Discrete Mathematics 344 (2021) 112177

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

On the line graph of a graph with diameter 2

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ARTICLE INFO

Article history:
Received 25 October 2019
Received in revised form 13 June 2020
Accepted 28 September 2020
Available online xxxx

Keywords:
Diameter
Line graph
Pancyclic
1-hamiltonian

ABSTRACT

A graph G is pancyclic if it contains cycles of all possible lengths. A graph G is 1-hamiltonian if the removal of at most 1 vertices from G results in a hamiltonian graph. In Veldman (1988) Veldman showed that the line graph $L(G)$ of a connected graph G with diameter at most 2 is hamiltonian. In this paper, we continue studying the line graph $L(G)$ of a connected graph G with $|E(G)| \geq 3$ and diameter at most 2 and prove the following:

(i) $L(G)$ is pancyclic if and only if G is not a cycle of length 4 or 5, and G is not the Petersen graph.

(ii) $L(G)$ is 1-hamiltonian if and only if $\kappa(L(G)) \geq 3$.

1. Introduction

We consider finite graphs without loops but permitting multiple edges, and follow [2] for undefined terms and notation. Let $G = (V(G), E(G))$ be a undirected graph with vertex set $V(G)$ and edge set $E(G)$. For a graph G, $\kappa(G)$, $\kappa'(G)$ and $\delta(G)$ denote the **connectivity, edge-connectivity** and **the minimum degree** of G, respectively. We shall use $d(u, v)$ to denote the **distance** between a vertex u and a vertex v in G. For subgraphs H_1 and H_2 in a connected G, the distance $d(H_1, H_2)$ is defined to be $\min\{d(v_1, v_2) : v_1 \in V(H_1) \text{ and } v_2 \in V(H_2)\}$. When H_1 is a vertex u (or edge e), we denote $d(H_1, H_2)$ by $d(u, H_2)$ (or $d(e, H_2)$). The **diameter** and the **edge diameter** of G, denoted by $diam(G)$ and $diam_e(G)$, are defined as $diam(G) = \max\{d(u, v) : u, v \in V(G)\}$, and $diam_e(G) = \max\{d(e_1, e_2) : e_1, e_2 \in E(G)\}$. The **girth** of a graph G, denoted by $g(G)$, is the length of a shortest cycle of G.

The **line** graph of a graph G, denoted by $L(G)$, is a simple graph with $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. Then $diam_e(G) = diam(L(G))$. In 1986, Thomassen initiated one of the most fascinating conjectures on hamiltonian line graphs, as stated in Conjecture 1.1. In [19], Ryjáček uses an ingenious argument to show that Conjecture 1.1(i) is equivalent to a seeming stronger statement in Conjecture 1.1(ii). Later, Ryjáček and Vrána in [20] indicated that all four statements in Conjecture 1.1 are mutually equivalent.

Conjecture 1.1. *(i) (Thomassen [21]) Every 4-connected line graph is hamiltonian.*
(ii) (Matthews and Sumner [18]) Every 4-connected claw-free graph is hamiltonian.
(iii) (Kučzel and Xiong [14]) Every 4-connected line graph is Hamilton-connected.
(iv) (Ryjáček and Vrána [20]) Every 4-connected claw-free graph is Hamilton-connected.

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https://doi.org/10.1016/j.disc.2020.112177

Towards Conjecture 1.1, Zhan proved the first result in this direction. The best known result is given by Kaiser and Vrána, as shown below.

Theorem 1.2. *Let G be a graph.*
(i) (Zhan, Theorem 3 in [24]) If $\kappa(L(G)) \geq 7$, then $L(G)$ is Hamilton-connected.
(ii) (Kaiser and Vrána [13]) If $\kappa(L(G)) \geq 5$ and $\delta(L(G)) \geq 6$, then $L(G)$ is Hamilton-connected.

A graph G with vertex set $V(G)$ and edge set $E(G)$ is **pancyclic** if it contains cycles of all lengths l, $3 \leq l \leq |V(G)|$. For an integer $s \geq 0$, a graph G of order $n \geq s+3$ is s-**hamiltonian** if for any $X \subseteq V(G)$ with $|X| \leq s$, $G - X$ is hamiltonian. Researchers also consider the necessary and sufficient condition version of Conjecture 1.1 by asking whether there exists an integer $s \geq 2$ such that every line graph $L(G)$ is s-hamiltonian if and only if $\kappa(L(G)) \geq s+2$, as seen in [4,8,16,17], among others.

While every conjecture in Conjecture 1.1 is till open, whether it is hard to find a counterexample remains to be answered. In [1], Blass and Harary indicated that using the Erdös–Rényi model [9,10] with any positive constant probability on the occurrence of an edge in the random graph, almost every graph has diameter 2. Thus a property possessed by the family of graphs of diameter 2 will have a higher probability to be a property for generic graphs. Gould and Veldman investigated the hamiltonian cycles in claw-free graphs of diameter 2 and the line graphs of a graph of diameter 2.

Theorem 1.3. *Let G be a graph with diameter at most 2.*
(i) (Gould [11]) If G is 2-connected and $K_{1,3}$-free, then G is hamiltonian.
(ii) (Veldman [22]) If $|E(G)| \geq 3$, then $L(G)$ is hamiltonian.

Let C_n be a cycle of length n and $P(10)$ denote the Petersen graph. In 1993, Xiong et al. [23] discussed the pancyclicity of the line graph and proved the following.

Theorem 1.4 ([23])**.** *Let G be a graph of order n with at least a cycle. If $diam(L(G)) \leq 2$ and $G \notin \{C_4, C_5\}$, then $L(G)$ is pancyclic.*

In this paper we consider the pancyclicity and 1-hamiltonicity of the line graph $L(G)$ when the diameter of G is at most 2. The main purpose of this research is to prove the following.

Theorem 1.5. *Let G be a graph with $|E(G)| \geq 3$ and $diam(G) \leq 2$. Then $L(G)$ is pancyclic if and only if $G \notin \{C_4, C_5, P(10)\}$.*

Theorem 1.6. *Let G be a graph with $diam(G) \leq 2$. Then $L(G)$ is 1-hamiltonian if and only if $\kappa(L(G)) \geq 3$.*

Let $P(10)'$ be the graph from $P(10)$ by adding an edge joining two neighbors of a vertex to form a 3-cycle. Then $diam(L(P(10)')) = 3$ and $diam(P(10)') = 2$. Thus whether $L(P(10)')$ is pancyclic or not cannot be decided by Theorem 1.4. However, as $P(10)'$ is not the Petersen graph, Theorem 1.5 can be applied to conclude that $L(P(10)')$ is pancyclic.

In Section 2, we introduce Catlin's reduction method and the related results. The proofs of the main results will be given in the last two sections.

2. Preliminaries

A graph G is **eulerian** if G is connected with $O(G) = \emptyset$, and is **supereulerian** if G has a spanning eulerian subgraph. A subgraph H of a graph G is **dominating** if $G - V(H)$ is edgeless. Harary and Nash–Williams proved a very useful connection between hamiltonian cycles in the line graph $L(G)$ and dominating eulerian subgraphs in G.

Theorem 2.1 (*Harary and Nash–Williams [12]*)**.** *For a connected graph G with $|E(G)| \geq 3$, $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

An edge cut X of G is **essential** if $G - X$ has at least two nontrivial components. For an integer $k > 0$, a graph G is **essentially** k-**edge-connected** if G does not have an essential edge cut X with $|X| < k$. In particular, the essential edge-connectivity of G, denote by $ess'(G)$, is the size of a minimum essential edge-cut, if one such cut exists; or infinity if no such cut exists. For any $v \in V(G)$ and an integer $i \geq 0$, define $D_i(G) = \{v \in V(G) : d_G(v) = i\}$.

Let $X \subseteq E(G)$ be an edge subset of G. The **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. If H is a subgraph of G, we write G/H for $G/E(H)$. If v_H is the vertex in G/H onto which H is contracted, then H is called the **preimage** of v, and denoted by $PI(v)$. Let $O(G)$ denote the set of odd degree vertices of G. A graph G is **eulerian** if $O(G) = \emptyset$ and G is connected. A graph G is **supereulerian** if G has a spanning eulerian subgraph. In [6] Catlin defined collapsible graphs. Given an even subset R of $V(G)$, a subgraph Γ of G is called an R-**subgraph** if $O(\Gamma) = R$ and $G - E(\Gamma)$ is connected. A graph G is **collapsible** if for any even subset R of $V(G)$, G has an R-subgraph. In particular, K_1 is collapsible. Catlin [6] showed that for any graph G, one can obtain the **reduction** G' of G by contracting all maximal collapsible subgraphs of G. A graph G' is **reduced** if G' has no nontrivial collapsible subgraphs. A vertex in G' is **nontrivial** (or **trivial**) if $|V(PI(x))| \geq 2$ (or $|V(PI(x))| = 1$). By definition, every collapsible graph is supereulerian.

For a graph G, let $F(G)$ be the minimum number of additional edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. The following theorem summarizes the useful results on collapsible graphs and reduced graphs needed in our arguments.

Theorem 2.2 (*Catlin, [6]*). *Let G be a connected graph. Then each of the following holds:*

(i) G is reduced if and only if G has no nontrivial collapsible subgraphs.

(ii) For $n \neq 2$, the complete graph K_n and the 2-cycle C_2 are collapsible.

(iii) If G is reduced, then G is simple, K_3-free, $g(G) \geq 4$ and $\delta(G) \leq 3$.

(iv) If H is a collapsible subgraph of G, then G is collapsible if and only if G/H is collapsible.

(v)If G is reduced, then $F(G) = 2|V(G)| - |E(G)| - 2$.

(vi) Let H be a collapsible graph of G and let v_H denote the vertex of G/H onto which H is contracted. If G/H has an eulerian subgraph L' containing v_H, then G has a eulerian subgraph L with $E(L') \subseteq E(L)$ and $V(H) \subseteq V(L)$.

Theorem 2.3 (*Catlin et al. Theorem 1.5 of [7]*). *Let G be a connected graph and let G' be the reduction of G. If $F(G) \leq 2$, then $G' \in \{K_1, K_2, K_{2,t}\}$ for some integer $t \geq 1$. Therefore, G is supereulerian if and only if $G' \notin \{K_2, K_{2,t}\}$ for some odd integer t.*

Let H is a subgraph of G, define

$$\partial_G(H) = \{uv \in E(G) : u \in V(H), v \in V(G) - V(H)\}.$$

The subscript G in the notation above might be omitted if G is understood from the context. From Theorem 2.1 one easily proves a more general result.

Theorem 2.4 (*[3]*). *The line graph $L(G)$ of a graph G contains a cycle of length $l \geq 3$ if and only if G has an eulerian subgraph H such that $|E(H)| \leq l \leq |E(H)| + |\partial_G(H)|$.*

A useful tool is introduced to investigate the pancyclicity of line graphs. Define

$$sp_L(G) = \{l : \text{there is an eulerian subgraph } H \subseteq G \text{ such that } |E(H)| \leq l \leq |E(H)| + |\partial_G(H)|\}.$$

Corollary 2.5. *Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is pancyclic if and only if for any integer l with $3 \leq l \leq m$, $l \in sp_L(G)$.*

Lemma 2.6. *Let G be spanned by a $K_{1,n-1}$ with $n \geq 2$ and $m = |E(G)| \geq 4$. Then the following statements hold.*

(i) $L(G)$ is pancyclic.

(ii) If G is essentially 3-edge-connected, then for any $e_0 \in E(G)$, $(G - e_0) - D_1(G - e_0)$ is supereulerian.

Proof. By assumption, G has $K_{1,n-1}$ as a spanning subgraph. Let v_0 be the vertex of degree $n - 1$ in this $K_{1,n-1}$. If $n = 2, 3$ or if $G = K_{1,n-1}$, then $L(G)$ is a complete graph and so both (i) and (ii) hold. Assume that $n \geq 4$, $m \geq n$.

(i) Since $m \geq n$, every edge of $G - D_1(G)$ lies in a cycle of length at most 3 that contains v_0. It follows that $G - D_1(G)$ has edge-disjoint subgraphs $S_1, S_2, \ldots, S_t$ each of which contains v_0 such that $2 \leq |E(S_i)| \leq 3$ $(1 \leq i \leq t)$, and $\cup_{i=1}^t S_i$ is a dominating eulerian subgraph of G. Let $s_0 = \sum_{i=1}^t |E(S_i)| = |E(\cup_{i=1}^t S_i)|$. For any integer l with $3 \leq l \leq m$, if $l \geq s_0$, then as $\cup_{i=1}^t S_i$ is a dominating eulerian subgraph, $l \in sp_L(G)$. Thus we assume that $3 \leq l < s_0$. Then there exist $S_1, S_2, \ldots, S_{t'}$ with $t' < t$ and an integer r such that $l = \sum_{i=1}^{t'} |E(S_i)| + r$ and $0 \leq r \leq 2$. Let $H = \cup_{i=1}^{t'} S_i$. Then $|E(H)| \leq l \leq |E(H)| + |\partial_G(H)|$. So $l \in sp_L(G)$. This completes the proof of (i).

(ii) By contraction, we assume that G is a counterexample with $|V(G)|+|E(G)|$ smallest. Then there exists some $e_0 \in E(G)$ such that $G^\star = (G - e_0) - D_1(G - e_0)$ is not supereulerian. If $G^\star$ contains a nontrivial collapsible subgraph H, then we set $n' = |V(G/H)|$. Since G is spanned by $K_{1,n-1}$, G/H is spanned by $K_{1,n'-1}$. By the minimality of G, $(G/H - e_0) - D_1(G/H - e_0)$ is supereulerian, and so by Theorem 2.2(vi), $(G - e_0) - D_1(G - e_0)$ is supereulerian. So we assume that $G - e_0$ is reduced. If $G - e_0 = K_{1,n-1}$, then $(G - e_0) - D_1(G - e_0) = K_1$ is supereulerian. Therefore, e_0 is incident to v_0 and G contains either a 3-cycle or a 4-cycle that contains e_0. Without loss of generality, we assume that $e_0 = v_0v_1$. If G contains a triangle, then this triangle must be $C_3 = v_0v_1v_2v_0$. Thus $\{v_0v_1, v_0v_2\}$ is an essential 2-edge-cut, a contradiction. If G contains a 4-cycle, then this 4-cycle must be $C_4 = v_0v_2v_1v_3v_0$ for some vertices $v_2, v_3 \in N_G(v_1)$. It follows that C_4 is a spanning cycle of $(G - e_0) - D_1(G - e_0)$, contrary to the assumption that G is a counterexample. ■

Definition 2.7. Let $C = x_1x_2y_1y_2x_1$ be a 4-cycle in G with a partition $\pi(C) = \langle\{x_1, y_1\}, \{x_2, y_2\}\rangle$. Following [5], we define $G/\pi(C)$ to be the graph obtained from $G - E(C)$ by identifying x_1 and y_1 to form a vertex v_1, by identifying x_2 and y_2 to form a vertex v_2, and by adding an edge $e_{\pi(C)} = v_1v_2$.

Theorem 2.8 (*Catlin, [5]*). *Let G be a graph that contains a 4-cycle C and let $G/\pi(C)$ be defined as above. Each of the following holds.*

(a) If $G/\pi(C)$ is collapsible, then G is collapsible.

(b) If $G/\pi(C)$ has a spanning eulerian subgraph, then G has a spanning eulerian subgraph.

Lemma 2.9. *Let G be a connected graph on $n \geq 4$ vertices with $diam(G) \leq 2$ and let $C = x_1x_2y_1y_2x_1$ be a 4-cycle of G. Using the notation in Definition 2.7, each of the following holds.*

(i) $diam(G/\pi(C)) \leq 2$.

(ii) Either $\kappa(G/\pi(C)) \geq 2$ or $G/\pi(C)$ is spanned by $K_{1,n-1}$.

Proof. (i) By contradiction, assume that $diam(G/\pi(C)) \geq 3$. Then there are two vertices $x \in N_{G/\pi(C)}(v_1) - \{v_2\}$, $y \in N_{G/\pi(C)}(v_2) - \{v_1\}$ such that $d_{G/\pi(C)}(x, y) \geq 3$. Without loss of generality, we assume that $x \in N_G(x_1)$ and $y \in N_G(x_2)$. Then $d_G(x, y) \geq 3$, a contradiction. So $diam(G/\pi(C)) \leq 2$.

(ii) Assume that $\kappa(G/\pi(C)) = 1$. By (i), $G/\pi(C)$ is spanned by $K_{1,n-1}$. ∎

3. Proof of Theorem 1.5

Let s, k be two positive integers. Let $H_1 \cong K_{2,s}$ and $H_2 \cong K_{2,k}$ be two complete bipartite graphs. Let v_1, u_1 be two nonadjacent vertices of degree s in H_1, and v_2, u_2 be two nonadjacent vertices of degree k in H_2. Let $S_{s,k}$ denote the graph obtained from H_1 and H_2 by identifying v_1 and v_2 and connecting u_1 and u_2 with a new edge u_1u_2. Note that $S_{1,1}$ is the same as C_5, the 5-cycle.

Theorem 3.1 (*Lai [15]*). *Let G be a reduced graph. If $diam(G) = 2$, then exactly one of the following holds:*
(a) $G \cong K_{1,t}$, $t \geq 2$;
(b) $G \cong K_{2,t}$, $t \geq 2$;
(c) $G \cong S_{s,k}$, $s, k \geq 1$;
(d) G is $P(10)$, the Petersen graph.

Lemma 3.2. *Let $G \notin \{C_4, C_5, P(10)\}$ be a graph with $m = |E(G)| \geq 3$ and $diam(G) \leq 2$, and let $s \geq 3$ be an integer. If G has a trail T with $|E(T)| \leq s \leq |E(T)| + |\partial(T)|$, then G has an eulerian subgraph H such that $|E(H)| \leq s \leq |E(H)| + |\partial_G(H)|$.*

Proof. By contradiction, we assume that

$$\text{there is an integer } s \geq 3 \text{ such that the conclusion of Lemma 3.2 is false.} \tag{1}$$

As $m \geq 3$, $G \notin \{C_4, C_5\}$ and $diam(G) \leq 2$, we have $\Delta(G) \geq 3$. By (1), $s \geq 4$. Let $T = v_0v_1v_2 \cdots v_{t-1}v_t$. We will apply the following operations in order on T.

(Step 1). If $d_T(v_0) > 1$, then delete the edge v_0v_1 from T to have the new trail $T_1 = v_1v_2 \cdots v_t$; if $d_T(v_t) > 1$, then delete the edge v_tv_{t-1} from T. Repeat this step until the two end vertices have degree one in the trail. After Step 1 is finished, we assume that $T_{l_1} = v_0^1v_1^1 \cdots v_{t_1}^1$.

(Step 2). If $N_G(v_0^1) - V(T_{l_1}) \neq \emptyset$ and $|E(T_{l_1})| < s$, then we assume that $y_0^1 \in N_G(v_0^1) - V(T_{l_1})$. Replace T by $T_2 = y_0^1v_0^1v_1^1 \cdots v_{t_1}^1$. Keep applying this operation on y_0^1 if $N_G(y_0^1) - V(T_2) \neq \emptyset$ and $|E(T_2)| < s$, and $v_{t_1}^1$ if $N_G(v_{t_1}^1) - V(T_2) \neq \emptyset$ and $|E(T_2)| < s$. After Step 2 is finished, we assume that $T_{l_2} = v_0^2v_1^2 \cdots v_{t_2}^2$.

(Step 3). If $d_{T_{l_2}}(v_1^2) \geq 4$, then replace T_{l_2} by $T_3 = v_2^2v_3^2 \cdots v_{t_2}^2$.

Repeat Steps 1–3 until the degree of the second and last second vertices have degree 2 in the trail.

Claim 1. *Assume that $T' = x_0x_1 \cdots x_k$ is the trail obtained from T by applying Steps 1–3. Then we have the following.*
(i) $|E(T')| \leq s \leq |E(T')| + |\partial(T')|$.
(ii) $d_{T'}(x_0) = d_{T'}(x_k) = 1$.
(iii) If $|E(T')| < s$, then $N_G(x_0) \subseteq V(T')$ and $N_G(x_k) \subseteq V(T')$.
(iv) $d_{T'}(x_1) = d_{T'}(x_{k-1}) = 2$.

Proof of Claim 1. If $d_T(v_0) > 1$, then v_1v_2 is not a cut edge of T. Then $|E(T_1)| = |E(T)| - 1$. As $d_T(v_0) > 1$, $|E(T_1)| + |\partial(T_1)| = |E(T)| + |\partial(T)|$. Keep applying this operation on the end vertices of trail if their degrees are greater than 1 in the trail. Although the number of edges would be smaller, $|E(H)| + |\partial(T)|$ cannot be changed. After Step 1 is finished, we assume that $T_{l_1} = v_0^1v_1^1 \cdots v_{t_1}^1$. Then $d_{T_{l_1}}(v_0^1) = d_{T_{l_1}}(v_{t_1}^1) = 1$ and $|E(T_{l_1})| \leq s \leq |E(T_{l_1})| + |\partial(T_{l_1})|$.

If $N_G(v_0^1) - V(T_{l_1}) \neq \emptyset$ and $|E(T_{l_1})| < s$, then $|E(T_2)| = |E(T_{l_1})| + 1$ and $|E(T_{l_1})| + |\partial(T_{l_1})| \leq |E(T_2)| + |\partial(T_2)|$. As $|E(T_{l_1})| < s$, $|E(T_2)| \leq s \leq |E(T_2)| + |\partial(T_2)|$. Keep applying this operation on y_0^1 if $N_G(y_0^1) - V(T_2) \neq \emptyset$ and $|E(T_2)| < s$, and $v_{t_1}^1$ if $N_G(v_{t_1}^1) - V(T_2) \neq \emptyset$ and $|E(T_2)| < s$. After Step 2 is finished, we have $d_{T_{l_2}}(v_0^2) = d_{T_{l_2}}(v_{t_2}^2) = 1$, $|E(T_{l_2})| \leq s \leq |E(T_{l_2})| + |\partial(T_{l_2})|$, and $N_G(v_0^2) \subseteq V(T_{l_2})$ and $N_G(v_{t_2}^2) \subseteq V(T_{l_2})$ if $|E(T_{l_2})| < s$.

If $d_{T_{l_2}}(v_1^2) \geq 4$, then, as $d_{T_{l_2}}(v_0^2) = 1$, $v_1^2v_2^2$ is not a cut edge of T_{l_2}. If $|E(T_{l_2})| < s$, as $N_G(v_0^2) \subseteq V(T_{l_2})$, we have $|E(T_3)| + |\partial(T_3)| = |E(T)| + |\partial(T)|$. Thus $|E(T_3)| \leq s \leq |E(T_3)| + |\partial(T_3)|$. If $|E(T_{l_2})| = s$, then, as $d_{T_{l_2}}(v_1^2) \geq 4$, we have $v_0^2v_1^2, v_1^2v_2^2 \in \partial(T_3)$. Thus $|E(T_3)| < s \leq |E(T_3)| + |\partial(T_3)|$. Repeat Steps 1–3 on this new trail T_3. Once this procedure cannot be performed, (i)-(iv) are true. Claim 1 holds. ∎

By (1), $x_0 \neq x_k$. By Claim 1(iv), $x_k \neq x_1$, $x_0 \neq x_{k-1}$, and $x_1 \neq x_{k-1}$. By (1) and Claim 1(iii),

$$x_1x_k, x_0x_{k-1} \notin E(G). \tag{2}$$

Claim 2. $s \geq 5$.

Proof of Claim 2. By contradiction, we assume that $s = 4$. By (1), we have

$$\Delta(G) \leq 3, G \text{ has no a 4-cycle, and if } G \text{ has a cycle } C_k (k = 2, 3), \text{ then } |\partial(C_k)| \leq 3 - k. \tag{3}$$

If $|E(T')| \leq 3$, by Claim 1(iii), $N_G(x_0) \subseteq V(T')$ and $N_G(x_k) \subseteq V(T')$. If $|E(T')| = 2$, then $d_G(x_1) \geq 4$ since $|E(T')|+|\partial(T')| \geq 4$, contrary to (3). If $|E(T')| = 3$, by (3), $x_0x_3, x_0x_2, x_1x_3 \notin E(G)$. Thus $d_G(x_0) = d_G(x_3) = 1$. This implies that $dist_G(x_1, x_3) = 3$, a contradiction. So $|E(T')| = 4$.

If $x_2 = x_4$, then $H_1 = x_2x_3x_2$ is an eulerian subgraph with $|E(H_1)| = 2$ and $x_1x_2 \in \partial(H_1)$. By (3), $\partial(H_1) = \{x_1x_2\}$. Thus $d_G(x_3) = 2$ and $d_G(x_2) = 3$. So $dist_G(x_0, x_3) = 3$, a contradiction. By symmetry, $x_0, x_1, \ldots, x_4$ are different vertices. Also we assume that $x_0x_4 \in E(G)$ (Otherwise, if $x_0x_4 \notin E(G)$, by (3), $x_1x_4 \notin E(G)$. Thus there is a vertex $w_1 \notin \{x_0, x_1, \ldots, x_4\}$ such that $w_1x_1, w_1x_4 \in E(G)$, and so $x_1x_2x_3x_4w_1x_1$ is a 5-cycle. Thus we use the new eulerian trail $T'' = x_1x_2x_3x_4w_1$ to discuss instead of T'.)

As $G \neq C_5$, there is a vertex $u_1 \notin \{x_0, x_1, \ldots, x_4\}$ such that $N_G(u_1) \cap \{x_0, \ldots, x_4\} \neq \emptyset$. Without loss of generality, we assume that $x_1u_1 \in E(G)$. By (3), $u_1x_3, u_1x_4 \notin E(G)$. As $dist_G(u_1, x_3) \leq 2$ and $dist_G(u_1, x_4) \leq 2$, there are vertices $u_3, u_4 \notin \{x_0, \ldots, x_4\}$ such that $x_4u_4, x_3u_3, u_1u_3, u_1u_4 \in E(G)$. By (3), $u_4x_2 \notin E(G)$. Thus there is a vertex $u_2 \notin \{u_1, u_3, u_4, x_0, \ldots, x_4\}$ such that $u_4u_2, u_2x_2 \in E(G)$. Similarly, there is a vertex $u_0 \notin \{u_1, u_2, u_3, u_4, x_0, \ldots, x_4\}$ such that $u_0x_0, u_0u_3 \in E(G)$. If $u_0u_2 \notin E(G)$, there is a vertex $w_2 \notin \{u_0, \ldots, u_4, x_0, \ldots, x_4\}$ such that $w_2u_0, w_2u_2 \in E(G)$. As $\Delta(G) \leq 3$, $dist_G(w_2, x_4) \geq 3$, a contradiction. So $u_0u_2 \in E(G)$. Therefore, $G = P(10)$, a contradiction. Claim 2 holds. ■

Notice that $|E(T')| + |\partial(T')| \geq s$. If $|E(T')| = 2$, then $d_G(x_1) \geq s$, contrary to (1). If $|E(T')| = 3$, then $|\partial(T')| \geq s - 3$. By (1) and Claim 1(iii), $x_0x_2, x_1x_3 \notin E(G)$ and $x_0x_3 \notin E(G)$. Thus $d_G(x_0) = d_G(x_3) = 1$. This implies that $dist_G(x_0, x_3) = 3$, a contradiction. If $|E(T')| = 4$, as $|E(T')| < s \leq |E(T')| + |\partial(T')|$ and Claim 1(iii), $x_0x_4 \notin E(G)$ and $x_0x_3, x_1x_4 \in E(G)$. As $dist_G(x_0, x_4) \leq 2$, we have $x_0x_2, x_2x_4 \in E(G)$. Therefore, the eulerian subgraph $H_2 = x_0x_1x_2x_0$ satisfies that $|E(H_2)| < s \leq |E(H_2)| + |\partial(H_2)|$ if $s = 5$, or $H_2 = x_2x_0x_1x_2x_4x_3x_2$ satisfies $|E(H_2)| \leq s \leq |E(H_2)| + |\partial(H_2)| = |E(T)| + |\partial(T)|$ if $s \geq 6$, contrary to (1). So $|E(T')| \geq 5$.

As $k = |E(T')| \geq 5$ and by Claim 1(iv), $x_1x_{k-1} \notin E(T)$. If $x_1x_{k-1} \in E(G)$, then the eulerian subgraph $H_3 = x_1x_2 \cdots x_{k-1}x_1$ satisfies $x_0x_1, x_{k-1}x_k \in \partial(H_3)$. By (1), $|E(T')| < s$. By Claim 1(iii), $|E(T')| + |\partial(T')| = |E(H_3)| + |\partial(H_3)|$ and so $|E(H_3)| < s \leq |E(H_3)|+|\partial(H_3)|$, contrary to (1). So $x_1x_{k-1} \notin E(G)$. As $dist_G(x_1, x_{k-1}) \leq 2$, there is a vertex w_2 such that $w_2x_1, w_2x_{k-1} \in E(G)$. By (1) and Claim 1(iii), $w_2 \in \{x_0, x_2, x_k, x_{k-2}\}$. By (2), $w_2 \in \{x_2, x_{k-2}\}$. Without loss of generality, we assume that $w_2 = x_2$. Thus $x_2x_{k-1} \in E(G)$. Let $H_4 = x_2x_3 \cdots x_{k-1}x_2$. Then $x_1x_2, x_{k-1}x_k \in \partial(H_4)$ and $|E(H_4)| = |E(T')| - 2$. If $|E(T')| = s$, then H_4 is an eulerian subgraph with $|E(H_4)| < s \leq |E(H_4)| + |\partial(H_4)|$, contrary to (1). So $|E(T')| \leq s - 1$. Thus $x_0x_k \notin E(G)$, otherwise, the eulerian subgraph $H_5 = x_0x_1 \cdots x_kx_0$ satisfies $|E(H_5)| = |E(T')| + 1 \leq s$ and $|E(H_5)| + |\partial(H_5)| = |E(T')| + |\partial(T')| \geq s$, contrary to (1).

Assume that $|E(T')| = s - 1$. As $|E(H_4)| = |E(T')| - 2 = s - 3$ and $x_1x_2, x_kx_{k-1} \in \partial(H_4)$, $|E(H_4)| + |\partial(H_4)| \geq s - 1$. By (1), $|E(H_4)|+|\partial(H_4)| = s-1$, and so $\partial(H_4) = \{x_1x_2, x_kx_{k-1}\}$. By Claim 1(iii), $d_G(x_0) = 1$ and so $dist(x_0, x_{k-1}) = 3$, a contradiction. So $|E(T')| \leq s - 2$. As $dist_G(x_0, x_k) \leq 2$, there is a vertex w_3 such that $w_3x_0, w_3x_k \in E(G)$. By (2), $w_3 \notin \{x_1, x_{k-1}\}$. By Claim 1(ii), $w_3x_0, w_3x_k \notin E(T')$. Thus the eulerian subgraph $H_6 = x_0x_1 \cdots x_kw_3x_0$ satisfies $|E(H_6)| = |E(T')| + 2 \leq s \leq |E(H_6)| + |\partial(H_6)|$, a contradiction. ■

Proof of Theorem 1.5. If $L(G)$ is pancyclic, then $L(G)$ contains C_k with $3 \leq k \leq |E(G)|$. But $L(C_4)$ has no 3-cycle, $L(C_5)$ has no 3-cycle and 4-cycle, $L(P(10))$ has no 4-cycle. Thus $G \notin \{C_4, C_5, P(10)\}$. It remains to prove the sufficiency of Theorem 1.5. Let G be a connected graph with order n. By Lemma 2.6, we assume that $n = |V(G)| \geq 4$. By contradiction, assume that

$$G \text{ is a counterexample with } |V(G)| + |E(G)| \text{ is minimized.} \tag{4}$$

Suppose $g(G) \leq 2$. Then G has a 2-cycle $\{e_1, e_2\}$. Since G is a counterexample, there is an integer l_0 with $3 \leq l_0 \leq m$ such that $l_0 \notin sp_L(G)$. By (4), we have $l_0 \in sp_L(G - e_1)$. By Theorem 2.4, $G - e_1$ has an eulerian subgraph H' and $\partial_{(G-e_1)}(H')$ such that $|E(H')| \leq l_0 \leq |E(H')| + |\partial_{(G-e_1)}(H')|$. As H' is a subgraph of G and $\partial_G(H') = \partial_{(G-e_1)}(H') \cup \{e_1\}$, $l_0 \in sp_L(G)$, a contradiction. So $g(G) \geq 3$.

If G has a dominating eulerian subgraph T with $t = |E(T)|$, then $t \leq |E(T)| + |\partial(T)| \leq m$. Thus for any integer $l \in \{t, t + 1, \ldots, m\}$, $l \in sp_L(G)$. For $l < t$, let T' be a section of T such that $|E(T')| = l$. By Lemma 3.2, $l \in sp_L(G)$, contrary to (4). So

$$G \text{ has no a dominating eulerian subgraph.} \tag{5}$$

Therefore, G is not collapsible. Let G' be the reduction of G. Then $diam(G') \leq 2$. By Theorem 3.1, $G' \in \{K_{1,t}, K_{2,t}, S_{s,k}, P(10)\}$. If $G' = K_{1,t}$, then G is spanned by $K_{1,n-1}$. By Lemma 2.6(i) we conclude that $L(G)$ is pancyclic, contrary to (4). If $G' \in \{S_{s,k}, P(10)\}$, as $diam(G) \leq 2$, we have $G = G'$. As $G \notin \{C_5, P(10)\}$, we have $G = S_{s,k}$, where $s + k \geq 3$. Thus G has a dominating eulerian subgraph, contrary to (5). If $G' = K_{2,t}$, then all vertices of degree 2 are trivial and at most one vertex of degree t is nontrivial. Thus G has a dominating eulerian subgraph, contrary to (5). ■

4. Proof of Theorem 1.6

Let $v_1, v_2 \in V(P(10))$ such that $v_1v_2 \notin E(P(10))$. Denote $P^+(10) = P(10) + v_1v_2$. To prove Theorem 1.6, it suffices to prove that if $\kappa(L(G)) \geq 3$, then $L(G)$ is 1-hamiltonian. If G is spanned by a $K_{1,n-1}$, then Theorem 1.6 holds by Theorem 2.1 and Lemma 2.6(ii). Thus we may assume that $\kappa(G) \geq 2$. If $G = P^+(10)$, then, for any $e \in E(P^+(10))$, $P^+(10) - e$ has a dominating eulerian subgraph. Thus $L(P^+(10))$ is 1-hamiltonian. In the next discussion, we will assume that $G \neq P^+(10)$. Define a ($\geq$ **3**)-**DES** of G to be a dominating eulerian subgraph of G that contains all vertices of degree at least 3. We will prove Theorem 1.6 by showing a slightly stronger result as follow.

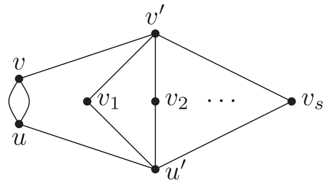

Fig. 1. The graph $G - e_0$ in Claim 1.

Theorem 4.1. *If $G \neq P^+(10)$ is a 2-connected graph with $ess'(G) \geq 3$ and $diam(G) \leq 2$, then for any edge $e \in E(G)$, $G - e$ has a (≥ 3)-DES.*

Proof. By contradiction, we assume that

$$G \text{ is a counterexample with } |V(G)| \text{ minimized.} \tag{6}$$

In particular,

$$\text{there exists an edge } e_0 \in E(G) \text{ such that } G - e_0 \text{ has no } (\geq 3) - DES. \quad \blacksquare \tag{7}$$

Claim 1. $G - e_0$ is reduced.

Proof of Claim 1. Let $H = G - e_0$. For proving this claim by contradiction, we assume that K is a nontrivial maximal collapsible subgraph in H. By (6), H/K has a (≥ 3)-DEST$'$. If $v_K \in V(T')$, by Theorem 2.2, H has a eulerian subgraph T with $E(T') \subseteq E(T)$ and $V(K) \subseteq V(T)$. Thus T is a (≥ 3)-DES of H, contrary to (7). So $v_K \notin V(T')$. Therefore, $d_{H/K}(v_K) = 2$.

Let $N_H(V(H) - K) \cap V(K) = \{u, v\}$. As K is a maximal collapsible subgraph, $N_H(u) \cap N_H(v) \subseteq V(K)$. As $ess'(G) \geq 3$, e_0 is incident to one of vertex in K. Assume that $|V(K)| \geq 3$. As $diam(G) \leq 2$, any vertex not in $V(K)$ is adjacent to either u or v. As $d_{H/K}(v_K) = 2$, $|V(H) - V(K)| = 2$. Let $V(H) - V(K) = \{a, b\}$ with $au, bv \in E(G - e_0)$. As G is 2-connected, $ab \in E(G - e_0)$. Thus H/K is a triangle abv_Ka, and so H has a spanning eulerian subgraph, contrary to (7). So K is a 2-cycle vuv. Thus $|V(H) - V(K)| \geq 2$.

Let $N_G(v) - \{u\} = \{v'\}$, $N_G(u) - \{v\} = \{u'\}$, $\{v_1, v_2, \ldots, v_s\} = N_G(v') - \{v\}$, $\{u_1, u_2, \ldots, u_t\} = N_G(u') - \{u\}$. As $diam(G) \leq 2$, $\{v_1, v_2, \ldots, v_s\} = \{u_1, u_2, \ldots, u_t\}$. Thus $G - e_0$ is the graph depicted in Fig. 1, and so $G - e_0$ must have a (≥ 3)-DES, contrary to (7). Hence Claim 1 holds. $\blacksquare$

Claim 2. G has no 4-cycles.

Proof of Claim 2. By contradiction, we assume that G has a 4-cycle $C_4 = x_1x_2y_1y_2x_1$. Define $G' = G/\pi(C_4)$ with a partition $\pi(C_4) = \langle\{x_1, y_1\}, \{x_2, y_2\}\rangle$. Following the notation in Definition 2.7, $e_\pi = e_{\pi(C_4)} = v_1v_2 \in E(G')$. By (6), $G' - e_0$ has a (≥ 3)-DES. If $ess'(G') = 1$, then there are two vertices $x, y \in V(G')$ such that $d_{G'}(x, y) \geq 3$. This contradicts to Lemma 2.9(i). Thus $ess'(G') \geq 2$. $\blacksquare$

Claim 2.1. $ess'(G') \geq 3$.

By contradiction, we assume that $ess'(G') = 2$. Then G' has a 2-edge-cut X such that $G' - X$ has two nontrivial components L_1 and L_2 with $|V(L_1)| \leq |V(L_2)|$. As $ess'(G) \geq 3$, $e_\pi = v_1v_2 \in X$. Let $X = \{v_1v_2, u_1u_2\}$ such that $v_1, u_1 \in V(L_1)$ and $v_2, u_2 \in V(L_2)$. As $diam(G) \leq 2$, we must have $V(L_1) = \{u_1, v_1\}$. Let $V(L_2) = \{v_2, u_2, w_1, w_2, \ldots, w_t\}$ and $W = \{w_1, \ldots, w_t\}$. Since X is an essential edge-cut of G', $N_G(u_1) \cap \{x_1, y_1\} \neq \emptyset$. Without loss of generality, we assume that $u_1y_1 \in E(G)$.

If $t = 0$, then $N_G(u_2) \cap \{x_2, y_2\} \neq \emptyset$. As $ess'(G) \geq 3$, we have either $|N_G(u_1) \cap \{x_1, y_1\}| = 2$ or $|N_G(u_2) \cap \{x_2, y_2\}| = 2$. Without loss of generality, we assume that $u_2x_2, u_2y_2 \in E(G)$. As $F(G - e_0) \geq 3$ and $|V(G - e_0)| = 6$, by Theorem 2.2(v), we have $|E(G - e_0)| \leq 7$. Thus $u_1x_1 \notin E(G)$. As $u_1u_2y_2x_1x_2y_1u_1$ and $u_1u_2x_2x_1y_2y_1u_1$ are hamiltonian cycles of G, $e_0 \notin \{y_1y_2, y_1x_2, u_2y_2, u_2x_2\}$. Thus $u_2y_2y_1x_2u_2$ is a (≥ 3)-DES in $G - e_0$, a contradiction. So $t \geq 1$.

As $diam(G) \leq 2$ and $diam(G') \leq 2$, $u_2w_i \in E(G)$ and $v_2w_i \in E(G')$ for $i = 1, \ldots, t$. Thus $N_G(w_i) \cap \{y_2, x_2\} \neq \emptyset$. Let $W_1 = \{x \in W | xy_2 \in E(G)\}$ and $W_2 = W - W_1$. Then for any $x \in W_2$, $xx_2 \in E(G)$. Let $E_1 = \{e \in E(G) | e = xy_2, x \in W_1\}$ and $E_2 = \{e \in E(G) | e = xx_2, x \in W_2\}$.

Assume that $y_2x_2 \in E(G)$, or $x_1y_1 \in E(G)$, or $y_2u_2 \in E(G)$ with $e_0 = y_2u_2$. By Claim 1, $e_0 = x_2y_2$ if $y_2x_2 \in E(G)$, and $e_0 = x_1y_1$ if $x_1y_1 \in E(G)$. If $|W_1|$ is odd and $|W_2|$ is even, then $G[E_1 \cup E_2 \cup \{u_2w | w \in W\} \cup \{u_2u_1, u_1y_1, y_1x_2, x_2x_1, x_1y_2\}]$ is a spanning eulerian subgraph of $G - e_0$; if $|W_1|$ is even and $|W_2|$ is odd, then $G[E_1 \cup E_2 \cup \{u_2w | w \in W\} \cup \{u_2u_1, u_1y_1, y_1y_2, y_2x_1, x_1x_2\}]$ is a spanning eulerian subgraph of $G - e_0$, contrary to (7). So either both $|W_1|$ and $|W_2|$ are odd, or both $|W_1|$ and $|W_2|$ are even. Notice that if $u_1x_1 \notin E(G)$, then $G[E_1 \cup E_2 \cup \{u_2w | w \in W\} \cup \{x_2x_1, x_1y_2, y_2y_1, y_1x_2\}]$ is a (≥ 3)-DES of $G - e_0$ if both $|W_1|$ and $|W_2|$ are even, and $G[E_1 \cup E_2 \cup \{u_2w | w \in W\} \cup \{x_2y_1, y_1y_2\}]$ is a (≥ 3)-DES of $G - e_0$ if both $|W_1|$ and $|W_2|$ are odd. By (7), $u_1x_1 \in E(G)$. Since $G[E_1 \cup E_2 \cup \{u_2w | w \in W\} \cup \{x_2x_1, x_1u_1, u_1y_1, y_1y_2\}]$ is a spanning eulerian subgraph of $G - e_0$ if both $|W_1|$ and $|W_2|$ are odd, we have both $|W_1|$ and $|W_2|$ are even. As $t \geq 1$, we

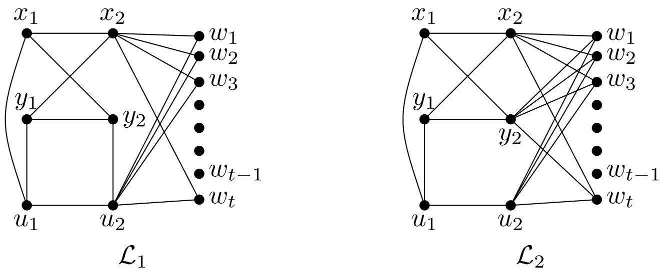

Fig. 2. The graph G in Claim 2.1.

may assume that $|W_1| \geq 2$. Then $G[E_1 \cup E_2 \cup \{u_2w|w \in W\} \cup \{x_2x_1, x_1u_1, u_1y_1, y_1x_2\}]$ is a spanning subgraph of $G - e_0$, contrary to (7). So $y_2x_2, y_1x_1 \notin E(G)$, and if $y_2u_2 \in E(G)$, then $e_0 \neq y_2u_2$. As $dist(x_1, u_1) \leq 2$, we have $u_1x_1 \in E(G)$.

Assume that $y_2u_2 \in E(G)$ and $W_1 \neq \emptyset$. Without loss of generality, we assume that $x_2w_1 \in E(G)$. Then $e_0 \in \{x_2w_1, u_2w_1\}$. Thus $|W_1| = 1$. So $|V(G)| = t + 6$ and $|E(G)| \geq 8 + 2(t-1) + d_G(w_1)$. By Theorem 2.2(v) and Claim 1, $F(G - e_0) = 2|V(G - e_0)| - |E(G - e_0)| - 2 \leq 5 - d_G(w_1)$. By Theorem 2.3, $F(G - e_0) \geq 3$. Thus $d_G(w_1) = 2$. If t is even, then $G[E_1 \cup E_2 \cup \{u_2w|w \in W - \{w_1\}\} \cup \{u_2u_1, u_1y_1, y_1y_2, y_2x_1, x_1x_2\}]$ is a (≥ 3)-*DES* in $G - e_0$, contrary to (7). So t is odd. Thus $G[E_1 \cup E_2 \cup \{u_2w|w \in W - \{w\}\} \cup \{u_2u_1, u_1y_1, y_1x_2, x_2x_1, x_1y_2, y_2u_2\}]$ is a (≥ 3)-*DES* in $G - e_0$, a contradiction. So, if $y_2u_2 \in E(G)$, then $W_1 = \emptyset$. Thus G is spanned by a graph $\mathcal{L}_1$ or $\mathcal{L}_2$ (see Fig. 2).

If $y_2u_2 \notin E(G)$, then $|E(G - e_0)| \geq 6 + 3t$. Thus $F(G - e_0) \leq 2(t+6) - (6+3t) - 2 = 4 - t$. By Theorem 2.3, $F(G - e_0) \geq 3$. Thus $t = 1$. So $G - e_0$ has a (≥ 3)-DES, contrary to (7). So $y_2u_2 \in E(G)$, and G is spanned by $\mathcal{L}_1$. As $|V(G - e_0)| = t + 6$, $|E(G - e_0)| \geq 2t + 7$, and $F(G - e_0) \geq 3$, we have $d_G(w_i) = 2$ for $w_i \in W$. Let $E_{x_2} = \{e \in E(G)|e = x_2w, w \in W\}$ and $E_{u_2} = \{e \in E(G)|e = u_2w, w \in W\}$. Notice that for any $w \in W$, $F(G - w) \leq 2$, implying that $G - w$ is collapsible. By Claim 1, $e_0 \notin E_{x_2} \cup E_{u_2}$. Since $G[\{w_1x_2, x_2x_1, x_1u_1, u_1y_1, y_1y_2, y_2u_2, u_2w_1\}]$ and $G[\{w_1x_2, x_2y_1, y_1y_2, y_2x_1, x_1u_1, u_1u_2, u_2w_1\}]$ are (≥ 3)-DES of G, we have $e_0 \in \{y_1y_2, u_1x_1\}$. So $G[\{x_2y_1, y_1u_1, u_1u_2, u_2y_2, y_2x_1, x_1x_2\}]$ is a (≥ 3)-DES of $G - e_0$, contrary to (7). So Claim 2.1 holds.

If $e_0 \in E(C_4)$, we use e_0 to denote v_1v_2 in G'. By the minimality of G, we assume that Γ is a (≥ 3)-*DES* of $G' - e_0$. Let $H = G[E(\Gamma) - \{e_\pi\}]$.

Claim 2.2. $e_0 \in E(C_4)$.

Suppose that $e_0 \neq e_\pi = v_1v_2$. If $e_\pi \in E(\Gamma)$, then $d_H(x_1) + d_H(y_1)$ is odd and $d_H(x_2) + d_H(y_2)$ is odd. Without loss of generality, we assume that $d_H(x_1)$ is odd. If $d_H(x_2)$ is odd, then $H_1 = H + \{x_1y_2, y_2y_1, y_1x_2\}$ is a (≥ 3)-*DES*; if $d_H(y_2)$ is odd, then $H_2 = H + \{x_1x_2, x_2y_1, y_1y_2\}$ is (≥ 3)-*DES*, a contradiction. So $e_\pi \notin E(\Gamma)$.

Assume that $v_1 \in V(\Gamma)$. If $v_2 \notin V(\Gamma)$, then $d_{G'-e_0}(v_2) \leq 2$. Thus $d_{G-e_0}(x_2) + d_{G-e_0}(y_2) \leq 5$. Without loss of generality, we assume that $d_{G-e_0}(y_2) = 2$ and $d_{G-e_0}(x_2) \leq 3$. If both $d_H(x_1)$ and $d_H(y_1)$ are odd, then $H_3 = H + \{x_1x_2, x_2y_1\}$ is a (≥ 3)-*DES*; if $d_H(x_1)$ and $d_H(y_1)$ are even, then $H_4 = H + \{x_1x_2, x_2y_1, y_1y_2, y_2x_1\}$ is a (≥ 3)-*DES*, a contradiction. So $v_2 \in V(\Gamma)$. If $d_H(x)$ is even for $x \in \{x_1, x_2, y_1, y_2\}$, then $H_5 = H + \{x_1x_2, x_2y_1, y_1y_2, y_2x_1\}$ is a (≥ 3)-*DES*, a contradiction; If $d_H(x)$ is odd for $x \in \{x_1, x_2, y_1, y_2\}$, as Γ is connected, we may assume that x_1, x_2 are same component of H, and y_1, y_2 are also on same component of H. Then $H_6 = H + \{x_1y_2, y_1x_2\}$ is a (≥ 3)-*DES*, a contradiction. So we may assume that $d_H(x_1)$ and $d_H(y_1)$ are odd and $d_H(x_2)$ and $d_H(y_2)$ are even. Since Γ is connected, we assume that x_1, y_1, x_2 are on the same component of H. Then $H_7 = H + \{x_1y_2, y_1y_2\}$ is a (≥ 3)-*DES*, a contradiction. So $v_1 \notin V(\Gamma)$. Similarly, $v_2 \notin V(\Gamma)$.

Therefore, $d_{G'-e_0}(v_1) \leq 2$ and $d_{G'-e_0}(v_2) \leq 2$. By Claim 2.1, $d_{G'-e_0}(v_1) = 2$ and $d_{G'-e_0}(v_2) = 2$. Let $N_{G'-e_0}(v_1) = \{v_2, w_1\}$ and $N_{G'-e_0}(v_2) = \{v_1, w_2\}$. Without loss of generality, we assume that w_1 is adjacent to y_1 and w_2 is adjacent to y_2. Since $\kappa'(G) \geq 3$, e_0 is incident to either x_1 or x_2. Without loss of generality, we assume that e_0 is incident to x_2. By Claim 1, $w_1x_2, w_1y_2, x_1y_1 \notin E(G)$. Thus $d_G(x_1) = 2$. So $dist_G(x_1, w_1) = 3$, a contradiction. Claim 2.2 holds.

By Claim 2.2, we have

$$girth(G - e_0) \geq 5. \tag{8}$$

By (8), $N_G(x_1) \cap N_G(y_1) \subseteq \{x_2, y_2\}$ and $N_G(x_2) \cap N_G(y_2) \subseteq \{x_1, y_1\}$. By Claim 2.1 and 2.2, $v_1v_2 \notin E(\Gamma)$ and $x_1y_1, x_2y_2 \notin E(G)$.

Claim 2.3. $d_G(x) \geq 3$ for $x \in \{x_1, x_2, y_1, y_2\}$.

Assume that $d_G(y_1) = 2$. If $d_G(x_1) = 2$, without loss of generality, we assume that $e_0 = x_1x_2$. By Claim 1, $G - x_1$ is reduced. As $diam(G) \leq 2$ and $y_1x_2, y_1y_2 \in E(G)$, we have $diam(G - x_1) \leq 2$. Notice that $d_{G-x_1}(y_1) = 2$ and $ess'(G) \geq 3$. By Theorem 3.1, $G - x_1 \in \{K_{2,n-3}, S_{t_1,t_2}\}$, where $n = |V(G)|$ and $t_1 + t_2 = n - 4$. Thus $G \in \{K_{2,n-2}, S_{t_1+1,t_2}, S_{t_1,t_2+1}\}$. So $G - e_0$ has a (≥ 3)-DES, contrary to (7). So $d_G(x_1) \geq 3$. Let $N_G(x_1) = \{x_2, y_2, a_1, \ldots, a_s\}(s \geq 1)$. For $i = 1, \ldots, s$, as $dist_G(a_i, y_1) \leq 2$ and as $d_G(y_1) = 2$ and $x_1y_1 \notin E(G)$, $N_G(a_i) \cap \{x_2, y_2\} \neq \emptyset$. Without loss of generality, we assume that $a_1x_2 \in E(G)$. Then $e_0 = x_1x_2$, $a_ix_2 \in E(G)$ and $a_iy_2 \notin E(G)$. By (8), $s = 1$. As $e_0 = v_1v_2$, $v_1 \notin V(\Gamma)$. Thus $a_1 \in V(\Gamma)$ and $d_\Gamma(v_2)$ is even. Therefore, both $d_H(x_2)$ and $d_H(y_2)$ are either even or odd. If $d_H(x_2)$ and $d_H(y_2)$ are even, then $H_1 = \begin{cases} H + \{a_1x_1, x_1y_2, y_2y_1, y_1x_2, x_2a_1\}, & \text{if } a_1x_2 \notin E(H) \\ H - a_1x_2 + \{a_1x_1, x_1y_2, y_2y_1, y_1x_2\}, & \text{if } a_1x_2 \in E(H) \end{cases}$ is a (≥ 3)-*DES* of $G - e_0$; if $d_H(x_2)$ and $d_H(y_2)$ are odd, then $H_2 = H + \{x_2y_1, y_1y_2\}$ is a (≥ 3)-*DES* of $G - e_0$, a contradiction. So Claim 2.3 holds.

By Claim 2.2, we may assume that $e_0 = x_1x_2$. By (8), we have $N_G(y_1) \cap N_G(y_2) = \emptyset$ and $|N_G(x_1) \cap N_G(x_2)| \leq 1$. Let $A = N_G(x_1) \cap N_G(x_2)$ (probably $A = \emptyset$). Let $A_1 = N_G(x_1) - (\{x_2, y_2\} \cup A)$, $B_1 = N_G(y_1) - \{x_2, y_2\}$, $A_2 = N_G(x_2) - (\{x_1, y_1\} \cup A)$ and $B_2 = N_G(y_2) - \{x_1, y_1\}$. Then any two of A, A_1, A_2, B_1, B_2 are disjoint. Let $S = A \cup A_1 \cup B_1 \cup A_1 \cup B_2 \cup \{x_1, y_1, x_2, y_2\}$. By (8), $A_1 \cup A$, $A_2 \cup A$, B_1, B_2 are independent, and for any $x, x' \in A_1$, $N_G(x) \cap N_G(x') = \{x_1\}$. By (8), if $z \in A$, then $N_G(z) \cap S = \{x_1, x_2\}$. Thus $|A| \leq 1$.

Let $x \in A_1$. Since $dist_G(x, y_1) \leq 2$, there is a vertex $y \in B_1$ such that $xy \in E(G)$. By (8), such the vertex y is unique. Thus $|A_1| \leq |B_1|$. Similarly, $|B_1| \leq |A_1|$. So $|A_1| = |B_1|$. Similarly, $|A_2| = |B_2|$. As $e_0 = x_1x_2$, by (8), $E(G[B_1 \cup B_2]) = \emptyset$. Let $A_1 = \{a_{11}, \ldots, a_{1s}\}$ and $B_1 = \{b_{11}, \ldots, b_{1s}\}$, and let $A_2 = \{a_{21}, \ldots, a_{2t}\}$ and $B_1 = \{b_{21}, \ldots, b_{2t}\}$. Then $E(G[A_1 \cup B_1])$ and $E(G[A_2 \cup B_2])$ consist of matchings of size s and t, respectively. Without loss of generality, we assume that $E(G[A_1 \cup B_1]) = \{a_{11}b_{11}, \ldots, a_{1s}b_{1s}\}$ and $E(G[A_2 \cup B_2]) = \{a_{21}b_{21}, \ldots, a_{2t}b_{2t}\}$.

Consider b_{1i} and b_{2j}, where $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, t\}$. As $dist_G(b_{1i}, b_{2j}) \leq 2$, there is a vertex w_{ij} such that $b_{1i}w_{ij}, w_{ij}b_{2j} \in E(G)$. By (8), w_{ij} are different vertices and $w_{ij} \notin S$. Let $i \in \{1, \ldots, s\}$. For $j = 1, \ldots, t$, as $dist_G(x_1, w_{ij}) \leq 2$, there exists a vertex $p_1 \in A_1 \cup A - \{a_{1i}\}$ such that $p_1w_{ij} \in E(G)$. By (8), $|N_G(p_1) \cap \{w_{i1}, w_{i2}, \ldots, w_{it}\}| = 1$. Thus $s \geq t$. Similarly, let $j \in \{1, \ldots, t\}$. Then for $i = 1, \ldots, s$, there exists a vertex $p_2 \in A_2 \cup A - \{a_{2j}\}$ such that $p_2w_{ij} \in E(G)$, $|N_G(p_2) \cap \{w_{1j}, w_{2j}, \ldots, w_{sj}\}| = 1$, and $t \leq s$. So $s = t$ and $A \neq \emptyset$. Therefore, $|A| = 1$. Assume that $A = \{z\}$. Let $Q = \{w_{ij} | i = 1, \ldots, t, j = 1, \ldots, t\}$ and let Y_1 be the subgraph of G induced by $S \cup Q$ and $Y = Y_1 - e_0$. Then $|V(Y)| = t^2 + 4t + 5$ and $|E(Y)| \geq 4t^2 + 6t + 5$, and so $F(Y) \leq 2(t^2 + 4t + 5) - (4t^2 + 6t + 5) - 2 = -2t^2 + 2t + 3$. As $F(Y) \geq 3$, $t = 1$ and $zw_{11} \in E(G)$.

Assume that $a_{11}a_{21} \notin E(G)$. As $dist_G(a_{11}, a_{21}) \leq 2$, there is a vertex w_2 such that $w_2a_{11}, w_2a_{21} \in E(G)$. By (8), $w_2 \notin S \cup \{w_{11}\}$ and $N_G(w_2) \cap \{x_2, y_2, b_{11}, y_1\} = \emptyset$. So $dist_G(w_2, y_1) = 3$, a contradiction. So $a_{11}a_{21} \in E(G)$.

Let $w_3 \in V(G) - (S \cup \{w_{11}\})$. Then $N_G(w_3) \cap \{x_1, x_2, y_1, y_2\} = \emptyset$. As $dist_G(w_2, x_1) \leq 2$, $N_G(w_3) \cap \{a_{11}, z\} \neq \emptyset$. As $dist_G(w_3, y_1) \leq 2$, $w_3b_{11} \in E(G)$. This would result in a 4-cycle in $G - e_0$, a contradiction. So $V(G) = S \cup \{w_{11}\}$, $G - e_0 = P(10)$ and $G = P^+(10)$, a contradiction. So Claim 2 holds. ∎

Let $e_0 = u_0v_0$. Also we assume that $d_G(u_0) \geq d_G(v_0)$. Since G is essentially 3-edge-connected, we have $d_G(u_0) \geq 3$. Let $d_G(u_0) = d$ and $N_G(u_0) = \{v_0, v_1, \ldots, v_{d-1}\}$. If G contains a triangle, we assume that this triangle is $u_0v_0v_1u_0$. By Claim 2, G has no 4-cycle. Therefore, we have the following observation.

Observation 4.2. *For each $i = 0, 1, \ldots, d-1$, let $N_i = N_G(v_i) - \{u_0\}$ and denote $N_i = \{z_1^i, z_2^i, \ldots, z_{t_i}^i\}$. Since $diam(G) \leq 2$ and G has no 4-cycle, the graph G has the following properties.*

(*a*) *If $i \neq i'$, then $N_i \cap N_{i'} = \emptyset$.*
(*b*) *Suppose that $i \neq d-1$. Since the distance between any vertex in N_i and v_{d-1} is at most 2, and since G has no 4-cycle, we conclude that $t_i = t_{d-1} = t$ is a constant. Thus $d \geq t+1$.*
(*c*) *Suppose that $(i, i') \neq (0, 1)$ when $v_0v_1 \in E(G)$. Since the distance between any vertex in N_i and $v_{i'}$ is at most 2, and since G has no 4-cycle, we conclude that there must be a permutation $\pi_{i,i'}$ on $\{1, 2, 3, \ldots, t\}$, such that for every $j \in \{1, 2, \ldots, t\}$, $z_j^i z_k^{i'} \in E(G)$, where $k = \pi_{i,i'}(j)$. Thus, for $x \in N_2 \cup \cdots \cup N_{d-1}$, $d_G(x) \geq d$. In addition, for $x \in N_0 \cup N_1$, we have $d_G(x) \geq d$ if $v_0v_1 \notin E(G)$ and $d_G(x) \geq d-1$ if $v_0v_1 \in E(G)$.*
(*d*) *Assume that $t = 1$. If $d \geq 4$, by Observation 4.2(c), we have $z_1^2z_1^0, z_1^2z_1^1, z_1^{d-1}z_1^0, z_1^{d-1}z_1^1 \in E(G)$. This would result in a 4-cycle, a contradiction. So $d = 3$. By Observation 4.2(c), $z_1^2z_1^0, z_1^2z_1^1 \in E(G)$. By Claim 1, $z_1^0z_1^1 \notin E(G)$. As $dist_G(z_1^0, v_1) \leq 2$, we have $v_0v_1 \in E(G)$. Thus $u_0v_2z_1^2z_1^0v_0v_1u_0$ is a spanning eulerian subgraph of $G - e_0$, a contradiction. So $t \geq 2$ and $d \geq 3$.*
(*e*) *Assume that $t = 2$. If $d = 3$, by Observation 4.2(c), we assume that $z_1^2z_1^0, z_1^2z_1^1, z_2^2z_2^0, z_2^2z_2^1 \in E(G)$. Since $dist_G(z_2^0, z_1^1) \leq 2$, $z_2^0z_1^1 \in E(G)$. Similarly, $z_1^0z_2^1 \in E(G)$. By Claim 2, $v_0v_1 \notin E(G)$. Thus G is the Petersen graph. So $G - e_0$ has a (≥ 3)-DES, a contradiction. So if $t = 2$, then $d \geq 4$.*

Claim 3. $v_0v_1 \in E(G)$.

Proof of Claim 3. Assume that $v_0v_1 \notin E(G)$. By Observation 4.2(c), $2|E(G - e_0)| \geq td^2 + d(t+2) - 2$. As $|V(G - e_0)| = 1 + d + td$, we have

$$2F(G - e_0) \leq 4td + 4d + 4 - (td^2 + d(t+2) - 2) - 4 = 3dt + 2d - td^2 + 2. \tag{9}$$

Since $G - e_0$ is reduced, $\delta(G - e_0) \leq 3$. Thus $t \in \{2, 3\}$. If $t = 2$, by (9), $F(G - e_0) \leq 4d - d^2 + 1 \leq 1$ since $d \geq 4$. So $G - e_0$ is collapsible, contrary to Claim 1. If $t = 3$, then $d \geq t + 1 \geq 4$. By (9), $F(G - e_0) \leq \frac{1}{2}(11d - 3d^2 + 2) \leq 1$. So $G - e_0$ is collapsible, contrary to Claim 1 again. So Claim 3 holds. ∎

By Claim 3, $v_0v_1 \in E(G)$. As G has no 4-cycles, $E(G[N_0 \cup N_1]) = \emptyset$. By Observation 4.2(c), $2|E(G - e_0)| \geq d + (t+1)d + (d-2)td + 2t(d-1) = dt + d^2t + 2d - 2t$. As $|V(G - e_0)| = 1 + d + td$, we have

$$2F(G - e_0) \leq 4 + 4d + 4dt - (dt + d^2t + 2d - 2t) - 4 = 2d + 3dt - d^2t + 2t \tag{10}$$

As $\delta(G - e_0) \leq 3$, $t \in \{2, 3\}$. If $t = 2$, then $F(G - e_0) \leq 4d - d^2 + 2 \leq 2$ since $d \geq 4$. Thus $G - e_0$ is collapsible, a contradiction. If $t = 3$, then $d \geq t + 1 \geq 4$. By (10), $F(G - e_0) \leq \frac{1}{2}(11d - 3d^2 + 6) \leq 1$. So $G - e_0$ is collapsible, contrary to Claim 1. ∎

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment

The research of Xiaoling Ma is supported by the Natural Science Foundation of China (grant number 11701490).

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