-matrix. The adjacency eigenvalues of G are the eigenvalues of $A(G)$. Denoted by $D(G) = \mathrm{diag}\{d_G(v_1), d_G(v_2), \ldots, d_G(v_n)\}$, the diagonal degree matrix of G, where $d_G(v_i)$ denotes the degree of v_i. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are called the Laplacian matrix and the signless Laplacian matrix of G, respectively. We use $\lambda_i(G)$, $\mu_i(G)$ and $q_i(G)$ to denote the ith largest eigenvalue of $A(G)$, $L(G)$ and $Q(G)$, respectively. The second smallest Laplacian eigenvalue $\mu_{n-1}(G)$ is called algebraic connectivity by Fiedler [6]. It is well known that $\mu_{n-1}(G) > 0$ if and only if G is connected.

For a connected graph G, *the spanning tree packing number*, denoted by $\tau(G)$, is the maximum number of edge-disjoint spanning trees in G. The *arboricity* $a(G)$ is the minimum number of edge-disjoint forests which covers all the edges of G. Fundamental theorems characterizing graphs G with $\tau(G) \geq k$ and with $a(G) \leq k$ have been obtained by Nash-Williams and Tutte, and by Nash-Williams, respectively.

Theorem 1.1. *Let G be a connected graph with $E(G) \neq \emptyset$. Each of the following holds.*

(*i*) *(Nash-Williams [19] and Tutte [24]) The spanning tree packing number* $\tau(G) \geq k$ *if and only if for any* $F \subseteq E(G)$, $|F| \geq k(c(G-F)-1)$.

(*ii*) *(Nash-Williams [20]) The arboricity* $a(G) \leq k$ *if and only if for any subgraph* H *of* G, $|E(H)| \leq k(|V(H)|-1)$.

Following the terminology in [3,21], the *strength* $\eta(G)$ and the *fractional arboricity* $\gamma(G)$ of a graph G is defined as

$$\eta(G) = \min_{F \subseteq E(G)} \frac{|F|}{c(G-F)-c(G)} \text{ and } \gamma(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)|-1},$$

where the optima are taken over all edge subsets F and all subgraphs H of G whenever the denominator is non-zero, respectively. Theorem 1.1 implies that for a connected graph G, $\tau(G) \geq k$ if and only if $\eta(G) \geq k$, and $a(G) \leq k$ if and only if $\gamma(G) \leq k$. Since $\tau(G)$ and $a(G)$ are integral, we have $\tau(G) = \lfloor \eta(G) \rfloor$ and $a(G) = \lceil \gamma(G) \rceil$. Therefore, $\eta(G)$ is also referred to as the *fractional spanning tree packing number* of G.

Motivated by Kirchhoff's matrix tree theorem [14] and by a problem of Seymour (see Reference [19] of [4]), Cioabă and Wong [4] initially conjectured an explicit relationship between $\tau(G)$ and $\lambda_2(G)$ of a regular graph. Afterwards, the conjecture was extended to general graphs.

Conjecture 1.2. *(Cioabă and Wong [4], Gu et al. [9], Li and Shi [15] and Liu et al. [16]) Let* $k \geq 2$ *be an integer and* G *be a graph with minimum degree* $\delta \geq 2k$. *If* $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, *then* $\tau(G) \geq k$.

Several researchers have made progress toward Conjecture 1.2, as seen in [4,9,15–17]. This conjecture was finally settled in [17].

Theorem 1.3. *(Liu et al. [17]) Let* $k \geq 2$ *be an integer and* G *be a graph with minimum degree* $\delta \geq 2k$. *If* $\mu_{n-1}(G) > \frac{2k-1}{\delta+1}$, *or* $\lambda_2(G) < \delta - \frac{2k-1}{\delta+1}$, *or* $q_2(G) < 2\delta - \frac{2k-1}{\delta+1}$, *then* $\tau(G) \geq k$.

In order to improve or extend the results in Theorem 1.3, Liu, Lai and Tian [18] considered certain graph families such as bipartite graphs or triangle-free graphs. As triangle-free graphs have girth at least four, they utilize the parameter girth to investigate directly more general graph families. To state their results, we need the following definition.

Definition 1.4. For integers δ, g with $\delta \geq 2$ and $g \geq 3$, let $t = \lfloor \frac{g-1}{2} \rfloor$. Define

$$N(\delta, g) = \begin{cases} 1 + \delta \sum_{i=0}^{t-1} (\delta-1)^i, & \text{if } g = 2t+1, \\ 2\sum_{i=0}^{t} (\delta-1)^i, & \text{if } g = 2t+2, \end{cases} \text{ and } f(\delta, g) = N(\delta, g) - \sum_{i=1}^{t-1} (\delta-1)^i.$$

Tutte [23] initiated the cage problem, which seeks, for any given integers d and g with $d \geq 2$ and $g \geq 3$, the smallest possible number of vertices $n(d, g)$ such that there exists a d-regular simple graph with girth g. The value $N(d, g)$ in Definition 1.4 is a tight lower bound (often called the Moore bound) on $n(d, g)$ which can be found in [5].

Theorem 1.5. *(Liu et al. [18]) Let g and k be integers with $g \geq 3$ and $k \geq 2$, and G be a simple graph of order n with minimum degree $\delta \geq 2k$ and girth g. If $\mu_{n-1}(G) > \frac{2k-1}{f(\delta,g)}$, or $\lambda_2(G) < \delta - \frac{2k-1}{f(\delta,g)}$, or $q_2(G) < 2\delta - \frac{2k-1}{f(\delta,g)}$, then $\tau(G) \geq k$.*

Since $\tau(G) = \lfloor \eta(G) \rfloor$, to extend Theorem 1.3, Hong et al. [11] investigated the relationship between $\eta(G)$ and the eigenvalues of G. They also discussed the relationship between the fractional arboricity $\gamma(G)$ and algebraic connectivity $\mu_{n-1}(G)$.

Theorem 1.6. *(Hong et al. [11]) Let G be a graph with minimum degree $\delta \geq 2s/t$. If $\mu_{n-1}(G) > \frac{2s-1}{t(\delta+1)}$, or $\lambda_2(G) < \delta - \frac{2s-1}{t(\delta+1)}$, or $q_2(G) < 2\delta - \frac{2s-1}{t(\delta+1)}$, then $\eta(G) \geq \frac{s}{t}$.*

Theorem 1.7. *(Hong et al. [11]) Let G be a graph of order $n \geq \lfloor \frac{2s}{t} \rfloor + 1$ with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$. Let $\beta = \frac{2s}{t} - \frac{1}{\lfloor \frac{2s}{t} \rfloor + 1} \sum_{i=1}^{\lfloor \frac{2s}{t} \rfloor + 1} d_i$.*

(i) If $\beta \geq 1$, then $\gamma(G) \leq \frac{s}{t}$.
(ii) If $0 < \beta < 1$, $n \geq \lfloor \frac{2s}{t} \rfloor + 1 + \frac{2s-2}{t\beta}$ and $\mu_{n-1}(G) > \frac{n(2s/t - 2/t - \beta(\lfloor 2s/t \rfloor + 1))}{(\lfloor 2s/t \rfloor + 1)(n - \lfloor 2s/t \rfloor - 1)}$, then $\gamma(G) \leq \frac{s}{t}$.

These results motivate the current research. The girth of a graph is larger than three if and only if its clique number is no more than two. By this fact and the former results, in this paper, we investigate the relationship between $\eta(G)$ and the eigenvalues of G with given girth or clique number, and the relationship between $\gamma(G)$ and the eigenvalues of G with given clique number. As can be seen in Section 4 of this paper or in Corollary 1.7 of [12], for any real number $p \geq 0$, if $q_2(G) < 2\delta - p$ or $\lambda_2(G) < \delta - p$, then $\mu_{n-1}(G) > p$. Therefore, we focus on establishing the lower bounds on $\mu_{n-1}(G)$. The main results of this paper are presented as Theorems 1.8, 1.9 and 1.13, where Theorems 1.8 and 1.9 extend Theorem 1.6, and Theorem 1.13 extends Theorem 1.7.

Theorem 1.8. *Let $r \geq 2$ be an integer, and G be a graph of order n with minimum degree $\delta \geq 2s/t$ and clique number $\omega(G) \leq r$. Let $\varphi(\delta, r) = \max\{\delta + 1, \left\lfloor \frac{r\delta}{r-1} \right\rfloor\}$. If $\mu_{n-1}(G) > \frac{2s-1}{t\varphi(\delta,r)}$, then $\eta(G) \geq \frac{s}{t}$.*

Theorem 1.9. *Let G be a graph of order n with minimum degree $\delta \geq 2s/t$ and girth $g \geq 3$. If $\mu_{n-1}(G) > \frac{2s-1}{tN(\delta,g)}$, then $\eta(G) \geq \frac{s}{t}$.*

By Theorem 1.8 and Theorem 1.9, we have the following two corollaries immediately.

Corollary 1.10. *Let $k \geq 2$ and $r \geq 2$ be integers, and G be a graph of order n with minimum degree $\delta \geq 2k$ and clique number $\omega(G) \leq r$. Let $\varphi(\delta, r) = \max\{\delta+1, \left\lfloor \frac{r\delta}{r-1} \right\rfloor\}$. If $\mu_{n-1}(G) > \frac{2k-1}{\varphi(\delta,r)}$, then $\tau(G) \geq k$.*

Corollary 1.11. *Let G be a graph of order n with minimum degree $\delta \geq 2k$ and girth $g \geq 3$. If $\mu_{n-1}(G) > \frac{2k-1}{N(\delta,g)}$, then $\tau(G) \geq k$.*

Remark 1.12. As $\tau(G) = \lfloor \eta(G) \rfloor$, Theorem 1.8 and Theorem 1.9 deduce Corollary 1.10 and Corollary 1.11, respectively. Since $\varphi(\delta, r) \geq \delta + 1$ and $N(\delta, g) \geq \delta + 1$, Theorems 1.8 and 1.9 extend Theorem 1.6 and Corollaries 1.10 and 1.11 extend Theorem 1.3. As $f(\delta, g) = N(\delta, g) - \sum_{i=1}^{t-1}(\delta-1)^i$, we have $N(\delta, g) > f(\delta, g)$ when $\delta \geq 2$ and $t = \lfloor \frac{g-1}{2} \rfloor \geq 2$, which implies $\frac{2k-1}{N(\delta,g)} < \frac{2k-1}{f(\delta,g)}$ and so Corollary 1.11 improves Theorem 1.5 when $g(G) \geq 5$.

Theorem 1.13. *Let $r \geq 2$ be an integer, and G be a graph of order $n \geq \lfloor \frac{2s}{t} \rfloor + 1$ with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n = \delta$ and clique number $\omega(G) \leq r$. Let $\beta = \frac{2s}{t} - \frac{1}{\lfloor \frac{2s}{t} \rfloor + 1} \sum_{i=1}^{\lfloor \frac{2s}{t} \rfloor + 1} d_i$ and $\theta = \max\{\lfloor \frac{2s}{t} \rfloor + 1, \lfloor \frac{r\delta}{r-1} \rfloor - 1\}$.*

(i) If $\beta > \frac{2s-2}{t\theta}$, then $\gamma(G) \leq \frac{s}{t}$.
(ii) If $0 < \beta \leq \frac{2s-2}{t\theta}$, $n \geq \theta + \frac{2s-2}{t\beta}$ and $\mu_{n-1}(G) > \frac{n(2s/t-2/t-\beta\theta)}{\theta(n-\theta)}$, then $\gamma(G) \leq \frac{s}{t}$.

In Section 2, we display some preliminaries and mechanisms, which will be applied in the proofs of the main results, to be presented in Section 3. As corollaries, adjacency and signless Laplacian eigenvalue conditions to characterize strength and fractional arboricity are obtained at the last section.

2. Preliminaries

In this section, we present some of the preliminaries to be used in the proof of main results. For $X \subseteq V(G)$, we use $\bar{d}_G(X)$ or simply $\bar{d}(X)$ to denote the average degree of all vertices of X in G, that is $\bar{d}(X) = \frac{1}{|X|}\sum_{v \in V(G)} d_G(v)$. The following result is the famous theorem of Turán [22].

Lemma 2.1. *(Turán [22]) Let $r \geq 1$ be an integer, and G be a graph of order n. If the clique number $\omega(G) \leq r$, then $|E(G)| \leq \left\lfloor \frac{r-1}{2r} \cdot n^2 \right\rfloor$.*

Lemma 2.2. *(Hong et al. [13]) Let $r \geq 2$ be an integer, and G be a graph with minimum degree δ and clique number $\omega(G) \leq r$, and X be a non-empty proper subset of $V(G)$. If $d(X) < \delta$, then $|X| \geq \max\{\delta+1, \left\lfloor \frac{r\delta}{r-1} \right\rfloor\}$.*

Lemma 2.3. *Let $r \geq 2$ be an integer, and G be a graph with clique number $\omega(G) \leq r$. Let X be a nonempty proper subset of $V(G)$, $Y = V(G) \setminus X$, and $\bar{d}(X)$ be the average degree of all vertices of X in G. If $e(X,Y) < \bar{d}(X)$, then $|X| > \max\{\bar{d}(X), \frac{r\bar{d}(X)}{r-1} - 2\}$.*

Proof. We first show that X contains more than $\bar{d}(X)$ vertices. Since each vertex in X is adjacent to at most $|X|-1$ vertices of X, we obtain

$$\bar{d}(X)|X| = \sum_{x\in X} d_G(x) \leq |X|(|X|-1) + e(X,Y) < |X|(|X|-1) + \bar{d}(X),$$

and so $(|X|-1)(|X|-\bar{d}(X)) > 0$, which means that $|X| > \bar{d}(X)$.

Next we show that $|X| > \frac{r\bar{d}(X)}{r-1} - 2$. By Lemma 2.1, we conclude that

$$|E(G[X])| \leq \frac{(r-1)|X|^2}{2r}. \tag{2.1}$$

Since $\sum_{x\in X} d_G(x) = 2|E(G[X])| + e(X,Y)$, by (2.1)

$$\bar{d}(X)|X| = \sum_{x\in X} d_G(x) \leq 2\frac{(r-1)|X|^2}{2r} + e(X,Y) < \frac{(r-1)|X|^2}{r} + \bar{d}(X)$$

and so $|X|^2 - \frac{r\bar{d}(X)}{r-1}|X| + \frac{r\bar{d}(X)}{r-1} > 0$. It follows that

$$(|X|-1)(|X| - \frac{r\bar{d}(X)}{r-1} + 1) > -1. \tag{2.2}$$

If $|X| = 1$, then $e(X,Y) = \bar{d}(X)$, a contradiction. Hence, $|X| \geq 2$ and so $\frac{1}{|X|-1} \leq 1$. By (2.2),

$$|X| > \frac{r\bar{d}(X)}{r-1} - 1 - \frac{1}{|X|-1} \geq \frac{r\bar{d}(X)}{r-1} - 2.$$

The result follows. □

Lemma 2.4. *(Hong et al. [13]) Let G be a simple connected graph with minimum degree $\delta \geq 2$ and girth $g \geq 3$, and X be a non-empty proper subset of $V(G)$. If $d(X) < \delta$, then $|X| \geq N(\delta, g)$.*

Let $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, and let G be a graph with vertex set $V(G) = \{1, 2, \ldots, n\}$. Then x can be considered as a function defined on $V(G)$, that is, for any vertex i, we map it to $x_i = x(i)$. Fiedler [7] derived a very useful expression for algebraic connectivity $\mu_{n-1}(G)$ as follows.

Lemma 2.5. *(Fiedler [7]) Let G be a graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E. Then*

$$\mu_{n-1}(G) = \min \frac{n \sum\limits_{ij\in E} (x_i - x_j)^2}{\sum\limits_{i,j\in V, i<j} (x_i - x_j)^2},$$

where the minimum is taken over all non-constant vectors $(x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$.

Given two non-increasing real sequences $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ with $n > m$, the second sequence is said to interlace the first one if $\mu_i \geq \lambda_i \geq \mu_{n-m+i}$ for $i = 1, \ldots, m$. The following result is known as the Cauchy Interlacing Theorem. A proof of this theorem can be found on page 27 of [2].

Theorem 2.6 *(Cauchy Interlacing). Let B be a principal submatrix of a symmetric matrix A. Then the eigenvalues of B interlace the eigenvalues of A.*

Let A be a symmetric matrix of order n and $V_1, \ldots, V_k$ be a partition of $\{1, \ldots, n\}$. For any $1 \leq i, j \leq k$, let b_{ij} denote the average number of neighbors in V_j of the vertices in V_i. The quotient matrix of this partition is the $k \times k$ matrix B whose (i, j)-th entry equals b_{ij}. Haemers [10] showed the eigenvalues of the quotient matrix B in fact interlace the eigenvalues of A.

Theorem 2.7. *(Haemers [10]) Let A be a symmetric matrix. Then the eigenvalues of every quotient matrix of A interlace the eigenvalues of A.*

3. The proof of main results

In this section, we present the proofs of Theorems 1.8, 1.9 and 1.13.

3.1. A unified way to prove Theorems 1.8 and 1.9

We need the following lemma, which comes from a special case of Lemma 3.2 in [17] when $a = -1$. In [8], Gu pointed out that this Lemma also holds for multigraphs. For the sake of completeness, we intuitively present a proof here.

Lemma 3.1. *(Liu et al. [17]) Let G be a graph and suppose that $X, Y \subset V(G)$ with $X \cap Y = \emptyset$. If $\mu_{n-1}(G) \geq \min\{\frac{d(X)}{|X|}, \frac{d(Y)}{|Y|}\}$, then $[e(X, Y)]^2 \geq (|X|\mu_{n-1}(G) - d(X))(|Y|\mu_{n-1}(G) - d(Y))$.*

Proof. Let L_{XY} be the principal submatrix of Laplacian matrix of G induced by the vertices in $X \cup Y$. Let $|X \cup Y| = m$. Then by Theorem 2.6, $\lambda_{m-1}(L_{XY}) \geq \mu_{n-1}(G)$. The quotient matrix of L_{XY} with respect to the partition (X, Y) is

$$L_2 = \begin{pmatrix} \frac{d(X)}{|X|} & -\frac{e(X,Y)}{|X|} \\ -\frac{e(X,Y)}{|Y|} & \frac{d(Y)}{|Y|} \end{pmatrix}.$$

By Theorem 2.7, the eigenvalues of L_2 interlace the eigenvalues of L_{XY}, and so $\lambda_1(L_2) \geq \lambda_{m-1}(L_{XY}) \geq \mu_{n-1}(G)$. Since $\mu_{n-1}(G) \geq \min\{\frac{d(X)}{|X|}, \frac{d(Y)}{|Y|}\}$, by $\det(\lambda_1(L_2)I - L_2) = 0$ we have

$$\begin{aligned}\frac{[e(X,Y)]^2}{|X||Y|} =&(\lambda_1(L_2)-\frac{d(X)}{|X|})(\lambda_1(L_2)-\frac{d(Y)}{|Y|})\\ \geq&(\mu_{n-1}(G)-\frac{d(X)}{|X|})(\mu_{n-1}(G)-\frac{d(Y)}{|Y|}).\end{aligned}$$

The result follows. □

Remark 3.2. Instead of $\mu_{n-1}(G) \geq \max\{\frac{d(X)}{|X|}, \frac{d(Y)}{|Y|}\}$ in Lemma 3.2 of [17] when $a = -1$, $\mu_{n-1}(G) \geq \min\{\frac{d(X)}{|X|}, \frac{d(Y)}{|Y|}\}$ in Lemma 3.1 is sufficient to guarantee the conclusion holds.

Theorem 3.3. *Let G be a graph of order n with minimum degree $\delta \geq \frac{2s}{t}$. If $|X|\mu_{n-1}(G) > \frac{2s-1}{t}$ for any $X \subset V(G)$ with $d(X) < \delta$, then $\eta(G) \geq \frac{s}{t}$.*

Proof. As $\mu_{n-1}(G) > \frac{2s-1}{t|X|} > 0$, G is connected and so $c(G) = 1$. By the definition of $\eta(G)$, it suffices to prove for any $F \subseteq E(G)$, $t|F| \geq s(c(G-F)-1)$.

Let $c = c(G-F)$ and V_i be the vertex set of each component of $G-F$ for each $i \in \{1,2\ldots,c\}$, which satisfy $d(V_1) \leq d(V_2) \leq \cdots \leq d(V_c)$. If $t \cdot d(V_2) \geq 2s$, then $t|F| \geq \frac{t}{2}\sum_{i=1}^{c} d(V_i) \geq s(c-1)$. Thus, we may assume that $t \cdot d(V_2) \leq 2s-1$.

Let q be the largest index such that $t \cdot d(V_q) \leq 2s-1$. Then $2 \leq q \leq c$ and $d(V_q) \leq \frac{2s-1}{t} < \delta$. By the hypothesis, we have $|V_i|\mu_{n-1}(G) > \frac{2s-1}{t} \geq d(V_i)$ for each $i \in \{1,2,\ldots,q\}$. Hence, by Lemma 3.1, for each $i \in \{2,\ldots,q\}$,

$$\begin{aligned}[e(V_1,V_i)]^2 \geq&(|V_1|\mu_{n-1}(G)-d(V_1))(|V_i|\mu_{n-1}(G)-d(V_i))\\ >&\left(\frac{2s-1}{t}-d(V_1)\right)\left(\frac{2s-1}{t}-d(V_i)\right)\\ \geq&\left(\frac{2s-1}{t}-d(V_i)\right)^2.\end{aligned}$$

It follows that $e(V_1,V_i) > \frac{2s-1}{t} - d(V_i)$, or equivalently $t \cdot e(V_1,V_i) \geq 2s - t \cdot d(V_i)$. Then $t \cdot d(V_1) \geq t\sum_{i=2}^{q} e(V_1,V_i) \geq \sum_{i=2}^{q}(2s - t\cdot d(V_i))$. Therefore $t\sum_{i=1}^{q} d(V_i) \geq 2s(q-1)$. Thus,

$$\begin{aligned}2t|F| \geq&t\sum_{i=1}^{c} d(V_i) = t\sum_{i=1}^{q} d(V_i) + t\sum_{i=q+1}^{c} d(V_i)\\ \geq&2s(q-1)+2s(c-q)\\ =&2s(c-1),\end{aligned}$$

which completes the proof. □

As $\tau(G) = \lfloor \eta(G) \rfloor$, by Theorem 3.3, we get the following corollary.

Corollary 3.4. *Let G be a graph of order n with minimum degree $\delta \geq 2k$. If $|X|\mu_{n-1}(G) > 2k-1$ for any $X \subset V(G)$ with $d(X) < \delta$, then $\tau(G) \geq k$.*

By Lemma 2.2, $|X| \geq \delta + 1$ for any $X \subset V(G)$ with $d(X) < \delta$ for any graph G. Combining this with the hypothesis of Theorem 1.3, we have $|X|\mu_{n-1}(G) > 2k-1$ and so $\tau(G) \geq k$. So Theorem 1.3 could be derived from Lemma 2.2 and Corollary 3.4.

Proofs of Theorems 1.8 and 1.9. If G is a graph with clique number $\omega(G) \leq r$, then by Lemma 2.2 we get $|X| \geq \varphi(\delta, r)$ for any $X \subset V(G)$ with $d(X) < \delta$; if G is a graph with girth $g \geq 3$, then by Lemma 2.4 we get $|X| \geq N(\delta, g)$ for any $X \subset V(G)$ with $d(X) < \delta$. Therefore, in both cases, we have $|X|\mu_{n-1}(G) > \frac{2s-1}{t}$. By Theorem 3.3, $\eta(G) \geq \frac{s}{t}$. □

It is easy to see that the condition of Theorem 3.3 is weaker than the one of Theorems 1.8 and 1.9. Meanwhile, Theorem 3.3 presents a unified way to deal with girth, clique number and other graph parameters together with eigenvalues, and Theorems 1.3, 1.8 and 1.9 are direct results of this unified viewpoint. For graphs with other structure parameter, we only need to discuss the lower bound of $|X|$ for any $X \subset V(G)$ with $d(X) < \delta$.

3.2. Proof of Theorem 1.13

Proof of Theorem 1.13. Suppose to the contrary that $\gamma(G) > s/t$. By the definition of $\gamma(G)$, there exists a nontrivial subgraph H of G with $V(H) = V_1$ and $E(H) = E_1$ satisfying $|E_1| > (|V_1|-1)s/t$. It follows that $t|E_1| > (|V_1|-1)s$, or equivalently $t|E_1| \geq (|V_1|-1)s + 1$. Thus,

$$|E_1| \geq (|V_1|-1)s/t + 1/t. \tag{3.1}$$

Since H is simple, $|V_1|(|V_1|-1) \geq 2|E_1| > 2(|V_1|-1)s/t$. Then $|V_1| > 2s/t$, and so

$$|V_1| \geq \lfloor 2s/t \rfloor + 1. \tag{3.2}$$

Let $\bar{d}_1 = \frac{1}{|V_1|}\sum_{v\in V_1} d_G(v)$. Since $|V_1| \geq \lfloor 2s/t \rfloor + 1$ and $\beta > 0$, we have

$$\frac{2s}{t} > \frac{2s}{t} - \beta = \frac{1}{\lfloor \frac{2s}{t} \rfloor + 1} \sum_{i=1}^{\lfloor \frac{2s}{t} \rfloor + 1} d_i \geq \bar{d}_1 \geq \frac{2|E_1|}{|V_1|} > \frac{2(|V_1|-1)s/t}{|V_1|} > \frac{2s}{t} - 1,$$

which yields to

$$2s/t - 1 < \bar{d}_1 \leq 2s/t - \beta < 2s/t. \tag{3.3}$$

Since $d(V_1) = \sum_{v\in V_1} d_G(v) - 2|E_1| = |V_1|\bar{d}_1 - 2|E_1|$, by (3.1), we have

$$d(V_1) \leq |V_1|\bar{d}_1 - 2(|V_1|-1)s/t - 2/t. \tag{3.4}$$

By (3.4) and (3.3),

$$d(V_1) \leq |V_1|(2s/t - \beta) - 2(|V_1|-1)s/t - 2/t = 2s/t - 2/t - \beta|V_1|. \tag{3.5}$$

By (3.5) and $d(V_1) \geq 0$, we get

$$|V_1| \leq \frac{2s-2}{t\beta}. \tag{3.6}$$

Moreover, by (3.4) and (3.3),

$$d(V_1) \leq |V_1|\bar{d}_1 - 2(|V_1|-1)s/t - 2/t < |V_1|\bar{d}_1 - \bar{d}_1(|V_1|-1) - 2/t < \bar{d}_1.$$

Hence, $d(V_1) < \bar{d}_1$. Thus, by Lemma 2.3,

$$|V_1| \geq \max\{\lfloor \bar{d}_1 \rfloor + 1, \left\lfloor \frac{r\bar{d}_1}{r-1} \right\rfloor - 1\}.$$

Combining this with (3.2) and $2s/t > \bar{d}_1 \geq \delta$, we have

$$|V_1| \geq \max\{\lfloor 2s/t \rfloor + 1, \lfloor \bar{d}_1 \rfloor + 1, \left\lfloor \frac{r\bar{d}_1}{r-1} \right\rfloor - 1\} \geq \max\{\lfloor 2s/t \rfloor + 1, \left\lfloor \frac{r\delta}{r-1} \right\rfloor - 1\} = \theta. \tag{3.7}$$

By (3.6) and (3.7),

$$\theta \leq |V_1| \leq \frac{2s-2}{t\beta}, \tag{3.8}$$

which implies $\beta \leq \frac{2s-2}{t\theta}$. This yields a contradiction to the condition of (i). Therefore, if $\beta > \frac{2s-2}{t\theta}$, then $\gamma(G) \leq \frac{s}{t}$. The proof of (i) is completed.

Combining $n \geq \theta + \frac{2s-2}{t\beta}$ with (3.8), we obtain

$$|V_1|(n - |V_1|) \geq \theta(n - \theta). \tag{3.9}$$

Let $(x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$. If $i \in V_1$, then set $x_i = \frac{1}{2}$; if $i \notin V_1$, then set $x_i = -\frac{1}{2}$. Thus,

$$\sum_{ij \in E} (x_i - x_j)^2 = d(V_1) \text{ and } \sum_{ij \in V, i<j} (x_i - x_j)^2 = |V_1|(n - |V_1|). \tag{3.10}$$

By (3.5), (3.8), (3.9) and (3.10),

$$\frac{n \sum\limits_{ij \in E} (x_i - x_j)^2}{\sum\limits_{i,j \in V, i<j} (x_i - x_j)^2} = \frac{n \cdot d(V_1)}{|V_1|(n - |V_1|)} \leq \frac{n \cdot (2s/t - 2/t - \beta|V_1|)}{|V_1|(n - |V_1|)} \leq \frac{n(2s/t - 2/t - \beta\theta)}{\theta(n - \theta)}.$$

By Lemma 2.5,

$$\mu_{n-1}(G) \leq \frac{n \sum_{ij\in E} (x_i - x_j)^2}{\sum_{i,j\in V, i<j} (x_i - x_j)^2} \leq \frac{n(2s/t - 2/t - \beta\theta)}{\theta(n-\theta)},$$

which yields a contradiction. Thus the proof of (ii) is completed. □

4. Adjacency eigenvalues and signless Laplacian eigenvalues

In this section, we present the relationship between $\eta(G)$ (or $\tau(G)$, $\gamma(G)$) and $\lambda_2(G)$ (or $q_2(G)$).

Theorem 4.1 *(Courant-Weyl Inequalities). Let A and B be Hermitian matrices of order n, and let $1 \leq i, j \leq n$. If $i + j \leq n + 1$, then $\lambda_i(A) + \lambda_j(B) \geq \lambda_{i+j-1}(A + B)$.*

For real numbers a, b with $b > 0$ and $a \geq -b$, let $\lambda_i(G, a, b)$ be the ith largest eigenvalue of the matrix $aD + bA$.

Corollary 4.2. *Let $p \geq 0$, $b > 0$ and $a \geq -b$ be real numbers and G be a graph of order n with minimum degree δ. If $\lambda_2(G, a, b) < (a + b)\delta - bp$, then $\mu_{n-1}(G) > p$. In particular, if $q_2(G) < 2\delta - p$ or $\lambda_2(G) < \delta - p$, then $\mu_{n-1}(G) > p$.*

Proof. Let A and D be the adjacency matrix and degree diagonal matrix of G. Since $b(D - A) + (aD + bA) = (a + b)D$, by Theorem 4.1, $\lambda_{n-1}(b(D - A)) + \lambda_2(aD + bA) \geq \lambda_n((a + b)D)$. As $b > 0$ and $a + b \geq 0$, $b\mu_{n-1}(G) + \lambda_2(G, a, b) \geq (a + b)\delta$. Therefore, if $\lambda_2(G, a, b) < (a + b)\delta - bp$, then $\mu_{n-1}(G) > p$. In particular, $\lambda_2(G, 1, 1) = q_2(G)$ and $\lambda_2(G, 0, 1) = \lambda_2(G)$. The result follows. □

By Corollary 4.2 and the sufficient conditions on $\mu_{n-1}(G)$ in Theorems 1.8, 1.9 and 1.13, we can obtain sufficient conditions on $\lambda_2(G, a, b)$, especially on $\lambda_2(G)$ and $q_2(G)$. We assume that $b > 0$ and $a \geq -b$ are real numbers and list the results as follows.

Theorem 4.3. *Let $r \geq 2$ be an integer, and G be a graph of order n with minimum degree $\delta \geq 2s/t$ and clique number $\omega(G) \leq r$. If $\lambda_2(G, a, b) < (a+b)\delta - \frac{b(2s-1)}{t\varphi(\delta,r)}$, then $\eta(G) \geq \frac{s}{t}$. In particular, if $\mu_{n-1}(G) > \frac{2s-1}{t\varphi(\delta,r)}$, or $\lambda_2(G) < \delta - \frac{2s-1}{t\varphi(\delta,r)}$, or $q_2(G) < 2\delta - \frac{2s-1}{t\varphi(\delta,r)}$, then $\eta(G) \geq \frac{s}{t}$.*

Corollary 4.4. *Let $k \geq 2$ and $r \geq 2$ be integers, and G be a graph of order n with minimum degree $\delta \geq 2k$ and clique number $\omega(G) \leq r$. If $\lambda_2(G, a, b) < (a + b)\delta - \frac{b(2k-1)}{\varphi(\delta,r)}$, then $\tau(G) \geq k$. In particular, if $\mu_{n-1}(G) > \frac{2k-1}{\varphi(\delta,r)}$, or $\lambda_2(G) < \delta - \frac{2k-1}{\varphi(\delta,r)}$, or $q_2(G) < 2\delta - \frac{2k-1}{\varphi(\delta,r)}$, then $\tau(G) \geq k$.*

Theorem 4.5. *Let G be a graph of order n with minimum degree $\delta \geq 2s/t$ and girth $g \geq 3$. If $\lambda_2(G,a,b) < (a+b)\delta - \frac{b(2s-1)}{tN(\delta,g)}$, then $\eta(G) \geq \frac{s}{t}$. In particular, if $\mu_{n-1}(G) > \frac{2s-1}{tN(\delta,g)}$, or $\lambda_2(G) < \delta - \frac{2s-1}{tN(\delta,g)}$, or $q_2(G) < 2\delta - \frac{2s-1}{tN(\delta,g)}$, then $\eta(G) \geq \frac{s}{t}$.*

Corollary 4.6. *Let G be a graph of order n with minimum degree $\delta \geq 2k$ and girth $g \geq 3$. If $\lambda_2(G,a,b) < (a+b)\delta - \frac{b(2k-1)}{N(\delta,g)}$, then $\tau(G) \geq k$. In particular, if $\mu_{n-1}(G) > \frac{2k-1}{N(\delta,g)}$, or $\lambda_2(G) < \delta - \frac{2k-1}{N(\delta,g)}$, or $q_2(G) < 2\delta - \frac{2k-1}{N(\delta,g)}$, then $\tau(G) \geq k$.*

Theorem 4.7. *Let $r \geq 2$ be an integer, and G be a graph of order $n \geq \lfloor \frac{2s}{t} \rfloor + 1$ with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n = \delta$ and clique number $\omega(G) \leq r$. Let $\beta = \frac{2s}{t} - \frac{1}{\lfloor \frac{2s}{t} \rfloor + 1} \sum_{i=1}^{\lfloor \frac{2s}{t} \rfloor + 1} d_i$ and $\theta = \max\{\lfloor \frac{2s}{t} \rfloor + 1, \lfloor \frac{r\delta}{r-1} \rfloor - 1\}$. If $0 < \beta \leq \frac{2s-2}{t\theta}$, $n \geq \theta + \frac{2s-2}{t\beta}$ and $\lambda_2(G,a,b) < (a+b)\delta - \frac{bn(2s/t-2/t-\beta\theta)}{\theta(n-\theta)}$, then $\gamma(G) \leq \frac{s}{t}$.*

Declaration of competing interest

There is no competing interests.

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