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On s-hamiltonicity of net-free line graphs

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ABSTRACT

For integers $s_1, s_2, s_3 > 0$, let N_{s_1,s_2,s_3} denote the graph obtained by identifying each vertex of a K_3 with an end vertex of three disjoint paths P_{s_1+1}, P_{s_2+1}, P_{s_3+1} of length s_1, s_2 and s_3, respectively. We prove the following results.

(i) Let $\mathcal{N}_1 = \{N_{s_1,s_2,s_3} : s_1 > 0, s_1 \geq s_2 \geq s_3 \geq 0 \text{ and } s_1+s_2+s_3 \leq 6\}$. Then for any $N \in \mathcal{N}_1$, every N-free line graph $L(G)$ with $|V(L(G))| \geq s+3$ is s-hamiltonian if and only if $\kappa(L(G)) \geq s+2$.

(ii) Let $\mathcal{N}_2 = \{N_{s_1,s_2,s_3} : s_1 > 0, s_1 \geq s_2 \geq s_3 \geq 0 \text{ and } s_1+s_2+s_3 \leq 4\}$. Then for any $N \in \mathcal{N}_2$, every N-free line graph $L(G)$ with $|V(L(G))| \geq s+3$ is s-Hamilton-connected if and only if $\kappa(L(G)) \geq s+3$.

1. Introduction

We consider finite graphs without loops but permitting multiple edges, and follow [1] for undefined terms and notations. In particular, for a graph G, $\kappa(G)$, $\kappa'(G)$, $\delta(G)$ and $\Delta(G)$ denote the connectivity, edge-connectivity, the minimum degree and the maximum degree of G, respectively. We use $c(G)$ and $g(G)$ to denote the **circumference** and the **girth** of G, which are the length of a longest cycle in G and the length of a shortest cycle of G, respectively. A graph is trivial if it has no edges. We write $H \subseteq G$ to mean that H is a subgraph of G. If $X \subseteq E(G)$, then $G[X]$ is the subgraph of G induced by X. If H and K are subgraphs of a graph G, then we define $H \cup K = G[E(H) \cup E(K)]$. Throughout this paper, we use P_k to denote a path of order k. For integers $s_1, s_2, s_3 \geq 0$, let N_{s_1,s_2,s_3} denote the graph formed by identifying each vertex of a K_3 with an end vertex of three disjoint paths P_{s_1+1}, P_{s_2+1}, P_{s_3+1} of length s_1, s_2, and s_3, respectively. A graph G is $\{H_1, H_2, \ldots, H_s\}$**-free** if G contains no induced subgraph isomorphic to any copy of H_i for any i. If $s = 1$, then an $\{H_1\}$-free graph is simply called an H_1-free graph. A **claw-free** graph is just a $K_{1,3}$-free graph. As in [1], a graph is hamiltonian if it has a spanning cycle and is Hamilton-connected if every pair of distinct vertices is joined by a spanning path.

The line graph of a graph G, denoted by $L(G)$, is a simple graph with vertex set $E(G)$, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. A few most fascinating problems in this area are presented below. By an ingenious argument of Z. Ryjáček [32], Conjecture 1.1(i) is equivalent to a seeming stronger conjecture of Conjecture 1.1(ii). In [33], it is shown that all conjectures stated in Conjecture 1.1 are equivalent to each other.

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Conjecture 1.1. *(i) (Thomassen [35]) Every 4-connected line graph is hamiltonian.*
(ii) (Matthews and Sumner [30]) Every 4-connected claw-free graph is hamiltonian.
(iii) (Kučel and Xiong [19]) Every 4-connected line graph is Hamilton-connected.
(iv) (Ryjáček and Vrána [33]) Every 4-connected claw-free graph is Hamilton-connected.

Towards Conjecture 1.1, Zhan gave a first result in this direction, and the best known result is given by Kaiser and Vrána, as shown below.

Theorem 1.2. *Let G be a graph.*
(i) (Zhan, Theorem 3 in [37]) If $\kappa(L(G)) \geq 7$, then L(G) is Hamilton-connected.
(ii) (Kaiser and Vrána [18]) Every 5-connected claw-free graph with minimum degree at least 6 is hamiltonian.
(iii) (Kaiser, Ryjáček and Vrána [17]) Every 5-connected claw-free graph with minimum degree at least 6 is 1-Hamilton-connected.

There have been many researches on hamiltonian properties in 3-connected claw-free graphs forbidding a $N_{k,0,0}$, as seen in the surveys in [2,11,13,14], among others. The following have been proved.

Theorem 1.3. *Let Q^* be the graph obtained from the Petersen graph by adding one pendant edge to each vertex. Let G be a 3-connected simple claw-free graph.*
(i) (Brousek, Ryjáček and Favaron, [4]) If G is $N_{4,0,0}$-free, then G is hamiltonian.
(ii) [24] If G is $N_{8,0,0}$-free, then G is hamiltonian. Moreover, the graph Q^* indicates the sharpness of this result.
(iii) (Fujisawa, [12], see also Ma et al. [29]) If G is $N_{9,0,0}$-free graph, then G is hamiltonian unless G is the line graph of Q^*.

It is natural to seek necessary and sufficient conditions for hamiltonicity of line graphs. For an integer $s \geq 0$, a graph G of order $n \geq s+3$ is ***s*-hamiltonian** (***s*-Hamilton-connected**, respectively), if for any $X \subseteq V(G)$ with $|X| \leq s$, $G - X$ is hamiltonian ($G - X$ is Hamilton-connected, respectively). It is well known that if a graph G is s-hamiltonian, then G is $(s+2)$-connected, and if G is s-Hamilton-connected, then G is $(s+3)$-connected. Broersma and Veldman in [3] initiated the problem of investigating graphs whose line graph is s-hamiltonian if and only if the connectivity of the line graph is at least $s+2$. They define, for an integer $k \geq 0$, a graph G to be ***k*-triangular** if every edge of G lies in at least k triangles of G. The following is obtained.

Theorem 1.4 (*Broersma and Veldman, [3]*)**.** *Let $k \geq s \geq 0$ be integers and let G be a k-triangular simple graph. Then L(G) is s-hamiltonian if and only L(G) is $(s+2)$-connected.*

Broersma and Veldman in [3] proposed an open problem of determining the range of an integer s such that within triangular graphs, $L(G)$ is s-hamiltonian if and only $L(G)$ is $(s+2)$-connected. This problem was first settled by Chen et al. in [10].

Theorem 1.5. *Each of the following holds.*
(i) (Chen et al. [10]) Let k and s be positive integers such that $0 \leq s \leq \max\{2k, 6k-16\}$, and let G be a k-triangular simple graph. Then L(G) is s-hamiltonian if and only L(G) is $(s+2)$-connected.
(ii) [21] Let G be a connected graph and let $s \geq 5$ be an integer. Then L(G) is s-hamiltonian if and only if L(G) is $(s+2)$-connected.

An hourglass is a graph isomorphic to $K_5 - E(C_4)$, where C_4 is a cycle of length 4 in K_5. The following are proved recently.

Theorem 1.6. *Each of the following holds.*
(i) (Kaiser, Ryjáček and Vrána [17]) Every 4-connected claw-free hourglass-free graph is 1-Hamilton-connected.
(ii) [25] For an integer $s \geq 2$, the line graph L(G) of a claw-free graph G is s-hamiltonian if and only if L(G) is $(s+2)$-connected.
(iii) [25] The line graph L(G) of a claw-free graph G is 1-Hamilton-connected if and only if L(G) is 4-connected.
(iv) (Hu and Zhang [16]) Every 3-connected $\{K_{1,3}, N_{1,2,3}\}$-free graph is Hamiltonian-connected.

In view of Conjecture 1.1 and motivated by Theorems 1.2, 1.3, 1.5 and 1.6, it is conjectured [21] that for any integer $s \geq 2$, $L(G)$ is s-hamiltonian if and only if $\kappa(L(G)) \geq s+2$. The main goal of this research is to investigate if $N_{8,0,0}$ in Theorem 1.3(ii) can be replaced by other N_{s_1,s_2,s_3} and if further evidences to support the conjecture in [21] can be found. The following results are obtained.

Theorem 1.7. *Let s be an integer.*
(i) Let $\mathcal{N}_1 = \{N_{s_1,s_2,s_3} : s_1 > 0, s_1 \geq s_2 \geq s_3 \geq 0$ and $s_1+s_2+s_3 \leq 6\}$. Then for any $N \in \mathcal{N}_1$, every N-free line graph L(G) with $|V(L(G))| \geq s+3$ is s-hamiltonian if and only if $\kappa(L(G)) \geq s+2$ for $s > 0$.
(ii) Let $\mathcal{N}_2 = \{N_{s_1,s_2,s_3} : s_1 > 0, s_1 \geq s_2 \geq s_3 \geq 0$ and $s_1+s_2+s_3 \leq 4\}$. Then for any $N \in \mathcal{N}_1$, every N-free line graph L(G) with $|V(L(G))| \geq s+3$ is s-Hamilton-connected if and only if $\kappa(L(G)) \geq s+3$ for $s \geq 0$.

Theorem 1.7 extends Theorem 1.3(i) in the context of line graph and furthers the main results in [36]. Let $O(G)$ be the set of odd degree vertices of a graph G. Following [1], a graph G is **eulerian** if G is connected with $O(G) = \emptyset$. A graph G is **supereulerian** if G contains a spanning eulerian subgraph. To prove Theorem 1.7, we prove an auxiliary theorem (Theorem 3.2 in Section 4), which leads to the following extension of Theorem 4 in [24].

Theorem 1.8. *Let G be a 2-edge-connected graph. Each of the following holds.*
(i) Let Γ be a graph with $\kappa'(\Gamma) \geq 3$ and $e \in E(\Gamma)$. If $G = \Gamma - e$ and $c(G) \leq 8$, then G is supereulerian.
(ii) If $c(G) \leq 8$ and G has at most two edge-cuts of size 2, then G is supereulerian.

Preliminaries and tools will be presented in the next section. In Sections 3 and 4, we assume the validity of a auxiliary theorem (Theorem 3.2 in Section 4) to prove Theorems 1.8 and 1.7, respectively. Theorem 3.2 will be proved in the last section.

2. Preliminaries

In [6] Catlin introduced collapsible graphs. It is shown in Proposition 1 of [22]) that a graph G is **collapsible** if for every subset $R \subseteq V(G)$ with $|R| = 0 \pmod 2$, G has a spanning connected subgraph Γ such that $O(\Gamma) = R$. See Catlin's survey [7] and it supplements [8,22] for further literature in this area. We use the notation that for a graph G and an integer $i \geq 0$, define $D_i(G) = \{v \in V(G) : d_G(v) = i\}$.

For a graph G and $X \subseteq E(G)$, the **contraction** G/X is the graph formed from G by contracting edges in X with resulting loops removed. We define $G/\emptyset = G$ and use G/e for $G/\{e\}$. When H is a subgraph of G, then we often use G/H for $G/E(H)$. If H is connected, then the vertex in G/H onto which H is contracted is denoted by v_H, and H is the **pre-image** of v_H in G. If $H_1, H_2, \ldots, H_k$ are all the maximal collapsible subgraphs of G, then $G' = G/(\cup_{i=1}^{k} H_i)$ is the **reduction** of G. A graph is **reduced** if it is the reduction of some graph. Let C_6^{++} denote the graph obtained from C_6 with $V(C_6) = \{v_1, v_2, \ldots, v_6\}$ by adding edges v_2v_5 and v_3v_6 and let $K_{3,3}^- = K_{3,3} - e$ for any edge $e \in E(K_{3,3})$. The next theorem briefs some of the useful results related to the reduction method of Catlin that would be used in this research.

Theorem 2.1. *Let G be a connected graph. Each of the following holds.*
(i) (Catlin, Theorem 8 of in [6]) If a connected graph G is reduced and not in $\{K_1, K_2\}$, then $|E(G)| \leq 2|V(G)| - 4$, $\delta(G) \leq 3$ and $g(G) \geq 4$.
(ii) (Catlin, Theorem 5 in [6]) G is reduced if and only if G has no nontrivial collapsible subgraphs. In particular, reduced graphs are simple graphs.
(iii) (Catlin, Corollary of Theorem 3 in [6]) Let H be a collapsible subgraph of G. Then G is supereulerian (collapsible, respectively) if and only if G/H is supereulerian (collapsible, respectively).
(iv) (Lemma 2.1 of [26]) Let G be a connected simple graph with $n \leq 8$ vertices and with $|D_1(G)| = 0$ and $|D_2(G)| \leq 2$, then the reduction of G is in $\{K_1, K_2, K_{2,3}\}$. Consequently, $K_{3,3}^-$ and C_6^{++} are collapsible.
(v) (Li et al. Lemma 2.2 of [26]) If G is collapsible, then for any $u, v \in V(G)$, G has a spanning (u, v)-trail.

A subgraph H of a graph G is **dominating** if $G - V(H)$ is edgeless. The following is well-known.

Theorem 2.2 (*Harary and Nash-Williams [15]*)**.** *For a connected graph G with $|E(G)| \geq 3$, $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

For a graph G and an integer $k > 0$, a k-edge-cut Y of G is an essential k-edge-cut of G if each component of $G - Y$ has an edge. If a connected graph G does not have an essential k'-edge-cut for any $k' < k$, then G is **essentially k-edge-connected**. The largest integer k such that a connected graph G is essentially k-edge-connected is denoted by $ess'(G)$. It is observed [34] that for a graph G, $\kappa(L(G)) \geq k$ if and only if either $L(G)$ is a complete graph of order at least $k + 1$ or $ess'(G) \geq k$.

Definition 2.3. Let $X_1(G) = \{e \in E(G) : e$ is incident with a vertex in $D_1(G)\}$. For each vertex $v \in D_2(G)$, let $E_G(v) = \{e_v, e'_v\}$ be the set of edges incident with v. The **core** of G is the graph G_0 defined below.

$$\begin{aligned} X_2(G) &= \{e_v : v \in D_2(G)\}, X'_2(G) = \{e'_v : v \in D_2(G)\}, \\ G_0 &= G/(X_1(G) \cup X'_2(G)). \end{aligned} \tag{1}$$

Following [1], for $u, v \in V(G)$, a uv-**trail** is a trail of G from u to v. For $e, e' \in E(G)$, an (e, e')-trail is a trail of G starting from e and ending at e'. An (e, e')-trail T is **dominating** if each edge of G is incident with at least one internal vertex of T, and T is **spanning** if T is a dominating trail with $V(T) = V(G)$. A graph G is **spanning trailable** if for each pair of edges e_1 and e_2, G has a spanning (e_1, e_2)-trail.

Suppose that $e = u_1v_1$ and $e' = u_2v_2$ are two edges of G. If $e \neq e'$, then the graph $G(e, e')$ is obtained from G by replacing $e = u_1v_1$ with a path $u_1v_ev_1$ and by replacing $e' = u_2v_2$ with a path $u_2v_{e'}v_2$, where $v_e, v_{e'}$ are two new vertices not in $V(G)$. If $e = e'$, then $G(e, e')$, also denoted by $G(e)$, is obtained from G by replacing $e = u_1v_1$ with a path $u_1v_ev_1$. As defined in [28], a graph G is **strongly spanning trailable (SST in short)** if for any $e, e' \in E(G)$, $G(e, e')$ has a $(v_e, v_{e'})$-trail T with $V(G) = V(T) - \{v_e, v_{e'}\}$. Since $e = e'$ is possible, SST graphs are both spanning trailable and supereulerian. The following former tools are useful.

Lemma 2.4 (*Shao, Lemma 1.4.1 and Proposition 1.4.2 of [34]*)**.** *Let G be a connected nontrivial graph such that $\kappa(L(G)) \geq 3$ and G_0 be the core of G. Then G_0 is uniquely determined by G with $\delta(G_0) \geq \kappa'(G_0) \geq 3$. Furthermore, each of the following holds.*
(i) $L(G)$ is hamiltonian if and only G_0 has a dominating eulerian subgraph containing the contraction preimages of the edges in $X_1(G) \cup X_2'(G)$. In particular, if G_0 is supereulerian, then $L(G)$ is hamiltonian.
(ii) (see also Lemma 2.9 of [23]) If G_0 is strongly spanning trailable, then $L(G)$ is Hamilton-connected.
(iii) (see also Proposition 2.2 of [23]) $L(G)$ is Hamilton-connected if and only if for any pair of edges $e, e' \in E(G)$, G has a dominating (e, e')-trail.

Let $X \subseteq E(G)$, which is also viewed as a vertex set in the line graph $L(G)$. Imitating the arguments in [15,34] and in Theorem 2.7 of [21], and by (1), we have the following observation.

Proposition 2.5. *Let $s \geq 0$ be an integer, G be a connected graph with $|E(G)| \geq s+3$ and $ess'(G) \geq 3$, and G_0 be the core of G.*
(i) (Theorem 2.7 of [21]) The line graph $L(G)$ is s-hamiltonian if and only if for any $X \subseteq E(G)$ with $|X| \leq s$, $G - X$ has a dominating eulerian subgraph.
...(ii) If for any $X \subseteq E(G_0)$ with $|X| \leq s$, $G_0 - X$ is supereulerian, then $L(G)$ is s-hamiltonian.

3. Auxiliary theorem and the proof of Theorem 1.8

We first present an auxiliary theorem, stated as Theorem 3.2. For notational convenience, we define $c(K_1) = 0$. We will assume the validity of Theorem 3.2 to prove Theorem 1.8. The justification of Theorem 3.2 will be postponed to the last section. We start with a lemma.

Lemma 3.1. *Let G be a connected graph with a 2-edge-cut X and let G_1 and G_2 be the two components of $G-X$. If both G/G_1 and G/G_2 are supereulerian, then G is also supereulerian.*

Proof. For $i \in \{1, 2\}$, let v_i denote the vertex in G/G_i onto which G_i is contracted. Then in G/G_i, the set of edges incident with v_i is X. Let L_i be a spanning eulerian subgraph of G/G_i. As $|X| = 2$ and as $v_i \in V(L_i)$, it follows that $X \subseteq E(L_i)$, and so $G[E(L_1) \cup E(L_2)]$ is a spanning eulerian subgraph of G. ■

Theorem 3.2. *Let G be a reduced graph. If $\kappa'(G) \geq 2$, $c(G) \leq 8$, $|D_2(G)| \leq 2$ and $ess'(G) \geq 3$, then G is collapsible.*

Thus Theorem 3.2 indicates that the only reduced graph satisfying the hypotheses of Theorem 3.2 is K_1 only.

Proof of Theorem 1.8. We argue by contradiction to prove Theorem 1.8(i), and assume that there exists a 3-edge-connected graph Γ, an edge $e = u_1u_2 \in E(\Gamma)$ such that $c(\Gamma - e) \leq 8$ and $G := \Gamma - e$ is not supereulerian with $|V(\Gamma)|$ minimized.

As $\kappa'(\Gamma) \geq 3$, we have $|D_2(G)| \leq 2$ and $\kappa'(G) \geq 2$. If $|V(G)| \leq 8$, then by Theorem 2.1(iv), G is supereulerian. Hence we assume that $|V(G)| \geq 9$. Suppose that G has an essential edge cut X with $|X| = 2$. Let G_1, G_2 be the two components of $G - X = \Gamma - (X \cup e)$ with $\min\{|E(G_1)|, |E(G_2)|\} \geq 1$. By the minimality of $|V(\Gamma)|$, G/G_i is supereulerian. By Lemma 3.1, G is supereulerian, contrary to the choice of G. Hence $ess'(G) \geq 3$.

If G is reduced, then by Theorem 3.2, G must be collapsible, and so supereulerian. Hence we assume that G contains a nontrivial collapsible subgraph H. Since $ess'(G) \geq 3$, we conclude that $|D_2(G/H)| \leq 2$. As $c(G/H) \leq c(G) \leq 8$ and $G/H = (\Gamma - e)/H = \Gamma/H - e$, it follows by the minimality of $|V(\Gamma)|$ that G/H has a spanning eulerian subgraph, and so by Theorem 2.1(iii), G is supereulerian, contrary to the assumption that G is a counterexample. This proves Theorem 1.8(i).

We again argue by contradiction to prove Theorem 1.8(ii) and assume that G is a counterexample to Theorem 1.8(ii) with $|V(G)|$ minimized. By the minimality of G and by Theorem 2.1(iii), we may assume that G is reduced. By the minimality of G and by Lemma 3.1, we may assume that G does not have any essential edge cut of size 2. If follows that there exist vertices u_1 and u_2 in $V(G)$ such that every 2-edge-cut of G must be the set of edges incident with u_1 or u_2. This implies that we can choose an edge $e = u_1u_2$ not in G such that adding e to G joining u_1 and u_2 will result in a graph Γ with $\kappa'(\Gamma) \geq 3$. As G is reduced, it follows by Theorem 1.8(i) that G is collapsible, and so supereulerian, contrary to the assumption that G is a counterexample. This completes the proof of the theorem. ■

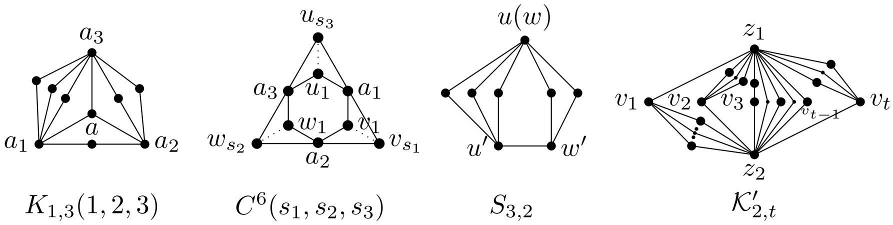

Fig. 1. Graphs in Definitions 4.2 and 4.3.

4. Proof of Theorem 1.7

In this section, we assume the validity of Theorem 3.2 to prove Theorem 1.7. For an integer $m > 0$, we use $\mathbb{Z}_m$ to denote the cyclic group of order m. For integers $s_1 \geq s_2 \geq s_3 \geq 1$, let Y_{s_1,s_2,s_3} be the graph obtained from disjoint paths P_{s_1+2}, P_{s_2+2} and P_{s_3+2} by identifying an end vertex of each of these three paths. (See Fig. 1 in [36] for an example.) By definition, $N_{s_1,s_2,s_3} = L(Y_{s_1,s_2,s_3})$. Define

$$\begin{aligned} \mathcal{Y}_1 &= \{Y_{s_1,s_2,s_3} : s_1 > 0, s_1 \geq s_2 \geq s_3 \geq 0, s_1+s_2+s_3 \leq 6\}. \\ \mathcal{Y}_2 &= \{Y_{s_1,s_2,s_3} : s_1 > 0, s_1 \geq s_2 \geq s_3 \geq 0, s_1+s_2+s_3 \leq 4\}. \end{aligned} \tag{2}$$

By definition of line graphs, a line graph $L(G)$ is N_{s_1,s_2,s_3}-free if and only if G does not have a Y_{s_1,s_2,s_3} as a subgraph. To complete the proof for Theorem 1.7, the following additional lemmas for a generic graph G will be needed.

Let $F(G)$ be the minimum number of additional edges that must be added to G to result in a graph with two edge-disjoint spanning trees. Catlin (Theorem 7 of [5], see also Corollary 2.13 of [27]) indicated that if G is connected, reduced and $G \notin \{K_1, K_2\}$, then

$$F(G) = 2|V(G)| - |E(G)| - 2. \tag{3}$$

Lemma 4.1 (*Theorem 2.4 of [9]*). *If G is a reduced graph with $\kappa'(G) \geq 2$, $|V(G)| \leq 11$, $F(G) \leq 3$ and $|D_2(G)| \leq 2$, then G is collapsible.*

Definition 4.2. Let $P(10)$ denote the Petersen graph and $P(10)^- = P(10) - e$ for an edge $e \in E(P(10))$, and let $K \cong K_{1,3}$ with $D_3(K) = \{a\}$ (the center of K) and $D_1(K) = \{a_1, a_2, a_3\}$. For integers $s_1, s_2, s_3, \ell, m, t$ with $\ell \geq 1$ and $m, t \geq 2$, we make the following definitions.
(i) Define $K_{1,3}(s_1, s_2, s_3)$ to be the graph obtained from K by adding s_i vertices with neighbors $\{a_i, a_{i+1}\}$, where $i \equiv 1, 2, 3 \pmod 3$.
(ii) Define $C^6(s_1, s_2, s_3) = K_{1,3}(s_1, s_2, s_3) - a$, where $s_2 \geq s_1 \geq 1$ and $s_3 \geq 2$. Furthermore, denote

$$\begin{aligned} N_{C^6(s_1,s_2,s_3)}(a_1) \cap N_{C^6(s_1,s_2,s_3)}(a_2) &= \{v_1, v_2, \ldots, v_{s_1}\}, \\ N_{C^6(s_1,s_2,s_3)}(a_2) \cap N_{C^6(s_1,s_2,s_3)}(a_3) &= \{w_1, w_2, \ldots, w_{s_2}\}, \\ N_{C^6(s_1,s_2,s_3)}(a_1) \cap N_{C^6(s_1,s_2,s_3)}(a_3) &= \{u_1, u_2, \ldots, u_{s_3}\}. \end{aligned} \tag{4}$$

(iii) Let $K_{2,t}(u, u')$ be a $K_{2,t}$ with u, u' being the nonadjacent vertices of degree t. Let $S_{m,\ell}$ be the graph obtained from a $K_{2,m}(u, u')$ and a $K_{2,\ell}(w, w')$ by identifying u with w, and joining u' and w' by an new edge $u'w'$.

Definition 4.3. Let $t \geq 2$, $r_1 \geq r_2 \geq \cdots \geq r_t \geq 0$ be integers such that $r_2 > 0$, K be a graph isomorphic to $K_{2,t}$ with $\{z_1, z_2\}$ and $\{v_1, v_2, \ldots, v_t\}$ being the bipartition of K. For each i with $1 \leq i \leq t$,

(i) denote $E_{K_{2,t}}(v_i) = \{e_i, e'_i\}$;

(ii) if $r_i > 0$, define $K_{2,r_i}(x_i, y_i)$ to be the bipartite graphs with x_i and y_i being the two nonadjacent vertices of degree r_i;

(iii) if $r_i = 0$, define $K_{2,0}(x_i, y_i) = K_2(x_i, y_i)$, which consists of an edge with end vertices x_i and y_i.

(K1) Define $K'_{2,t}(r_1, r_2, \ldots, r_t)$ to be a graph formed by, for each $i \in \{1, 2, \ldots, t\}$, replacing exactly one of e_i, e'_i by a $K_{2,r_i}(x_i, y_i)$ by identifying x_i and v_i and by identifying y_i with exactly one of z_1 or z_2. (See the fourth graph in Fig. 1 for an example). Let $\mathcal{K}'_{2,t}$ denote the family of all such defined $K'_{2,t}(r_1, r_2, \ldots, r_t)$'s. For notational convenience, when there is no confusion arises, we often use $K'_{2,t}$ to denote an arbitrary member in $\mathcal{K}'_{2,t}$.

(K2) Let $\mathcal{B}_t = \{K'_{2,t}(r_1, r_2, \ldots, r_t) + z_1z_2 : K'_{2,t}(r_1, r_2, \ldots, r_t) \in \mathcal{K}'_{2,t}\}$.

By definition, the 6-cycle $C_6 = K'_{2,2}(1, 1)$ is a member in $\mathcal{K}'_{2,2}$. Following [1], for a given graph $K'_{2,t}$, a (z_1, z_2)-**component** of this $K'_{2,t}$ is a subgraph of the form $K'_{2,t}[\{z_1, z_2, v_i\} \cup N_{K'_{2,t}}(v_i)]$ for some i with $1 \leq i \leq t$. Throughout the rest of the paper, we define

$$\begin{aligned} \mathcal{G} = {} & \{K_{2,t} : t \geq 1\} \cup \{S_{m,\ell} : \ell \geq m \geq 1\} \cup \{K_{1,3}(s_1, s_2, s_3) : s_1 \geq s_2 > 0 \text{ and } s_3 \geq 0\} \\ & \cup \{K_1\} \cup \{C^6(s_1, s_2, s_3) : s_1 \geq s_2 \geq 1, s_3 \geq 2\} \cup (\cup_{t \geq 2}(\mathcal{B}_t \cup \mathcal{K}'_{2,t})). \end{aligned} \tag{5}$$

Lemma 4.4. *Let G be a noncollapsible reduced graph with $\kappa(G) \geq 2$. Then each of the following holds.*
(i) $c(G) \leq 6$ if and only if $G \in \mathcal{G}$.
(ii) If $c(G) \leq 6$, then $|D_2(G)| \geq 3$. Furthermore $|D_2(G)| = 3$ if and only if $G \in \{K_{2,3}, K_{1,3}(1,1,1)\}$.

The proof of Lemma 4.4 will be postponed in the last section.

4.1. Proof of Theorem 1.7(i)

Lemma 4.5. *For any $Y \in \mathcal{Y}_1$. Let G be a connected graph with $\kappa'(G) \geq 3$ and $|E(G)| \geq 4$. If G does not contain Y as a subgraph, then for any $e \in E(G)$, $G - e$ is supereulerian.*

Proof. The lemma holds trivially if $n = |V(G)| \leq 3$ and $|E(G)| \geq 4$. We argue by contradiction and assume that

$$G \text{ is a counterexample graph with } |V(G)| \text{ minimized.} \tag{6}$$

Claim 1. *There exists an edge $e_0 \in E(G)$ such that*
(i) $G - e_0$ is not supereulerian.
(ii) $G - e_0$ is reduced, $g(G - e_0) \geq 4$ and $c(G - e_0) \geq 9$.

Claim 1(i) follows from (6). If $G - e_0$ has a nontrivial collapsible subgraph H, then $|V(G/H)| < |V(G)|$ and so by (6), $(G - e_0)/H = G/H - e_0$ is supereulerian, By Theorem 2.1(iii), $G - e_0$ is supereulerian, contrary to (6). Hence $G - e_0$ must be reduced. By Theorem 2.1(i), $g(G - e_0) \geq 4$. By Theorem 1.8(i), $c(G - e_0) \geq 9$. This proves the claim.

Let $C = v_1v_2...v_cv_1$ with $c = |E(C)| \geq c(G - e_0) \geq 9$ be a longest cycle of $G - e_0$. Since C is not spanning G, we assume that there exists a vertex $u_1 \in V(G) - V(C)$ such that $u_1v_1 \in E(G - e_0)$. By definition, we observe that, as $c \geq 9$, the subgraph $G[E(C) \cup \{u_1v_1\}]$ contains every member in $\{Y_{s_1,s_2,0} : s_1 \geq s_2 \geq 0, 1 \leq s_1 + s_2 \leq 6\}$ with v_1 being the unique vertex of degree 3 in these subgraphs. In the following, we shall show that either G has a longer cycle than C, or G contains every member in $\mathcal{Y}_1$ as defined in (2). These contradictions will then justify the lemma.

If there exists a $u_2 \in N_G(u_1) - V(C)$, then as $c \geq 9$, the subgraph $G[E(C) \cup \{u_1v_1, u_1u_2\}]$ contains every member in $\mathcal{Y}_1$ with $1 \in \{s_1, s_2, s_3\}$ and with v_1 being the unique vertex of degree 3 in these subgraphs. It remains to show that G also contains a $Y_{2,2,2}$. If $N_G(u_2) - V(C)$ has at least two vertices, then there exists a $u_3 \in N_G(u_2) - (V(C) \cup \{u_1\})$, and so $G[E(C) \cup \{u_1v_1, u_1u_2, u_2u_3\}]$ contains a $Y_{2,2,2}$ as a subgraph. Hence $N_G(u_2) - \{u_1\} \subseteq V(C)$. By $\kappa'(G) \geq 3$, $g(G - e_0) \geq 4$ and the choice of C, we assume that $v_{j_1}, v_{j_2} \in N_G(u_2) \cap V(C)$ with $5 \leq j_1 + 1 < j_2 \leq c - 2$. Then $C^1 = v_1u_1u_2v_{j_1}v_{j_1+1} \cdots v_cv_1$ is a cycle of length $c - (j_1 - 1) + 3 = (c - j_2) + (j_2 - j_1) + 4 \geq 2 + 2 + 4 = 8$. If $j_1 \geq 5$, $G[E(C^1) \cup \{v_1v_2, v_2v_3, v_3v_4\}]$ contains every member in $\mathcal{Y}_1$ with v_1 being the unique vertex of degree 3 in these subgraphs. By symmetry, we assume that $j_1 = 4$ and $j_2 = c - 2$. Thus $G[(E(C) - \{v_1v_2, v_{j_1}v_{j_1+1}, v_{j_2}v_{j_2+1}\}) \cup \{v_1u_1, u_1u_2, u_2v_{j_1}, u_2v_{j_2}\}] = Y_{2,2,2}$. Thus in any case, G contains every member in $\mathcal{Y}_1$. This contradiction shows that

$$\text{for any } u \in V(G) - V(C),\ N_G(u) \subseteq V(C). \tag{7}$$

Let $v_1, v_i, v_j \in N_{G_0}(u_1)$ with $1 < i < j$. By $g(G - e_0) \geq 4$, we have $3 < i + 1 < j \leq c - 1$. Let $k = \max\{i - 1, j - i, c - j + 1\}$.

Assume that $k \geq 4$. Without loss of generality, we assume that $i - 1 \geq 4$. Then $C^4 = G[E(C - \{v_2, \ldots, v_{i-1}\}) \cup \{v_1u_1, u_1v_i\}]$ is a cycle of length at least 6. If the length of C^4 is at least 8, then $G[(E(C) - \{v_{i-1}v_i, v_iv_{i+1}, u_1v_j\}) \cup \{v_1u_1, u_1v_i\}]$ contains $Y_{4,1,1}$ and $Y_{3,2,1}$, while $G[(E(C) - \{v_{i-1}v_i, v_{j-1}v_j, u_1v_j\}) \cup \{v_1u_1, u_1v_i\}]$ contains $Y_{2,2,2}$. If the length of C^4 is 7, without loss of generality, we assume that $j = i + 2$ and $c - j = 2$. Then $G[(E(C) - \{v_iv_{i-1}, v_jv_{j-1}\}) \cup \{v_1u_1, u_1v_i\}]$ contains the subgraph $Y_{2,2,2}$, $G[(E(C) - \{v_iv_{i-1}, v_iv_{i+1}\}) \cup \{v_1u_1, u_1v_i\}]$ contains the subgraph $Y_{3,2,1}$, and $G[(E(C) - \{v_iv_{i-1}, v_1v_c\}) \cup \{v_1u_1, u_1v_j\}]$ contains the subgraph $Y_{4,1,1}$. If the length of C^4 is 6, then $c = j + 1$, $j = i + 2$ and $i \geq 6$. Thus $G[(E(C) - \{v_jv_{j-1}, v_1v_c, v_iv_{i-1}\}) \cup \{u_1v_1, u_1v_i, u_1v_j\}]$ contains the subgraph $Y_{4,1,1}$, $G[(E(C) - \{v_iv_{i+1}, v_{i-1}v_{i-2}\}) \cup \{v_1u_1, u_1v_i\}]$ contains the subgraph $Y_{2,2,2}$, and $G[(E(C) - \{v_iv_{i+1}, v_iv_{i-1}\}) \cup \{v_1u_1, u_1v_i\}]$ contains the subgraph $Y_{3,2,1}$. Therefore, $k \leq 3$. As $c \geq 9$, we have $i = 4, j = 7$ and $c = 9$. Then $G[(E(C) - \{v_3v_4\}) \cup \{u_1v_1, v_1v_i\}]$ contains every member in $\mathcal{Y}_1$ with $1 \in \{s_1, s_2, s_3\}$ and with v_1 being the unique vertex of degree 3 in these subgraphs, and $G[(E(C) - \{v_1v_2, v_iv_{i+1}, v_jv_{j+1}\}) \cup \{u_1v_1, u_1v_i, u_1v_j\}] \cong Y_{2,2,2}$. All these contradictions indicate the truth of the lemma. ∎

Proof of Theorem 1.7(i). By the definition of line graph, a graph Γ has a subgraph in $\mathcal{Y}_1$ if and only if $L(G)$ has a member as an induced subgraph in $L(G)$. Therefore, to prove Theorem 1.7(i), it suffices to show that, for any fixed $Y \in \mathcal{Y}_1$ and for an integer $s \geq 1$, if G does not have Y as a subgraph, then

$$\kappa(L(G)) \geq s + 2 \text{ implies that } L(G) \text{ is } s\text{-hamiltonian.} \tag{8}$$

We argue by induction on s to prove (8), and assume that $s = 1$. Let G be a graph with $\kappa(L(G)) \geq 3$, and let G_0 be the core of G. Since G does not have Y as a subgraph, G_0 also contains no subgraph isomorphic to Y. By Lemma 2.4, $\kappa'(G_0) \geq 3$ and so by Lemma 4.5, for any $e_0 \in E(G_0)$, $G - e_0$ is supereulerian. By Proposition 2.5(ii), (8) holds for $s = 1$.

Assume that $s \geq 2$ and (8) holds for smaller values of s. For any edge subset $X \subseteq E(G)$ with $|X| = s$. Pick $e_0 \in X$. Define $G_1 = G - e_0$ and $X_1 = X - \{e_0\}$. As G does not have Y as a subgraph, G_1 also contains no subgraph isomorphic to Y, with $\kappa(L(G_1)) = \kappa(G - e_0) \geq (s + 2) - 1 = (s - 1) + 2$. By induction, G_1 is $(s - 1)$-hamiltonian, and so $L(G) - X = L(G_1) - X_1$ is hamiltonian. Thus (8) holds for all integer $s \geq 1$, and so Theorem 1.7(i) is justified. ∎

4.2. Proof of Theorem 1.7(ii)

Throughout the rest of this paper, suppose that $P = v_1v_2...v_n$ denotes a v_1v_n-path and $1 \le i < j \le n$. We define $P[v_i, v_j] = v_iv_{i+1}...v_j$ and $P^{-1}[v_i, v_j] = v_jv_{j-1}...v_i$. Thus $P = P[v_1, v_n]$. Similarly, suppose that $C = v_1v_2...v_nv_1$ denotes a cycle and $1 \le i < j \le n$. Define $C[v_i, v_j] = v_iv_{i+1}...v_j$ and $C^{-1}[v_i, v_j] = v_jv_{j+1}...v_nv_1...v_i$ to be the subpaths of C. Let $\mathbb{Z}_n$ be the additive group of integers modulo n. Define H_8 to be the graph with $V(H_8) = \{v_i : i \in \mathbb{Z}_8\}$ and $E(H_8) = \{v_iv_{i+1}, v_iv_{i+4} : i \in \mathbb{Z}_8\}$. The graph H_8 is known as the Wagner graph [31] in the literature. It is routine to verify that

for any $Y \in \mathcal{Y}_2$, H_8 contains Y as a subgraph. (9)

Lemma 4.6. *Let $Y \in \mathcal{Y}_2$ and G be a graph with $\kappa'(G) \ge 3$ such that*

G does not contain Y as a subgraph. (10)

Then each of the following holds.
(i) For any $e', e'' \in E(G)$, $G(e', e'')$ is collapsible.
(ii) G is strongly spanning trailable.

Proof. By Theorem 2.1, (i) implies (ii) and so it suffices to prove (i). We argue by contradiction and assume that

G is a counterexample to Lemma 4.6(i) with $|V(G)|$ minimized. (11)

Then there must be edges $e^1, e^2 \in E(G)$ such that $G(e^1, e^2)$ is not collapsible. Let $J = G(e^1, e^2)$. By (11), we may assume J is reduced. Since $G(e^1, e^2)$ is not collapsible, $J \ne K_1$.

Suppose that $c(J) \le 8$. By Theorem 3.2, J must have an essential edge-cut X with $X = \{f_1, f_2\}$. For each $i \in \{1, 2\}$, if f_i is incident with v_{e^j}, for some $j \in \{1, 2\}$, then define $f_i' = e^j$, otherwise set $f_i' = f_i$. By definition, $f_1', f_2' \in E(G)$ and so $\{f_1', f_2'\}$ would be an essential 2-edge-cut of G, contrary to $\kappa'(G) \ge 3$. Hence we must have $|c(J)| \ge 9$. Let C' be a longest cycle of J. We lift C' to a cycle C'' in $G(e^1, e^2)$ and convert C'' to a cycle C of G by undoing the subdivisions on e^1 and e^2 if $\{v_{e^1}, v_{e^2}\} \cap V(C') \ne \emptyset$. As v_{e^1}, v_{e^2} might be in $V(C')$, we have $|E(C)| \ge 7$.

Assume first that $V(G) - V(C) \ne \emptyset$. Since G is connected, there must be a vertex $v \in V(G) - V(C)$ with $uv \in E(G)$ for some $u \in V(C)$. Since $|E(C)| \ge 7$, $G[E(C) \cup \{uv\}]$ contains $Y_{4,0,0}$, $Y_{3,1,0}$ and $Y_{2,2,0}$ as subgraphs, in each of which u is the only degree 3 vertex. We are to show that G also contains $Y_{2,1,1}$ as a subgraph to find a contradiction to (10). Suppose there exists $w \in N_G(v) - V(C)$. Then $G[E(C) \cup \{uv, vw\}]$ contains $Y_{2,1,1}$ as a subgraph that takes u as the only degree 3 vertex. Hence we have $N_G(v) \subseteq V(C)$. Since $\kappa'(G) \ge 3$, we may assume $\{v_1, v_2, v_3\} \subseteq N_G(v)$. As C is the longest cycle of G, we have $2 \le d_C(v_i, v_j) \le 3$ for $1 \le i < j \le 3$. Then $G[E(C) \cup \{vv_1, vv_2, vv_3\}]$ contains $Y_{2,1,1}$ as a subgraph that takes v as the only degree 3 vertex, a contradiction. Hence we must have $V(G) = V(C)$. Let $n = |V(G)|$. Denote $V(C) = \{v_i : i \in \mathbb{Z}_n\}$ with $E(C) = \{v_iv_{i+1} : i \in \mathbb{Z}_n\}$. If $n = 7$, then by (3), J satisfies the hypotheses of Lemma 4.1, and so J is collapsible, a contradiction. Therefore we must have $n \ge 8$.

Claim 2. *If $n = 8$, then (10) is violated.*

We assume that $n = 8$ to justify the claim. By (3), if $\Delta(G) \ge 4$, then $F(J) \le 3$, and so by Lemma 4.1, J must be collapsible, a contradiction. Thus G must be a 3-regular graph with C being a Hamilton cycle of G. For any $t \in \mathbb{Z}_8$, there exists an $i(t) \in \mathbb{Z}_8 - \{t\}$ such that $v_tv_{i(t)} \in E(G) - E(C)$. Since $\kappa'(G) \ge 3$ and G is 3-regular, G cannot have parallel edges, and so $i(t) \notin \{t-1, t+1\}$ in $\mathbb{Z}_8$.

If there is a $t \in \mathbb{Z}_8$ with $i(t) = t+2$ in $\mathbb{Z}_8$, then by symmetry, we may assume that $i(1) = 3$. If, in addition, $i(2) = 4$, then as G is 3-regular, $\{v_8v_1, v_4v_5\}$ is a 2-edge-cut of G, contrary to $\kappa'(G) \ge 3$. Thus in $\mathbb{Z}_8$, by symmetry $i(2) \notin \{4, 8\}$, and so $i(2) \in \{5, 6, 7\}$. Suppose $i(2) = 5$. Then as $\mathcal{Y}_2 = \{Y_{4,0,0}, Y_{3,1,0}, Y_{2,2,0}, Y_{2,1,1}\}$, for each $Y \in \mathcal{Y}_2$, $G[E(C) \cup \{v_1v_3, v_2v_5\}]$ contains a Y as a subgraph with v_1 being the only vertex of degree 3, contrary to (10). Hence by symmetry, $i(2) \notin \{5, 7\}$, forcing $i(2) = 6$. It follows that for any $Y \in \{Y_{4,0,0}, Y_{3,1,0}, Y_{2,2,0}\}$, $G[E(C) \cup \{v_1v_3, v_2v_6\}]$ contains Y as a subgraph with v_1 being the only vertex of degree 3. Furthermore, $G[E(C) \cup \{v_1v_3, v_2v_6\}]$ contains $Y_{2,1,1}$ as a subgraph with v_6 being the only degree 3 vertex, and so (10) is violated. We conclude that by symmetry, for any $t \in \mathbb{Z}_8$, $i(t) \notin \{t-2, t-1, t, t+1, t+2\}$, or equivalently,

for any $t \in \mathbb{Z}_8$, $i(t) \in \{t+3, t+4, t+5\}$ in $\mathbb{Z}_8$. (12)

If there is a $t \in \mathbb{Z}_8$ with $i(t) = t+3$ in $\mathbb{Z}_8$, then by symmetry, we may assume that $i(1) = 4$. If $i(2) = 5$, then by (12) and as G is 3-regular, we must have $i(3) = 7$, forcing $i(6) = 8$ violating (12). Thus by symmetry, in $\mathbb{Z}_8$, we must have $i(2) \notin \{5, 7\}$, and so $i(2) = 6$. It follows that for any $Y \in \{Y_{4,0,0}, Y_{3,1,0}, Y_{2,2,0}\}$, $G[E(C) \cup \{v_1v_4, v_2v_6\}]$ contains Y as a subgraph with v_1 being the only vertex of degree 3. As $G[E(C) \cup \{v_1v_4, v_2v_6\}]$ also contains $Y_{2,1,1}$ as a subgraph with v_6 being the only degree 3 vertex, (10) is violated. We now conclude that by symmetry, we must have $i(t) = t+4$ for any $t \in \mathbb{Z}_8$, and so $G \cong H_8$. By (9), (10) is violated. This completes the proof for the claim.

By Claim 2, we must have $n \geq 9$. We first prove that

$$G \text{ always contains } Y_{4,0,0} \text{ as a subgraph.} \tag{13}$$

Since $\kappa'(G) \geq 3$, we may assume that $v_1v_j \in E(G)$ for some j with $1 < j \leq n/2 + 1$. If $n \geq 11$, then $G[E(C[v_{j-1}, v_{j+5}]) \cup \{v_1v_j\}] \cong Y_{4,0,0}$. Assume that $n = 10$. If there exists an $t \in \mathbb{Z}_{10}$, and a $t' \in \mathbb{Z}_{10} - \{t-1, t, t+1\}$ with $v_tv_{t'} \in E(G) - E(C)$, such that v_t and $v_{t'}$ are of distance at most 4 on C, then as $G[E(C) \cup \{v_tv_{t'}\}]$ contains a cycle of length at least 7 other than C, $Y_{4,0,0}$ is a subgraph of $G[E(C) \cup \{v_tv_{t'}\}]$. It follows that we must have $t' = t + 5$ in $\mathbb{Z}_{10}$, whence $G[(E(C - \{v_8, v_9, v_{10}\}) - \{v_3v_4\}) \cup \{v_2v_7, v_4v_9\}] \cong Y_{4,0,0}$. Now assume that $n = 9$. We observe that to avoid a $Y_{4,0,0}$, any chord of C must have the form v_iv_{i+4}, and so $G[(E(C - \{v_8\}) - \{v_2v_3, v_5v_6\}) \cup \{v_1v_5, v_3v_7\}] \cong Y_{4,0,0}$. Hence (13) must hold.

By (13), it suffices to show that any $Y \in \{Y_{3,1,0}, Y_{2,2,0}, Y_{2,1,1}\}$ is a subgraph of G. Let $e \in E(G) - E(C)$ be an edge. Since C is a Hamilton cycle of G, e is a chord of C. Let $g(C + e)$ be the length of a shortest cycle of $G[E(C) \cup \{e\}]$. Since $J = G(e^1, e^2)$ is reduced, and since cycles of length at most 3 is collapsible, it follows that every cycle of length at most 3 contains either e^1 or e^2, and a cycle of length 2 in G must be induced by $\{e^1, e^2\}$. Since $n \geq 9$ and $\kappa'(G) \geq 3$, C has at least $\lceil \frac{n}{2} \rceil = 5$ chords. It follows that there must be a chord $e \in E(G) - E(C)$ such that $g(C + e) \geq 4$. By symmetry, assume that $e = v_1v_j$ with $4 \leq j \leq 7$. Then for any $Y \in \{Y_{3,1,0}, Y_{2,2,0}, Y_{2,1,1}\}$, $G[E(C) \cup \{v_1v_j\}]$ contains Y as a subgraph with v_1 being the only vertex of degree 3 in Y. This, together with (13), implies that (10) is violated. This completes the proof of the Lemma. ■

The following corollary can be partially proved by Theorem 1.6(iv). For the sake of completeness, we present a formal proof.

Corollary 4.7. *Every 3-connected N_{s_1,s_2,s_3}-free line graph $L(G)$ with $s_1 + s_2 + s_3 \leq 4$ is hamiltonian-connected where $s_1 > 0, s_1 \geq s_2 \geq s_3 \geq 0$.*

Proof. Let G be a graph with $\kappa(L(G)) \geq 3$, and let G_0 be the core of G. By Lemma 2.4(ii), it suffices to show that G_0 is strongly spanning trailable. By Lemma 2.4(i), $\kappa'(G_0) \geq 3$. By (10), G_0 also does not contain any $Y \in \mathcal{Y}_2$ as a subgraph. It follows by Lemma 4.6 that G_0 is strongly spanning trailable. Hence the corollary holds. ■

Proof of Theorem 1.7(ii). It suffices to prove that for $s \geq 0$,

$$\text{if (10) and } \kappa(L(G)) \geq s + 3, \text{ then } L(G) \text{ is } s\text{-Hamilton-connected.} \tag{14}$$

We argue by induction on s to prove (14), and assume that $s = 0$. By Corollary 4.7, (14) holds for $s = 0$.

Assume that $s > 0$ and (14) holds for smaller values of s. For any edge subset $X \subseteq E(G)$ with $0 < |X| \leq s$. Pick $e_0 \in X$. Define $G_1 = G - e_0$ and $X_1 = X - \{e_0\}$. By (10), G_1 does not have any $Y \in \mathcal{Y}_2$ as a subgraph, with $\kappa(L(G_1)) = \kappa(G - e_0) \geq (s+3) - 1 = (s-1) + 3$. By induction, G_1 is $(s-1)$-Hamilton-connected, and so $L(G) - X = L(G_1) - X_1$ is Hamilton-connected. This completes the proof of Theorem 1.7(ii).

5. Proofs of Lemma 4.4 and Theorem 3.2

The arguments in this section do not depend on any result in Sections 3 and 4, it develops the needed tools to prove Lemma 4.4 and Theorem 3.2. We shall use the notation in Definition 4.2 and develop some more tools. For sets X and Y, the **symmetric difference** of X and Y is $X \Delta Y = (X \cup Y) - (X \cap Y)$. If an edge $e = uv \notin E(G)$ but $u, v \in V(G)$, then let $G + e$ be the graph containing G as a spanning subgraph with edge set $E(G) \cup \{e\}$. For $v \in V(G)$ and $e \in E(G)$, we first study reduced graphs with circumferences at most 6.

5.1. Nontrivial 2-connected reduced graph with circumference at most 6

We have the following observations and facts. The first two are from the definition of $\mathcal{G}$ in (5).

Observation 5.1. *Let G be a nontrivial connected graph.*
(i) If $|D_2(G)| \leq 2$ or $|D_2(G)| = 3$ and G contains two adjacent degree 2 vertices, we have $G \notin \mathcal{G}$.
(ii) If $|D_2(G)| = 3$, then $G \in \mathcal{G}$ if and only if $G \in \{K_{2,3}, K_{1,3}(1, 1, 1)\}$.
(iii) (Theorem 3 of [20]) If G is reduced with diameter 2, then $G \in \{K_{1,t}, K_{2,t}, S_{m,l}, P(10)\}$ where $t \geq 2$.

Lemma 5.2. *Suppose $G \in \mathcal{K}'_{2,t}$. Let $x, y \in V(G)$ such that $d_G(x, y) \geq 2$ and $\{x, y\} \cap \{z_1, z_2\} = \emptyset$. Then there exists a cycle C of G such that $|E(C)| \geq 5$ and $|V(C) \cap \{x, y\}| = 1$ unless, up to isomorphism, $G \in K'_{2,2}$ and $\{x, y\} = \{v_1, v_2\}$.*

Proof. Suppose first that x, y are in the same (z_1, z_2)-component of G, then by Definition 4.3, $G - x$ is also in $\mathcal{K}'_{2,t}$, and so a cycle C of length at least 5 containing y exists in $G - x$. If $t \geq 3$ and x and y are in different (z_1, z_2)-component of G, then by Definition 4.3, the graph G' formed by deleting the component of $G - \{z_1, z_2\}$ containing x is also in $\mathcal{K}'_{2,t}$, and so a cycle C of length at least 5 containing y exists in G'. Therefore, we may assume that $t = 2$, $G \neq C_6$, and x and y

are in different (z_1, z_2)-component of G. By symmetry, we may further assume that $d_G(v_1) \geq 3$, x and v_1 are in the same (z_1, z_2)-component and y and v_2 are in the same (z_1, z_2)-component. If $x \neq v_1$, then $G - x$ is also in $\mathcal{K}'_{2,t}$ and so a cycle of length at least 5 containing y but not x exists. Hence $x = v_1$. Similarly, $y = v_2$. ■

We have the following observation.

Observation 5.3. *Let $C = v_1v_2 \ldots v_nv_1$ be a cycle of G, P_1 be a v_iv_k-path of G satisfying $V(P_1) \cap V(C) = \{v_i, v_k\}$, and P_2 be a v_jv_ℓ-path of G satisfying $V(P_2) \cap V(C) = \{v_j, v_\ell\}$. Suppose that $1 \leq i < j < k < \ell < n$. If $|E(P_1)| + |E(P_2)| > |E(C[v_k, v_\ell])| + |E(C[v_i, v_j])|$, then C is not a longest cycle of G.*

Proof of Lemma 4.4. As (ii) follows immediately from (i) and Observation 5.1, it suffices to justify (i). It is routine to verify that graphs in $\mathcal{G}$ are reduced and if $G \in \mathcal{G}$, then $c(G) \leq 6$. If $c(G) \leq 5$, then the diameter of G is at most 2 and so by Observation 5.1(iii), $G \in \mathcal{G}$. Hence we assume that $c(G) = 6$.

Claim 3. *The graph G is spanned by H where $H \in \{C^6(s_1, s_2, s_3) : s_1 \geq s_2 \geq 1, s_3 \geq 2\} \cup (\cup_{t\geq 2}\mathcal{K}'_{2,t})$.*

Since a cycle of order 6 is in $\mathcal{K}'_{2,2}$ and $c(G) = 6$, we conclude that G contains a member in $\mathcal{K}'_{2,t}$ as a subgraph. Choose an $H \in \{C^6(s_1, s_2, s_3) : s_1 \geq s_2 \geq 1, s_3 \geq 2\} \cup (\cup_{t\geq 2}\mathcal{K}'_{2,t})$ such that

$$H \text{ is a subgraph of } G \text{ with } |V(H)| + |E(H)| \text{ maximized.} \tag{15}$$

If $V(G) = V(H)$, then done. Therefore there must be a vertex $u \in V(G) - V(H)$. As $\kappa(G) \geq 2$, G has a uv-path P_1 and a uw-path P_2 with $V(P_1) \cap V(P_2) = \{u\}$, $V(P_1) \cap V(H) = \{v\}$, $V(P_2) \cap V(H) = \{w\}$ for distinct vertices v and w. If $vw \in E(H)$, then since each edge of H lies in a cycle with length at least 5, $H \cup P_1 \cup P_2$ contains a cycle with length greater than 6, contrary to $c(G) = 6$. Hence $d_H(v, w) \geq 2$. In the arguments below, we will use the notations in Fig. 1.

Assume first that $H \in \{C^6(s_1, s_2, s_3) : s_1 \geq s_2 \geq 1, s_3 \geq 2\}$. By (15) and Observation 5.3, we have $\{v, w\} \in \{\{u_p, u_q\}, \{v_p, v_q\}, \{w_p, w_q\} : p \neq q\}$. If $\{v, w\} = \{u_p, u_q\}$, then $G[\{u_pa_1, a_1v_1, v_1a_2, a_2w_1, w_1a_3, a_3u_q\} \cup E(P_1) \cup E(P_2)]$ contains a cycle of length longer than 6, contrary to $c(G) = 6$. Hence $\{v, w\} \neq \{u_p, u_q\}$. By symmetry, we also conclude that $\{v, w\} \neq \{w_p, w_q\}$ and $\{v, w\} \neq \{v_p, v_q\}$.

Therefore, we may assume that $H \in \mathcal{K}'_{2,t}$. If $\{v, w\} = \{z_1, z_2\}$, then $G[H + u] \in \mathcal{K}'_{2,t+1}$, violating (15). Hence we must have $\{v, w\} \neq \{z_1, z_2\}$.

Suppose that $\{v, w\} \cap \{z_1, z_2\} = \emptyset$. By Lemma 5.2, either $t = 2$ and $\{v, w\} = \{v_1, v_2\}$, whence $G[H + u] \in K'_{2,3}$ or $G[H + u] \cong C^6(s_1, s_2, s_3)$, contrary to (15); or there exists a cycle C with $|V(C)| \geq 5$ such that (by symmetry) $V(C) \cap \{v, w\} = \{w\}$. As such a cycle C must contain both z_1 and z_2, we may assume that $wz_1 \in E(H)$. Let P be the shortest vz_1-path in H. Then $C' = P[v, z_1]z_1wP_2^{-1}[w, u]$ is a cycle of length at least 4 and $|E(C \cap C')| = 1$. Thus $C \triangle C'$ is a cycle of length greater than 6, contrary to $c(G) = 6$. These contradictions indicate that we must have $|\{v, w\} \cap \{z_1, z_2\}| = 1$.

By symmetry, we assume $w = z_1$. Since G is reduced and $c(G) = 6$, both $uw, uv \in E(G)$. If $v = v_i$ for some $i \in \{1, 2, \ldots, t\}$, then $G[H+u] \in K'_{2,t}$, violating (15). Hence we have $v \in V(H) - \{v_1, \ldots, v_t, z_1, z_2\}$, whence $N_H(v) = \{z_2, v_i\}$ for some $1 \leq i \leq t$. If $d_H(v_i) = 2$, then $G[H + u] \in K'_{2,t}$, again violating (15). Therefore we have $d_H(v_i) \geq 3$, and so by Observation 5.3, $G[H + u]$ contains cycle with length greater than 6. This justifies Claim 3.

By Claim 3, G is spanned by an H where $H \in \{C^6(s_1, s_2, s_3) : s_1 \geq s_2 \geq 1, s_3 \geq 2\} \cup K'_{2,t}$. Suppose $xy \in E(G) - E(H)$. Then since G is reduced, we have $d_H(x, y) \geq 3$.

Assume first that $H \in \{C^6(s_1, s_2, s_3) : s_1 \geq s_2 \geq 1, s_3 \geq 2\}$. Since $d_H(x, y) \geq 3$, we have $\{x, y\} \in \{\{a_2, u_p\}, \{a_1, w_p\}, \{a_3, v_p\} : p \geq 1\}$. But any such case implies that $G[H + xy] \cong K_{1,3}(s'_1, s'_2, s'_3)$ where $s_1 + s_2 + s_3 = s'_1 + s'_2 + s'_3 + 2$. If $G = G[H + xy]$, then $G \in \mathcal{G}$ and we are done. Assume that there exists an edge $e' \in E(G) - E(G[H + xy])$. By the definition of $K_{1,3}(s'_1, s'_2, s'_3)$, the graph $K_{1,3}(s'_1, s'_2, s'_3) + e'$ must create a cycle of length at most 3 or an C_6^{++}. By Theorem 2.1(iv), G is not reduced, contrary to the assumption that G is reduced.

Thus we must have $H \in K'_{2,t}$. Recall that $xy \in E(G) - E(H)$ is an edge not in H. By Definition 4.3, any $z' \in V(G) - \{z_1, z_2\}$ has distance at most two to z_1 and z_2. Thus if $x \in \{z_1, z_2\}$ and $y \notin \{z_1, z_2\}$, then $H + xy$ contains a cycle of length at most 3, contrary to the assumption that G is reduced. Thus either $\{x, y\} = \{z_1, z_2\}$ or $\{x, y\} \cap \{z_1, z_2\} = \emptyset$.

If $\{x, y\} = \{z_1, z_2\}$, then since G is reduced, by (15) and by Theorem 2.1(iv), we have $G[E(H) \cup xy] \in \mathcal{B}$. Assume that $\{x, y\} \cap \{z_1, z_2\} = \emptyset$. If $H \in \mathcal{K}'_{2,2}$ with $\{x, y\} = \{v_1, v_2\}$, then $G[E(H) \cup xy] \in \mathcal{B}$. As any additional edge added to a graph in $\mathcal{B}$ will result in a cycle of length at most 3, contrary to the assumption that G is reduced.

Hence we must have that $\{x, y\} \cap \{z_1, z_2\} = \emptyset$ and if $H \in \mathcal{K}'_{2,2}$, then $\{x, y\} \neq \{v_1, v_2\}$. By Lemma 5.2, there exists a cycle C of H such that $|V(C)| \geq 5$ and $|V(C) \cap \{x, y\}| = 1$. Assume first that $t \geq 3$. Then we may assume that for some i, $v_i \notin V(C)$ and $V(C) \cap \{x, y\} = \{y\}$. By the definition of $K'_{2,t}$, as $|V(C)| \geq 5$ and as any cycle of a $K'_{2,t}$ with length at least 5 must contain both z_1 and z_2, it follows that $\{z_1, z_2\} \subseteq V(C)$. Since $d_H(y, z_1) + d_H(y, z_2) \leq 3$, we may assume $d_H(y, z_1) = 1$, and so $yz_1 \in E(H)$. Let Q be the shortest xz_1-path in H. As G is reduced, $|V(Q)| \geq 3$ and so $C'' = Q[x, z_1]z_1yx$ is a cycle with length at least 4 with $|E(C \cap C'')| = 1$. It follows that $C \triangle C''$ is a cycle with length at least 7, contrary to assumption of $c(G) = 6$.

Hence we must have $t = 2$ but $y \notin \{v_1, v_2\}$. Again by Definition 4.3 and by $|V(C)| \geq 5$, we have $\{z_1, z_2\} \subseteq V(C)$ and we by symmetry may assume that $v_2y, z_2y \in E(H)$ with $N_G(z_2) \cap N_G(v_2) - \{y\} \neq \emptyset$. Since G is reduced, we may assume that either $x = v_1$ or $N_G(z_1) \cap N_G(v_1) = \{x\}$, whence G contains a $K_{1,3}(1, s_2, s_3)$, violating (15); or there exist distinct $x, x' \in N_G(z_1) \cap N_G(v_1)$, whence for any $y' \in N_G(z_2) \cap N_G(v_2) - \{y\}$, the cycle $y'v_2yxz_1x'v_1v_2y'$ has length at least 7, contrary to the assumption. This completes the proof of the lemma. ■

Definition 5.4. Let $C = x_1x_2y_1y_2x_1$ be a 4-cycle in G with a partition $\pi(C) = \langle\{x_1, y_1\}, \{x_2, y_2\}\rangle$.
(i) (Catlin [5]) Let $G/\pi(C)$, the $\pi(C)$-**reduction of** G, be the graph obtained from $G - E(C)$ by identifying x_1 and y_1 to form a vertex v_1, by identifying x_2 and y_2 to form a vertex v_2, and by adding an edge $e_{\pi(C)} = v_1v_2$.
(ii) The 4-cycle C is a **reducible 4-cycle of** G if $G/\pi(C)$ has a cycle containing the edge $e_{\pi(C)} = v_1v_2$. (In other words, $e_{\pi(C)}$ is not a cut edge of $G/\pi(C)$.)

Theorem 5.5. *Let G be a graph containing a 4-cycle C and let $G/\pi(C)$ be defined as above. Each of the following holds.*
(i) (Catlin, Corollary 1 of [5]) If $G/\pi(C)$ is collapsible, then G is collapsible.
(ii) (Catlin, Corollary 2 of [5]) If $G/\pi(C)$ is supereulerian, then G is supereulerian.
(iii) $c(G/\pi(C)) \leq c(G)$.

Proof. We adopt the notation in Definition 5.4 to justify (iii). Let C' be a longest cycle of $G/\pi(C)$. If $e_{\pi(C)} = v_1v_2$ is not an edge of C', then C' is a cycle of G and so $c(G/\pi(C)) \leq c(G)$. Assume that $e_{\pi(C)}$ is an edge of C'. Then by the definition of $e_{\pi(C)} = v_1v_2$, C' can be modified into a cycle of G of length at least $|E(C')|$ by adding a path joining a vertex in $\{x_1, y_1\}$ to a vertex in $\{x_2, y_2\}$ to $C' - v_1v_2$. Again we have $c(G/\pi(C)) \leq c(G)$, and so (iii) must hold. ■

5.2. Proof of Theorem 3.2

By contradiction, we assume that

$$G \text{ be a counterexample to Theorem 3.2 with } |V(G)| \text{ minimized.} \tag{16}$$

We shall make a number of claims in our proofs.

Claim 4. *Each of the following holds.*
(i) G is simple, $\kappa(G) \geq 2$, $c(G) \leq 8$, $|D_2(G)| \leq 2$, $g(G) \geq 4$, and G does not have essential 2-edge-cuts.
(ii) $|V(G)| \geq c(G) \geq 7$.
(iii) G does not contain a reducible 4-cycle.

As Claim 4(i) and (ii) follow from assumption of Theorem 3.2, Theorem 2.1 and Lemma 4.4, it remains to prove Claim 4(iii). By contradiction, assume that G has a reducible 4-cycle $C' = x_1x_2y_1y_2x_1$. In the arguments below, let $G_\pi = G/\pi(C')$, G'_π be the reduction of G_π and we adopt the notation in Definition 5.4 with $e_{\pi(C')} = v_1v_2$, and view $E(G_\pi) = (E(G) - E(C')) \cup \{v_1v_2\}$ and $V(G_\pi) = (V(G) - V(C')) \cup \{v_1, v_2\}$. Then for each $i \in \{1, 2\}$, $d_{G_\pi}(v_i) = d_G(x_i) + d_G(y_i) - 3$. As C' is a reducible 4-cycle, $d_{G_\pi}(v_i) \geq 2$, where equality holds if and only if exactly one of $d_G(x_i)$ and $d_G(y_i)$ equals 2 and the other equals 3. We have the following subclaims.

(**2A**) $|D_2(G_\pi)| \leq |D_2(G)|$.
If $d_{G_\pi}(v_i) > 2$, then $D_2(G_\pi) \subseteq D_2(G)$, and so (2A) holds. Assume that $d_{G_\pi}(v_i) = 2$. By symmetry, we may assume that $i = 2$ and $d_G(x_2) = 2$ and $d_G(y_2) = 3$. Then $D_2(G_\pi) = (D_2(G) - \{x_2\}) \cup \{v_2\}$, and so (2A) follows.
By Theorems 2.1, 5.5 and Claim 4(i), we have the following observation (2B).

(**2B**) Each of the following holds.
(i) The edge e_π cannot be contained in any collapsible subgraph of G_π and G'_π is nontrivial.
(ii) Any essential 2-edge-cut of G'_π must contain e_π.
Thus $e_\pi = v'_1v'_2 \in E(G'_\pi)$, where v'_i denotes the vertex of the contraction image in G_π that contains v_i. If $ess'(G'_\pi) \geq 3$, then G'_π satisfies the hypotheses of Theorem 3.2, and so by (16), G'_π is collapsible. By Theorems 2.1(iii) and 5.5(i), G is collapsible, contrary to (16). Hence

$$G'_\pi \text{ has an essential edge cut } X \text{ with } |X| = 2. \tag{17}$$

By Claim 4(i), we may assume that $X = \{v'_1v'_2, w_1w_2\}$ for some vertices $w_1, w_2 \in V(G_\pi)$. Let L_1, L_2 be the two components of $G - (E(C') \cup \{w_1w_2\})$ and we assume that for $i \in \{1, 2\}$, $w_i, x_i, y_i \in V(L_i)$. Thus $|V(L_i)| \geq 2$ where equality holds if and only if $w_i \in \{x_i, y_i\}$. By symmetry, assume that $|E(L_1 - \{w_1, x_1, y_1\})| \geq |E(L_2 - \{w_2, x_2, y_2\})|$. Throughout the rest of the proof, B' denotes the block of G'_π with $e_\pi \in E(B')$.

(**2C**) Each of the following holds.
(i) Any vertex $v \in D_2(B') - D_2(G)$ must be adjacent to e_π, and $|D_2(B')| \leq 4$.
(ii) $|\{v'_1, v'_2\} \cap D_2(B')| \leq 1$.

As (2C)(i) follows from $|D_2(G)| \le 2$, it suffices to show (2C)(ii). Suppose $d_{B'}(v_1') = d_{B'}(v_2') = 2$. Let $E_{B'}(v_1') = \{e_{v_1'}, e_\pi\}$, $E_{B'}(v_2') = \{e_{v_2'}, e_\pi\}$. By (2B)(i), $\{e_{v_1'}, e_{v_2'}\}$ cannot be an essential 2-edge-cut of B', and so $G[\{v_1'v_2', e_{v_1'}, e_{v_2'}\}]$ is a 3-cycle, contrary to (2B)(i).

(**2D**) Each of the following holds.
(i) $|E(L_1 - \{w_1, x_1, y_1\})| \neq 0$.
(ii) $|E(L_2 - \{w_2, x_2, y_2\})| = 0$.
(iii) x_i, y_i, w_i are mutually distinct.

Suppose $|E(L_2 - \{w_2, x_2, y_2\})| = |E(L_1 - \{w_1, x_1, y_1\})| = 0$. Since G is reduced, it cannot contain $K_{3,3}^-$ as a subgraph, and so we must have $V(L_2) - \{w_2, x_2, y_2\} \subseteq D_2(G)$. Furthermore, $|V(L_2) - \{w_2, x_2, y_2\}| = |D_2(G)| = 2$ by $|D_2(G)| \le 2$. Then $V(L_1) - \{w_1, x_1, y_1\} \subseteq D_3(G)$ and $|V(L_1) - \{w_1, x_1, y_1\}| \ge 2$. Let $\{u, v\} \subseteq V(L_1) - \{w_1, x_1, y_1\}$. Then $G[\{x_1x_2, x_2y_1, ux_1, uy_1, vx_1, vy_1, uw_1, vw_1\}]$ is a $K_{3,3}^-$, a contradiction. This proves (2D)(i).

Hence $L_1 - \{w_1, x_1, y_1\}$ must contain an edge $e_1 = z_1'z_2'$. To prove (2D)(ii), assume that $L_2 - \{w_2, x_2, y_2\}$ has an edge $e_2 = z_1''z_2''$. As $\kappa(G) \ge 2$, G has a cycle C_1 containing e_1 and e_2. By the choice of e_1 and e_2, $|E(C_1)| \ge 8$. Moreover, if $|E(C_1) \cap E(C')| = 1$, then $C_1 \triangle C'$ is a cycle of length $|E(C_1)| + 2 > 8$. Since $c(G) \le 8$, we may assume that $C_1 = z_1'z_2'y_1y_2z_2''z_1''x_2x_1z_1'$ is a cycle of length 8. Again by $\kappa(G) \ge 2$, G has a cycle C_1' containing $z_1'z_2'$ and w_1w_2. We may assume by symmetry that C_1' has a w_1z_2'-path Q_1 not containing z_1' and a w_2z_2''-path Q_2 not containing z_1''. But then G contains a cycle containing e_1 and e_2, and intersecting C' at only one edge, implying the existence of a cycle of length at least 9 in G, contrary to $c(G) \le 8$. This proves (2D)(ii).

As $x_2 \ne y_2$, we first assume that $w_2 \in \{x_2, y_2\}$ (say $w_2 = y_2$). Since G is reduced and since $E(L_2) \ne \emptyset$, there must be a vertex $w \in V(L_2) - \{x_2, y_2\}$ satisfying $wx_2, wy_2 \in E(G)$. Then $w \in D_2(G)$, and $G_\pi[\{v_2', w\}]$ contains a 2-cycle, and so $d_{G_\pi'}(v_2') = 2$. By (2B)(ii), $ess'(G_\pi') \ge 3$, contrary to (17). This proves that $|\{w_2, x_2, y_2\}| = 3$. Next, as $x_1 \ne y_1$, we assume that $w_1 \in \{x_1, y_1\}$ (say $w_1 = y_1$). If there exists $u \in V(L_2) - \{x_2, y_2, w_2\}$ such that $d_G(u) = 3$, then $G_\pi[u, v_1', v_2']$ is a 3-cycle, a collapsible subgraph containing e_π, contrary to (B)(i). Hence $V(L_2) - \{x_2, y_2, w_2\} \subseteq D_2(G)$, and so $|V(L_2) - \{x_2, y_2, w_2\}| \le |D_2(G)| \le 2$. This implies that $D_2 \subseteq V(L_2)$. As every vertex in $V(L_2) - \{x_2, y_2, w_2\}$ must be adjacent to two vertices in $\{x_2, y_2, w_2\}$, that $|V(L_2) - \{x_2, y_2, w_2\}| = 1$ would imply that $|D_2(G)| > 2$. Hence $|V(L_2) - \{x_2, y_2, w_2\}| = |D_2(G)| = 2$. Let $\{u, v\} = V(L_2) - \{x_2, y_2, w_2\} = D_2(G)$. We may assume $\{ux_2, uw_2, vy_2, vw_2\} \in E(G)$. Let $G_1 = G[V(G) - \{u, v, w_2\}]$. Then G_1 satisfies the hypotheses of Theorem 3.2, and so by (16), G_1 is collapsible. By Theorem 2.1, G is also collapsible, contrary to (16). This completes the proof of (2D).

By (2D)(i), in the rest of the arguments, we assume that $z_1', z_2' \in V(L_1 - \{w_1, x_1, y_1\})$ such that $z_1'z_2' \in E(L_1)$.

(**2E**) $c(B') \le 6$.

Let H be the block of G_π with $e_\pi \in E(H)$. Choose a longest cycle C in H such that $|\{e_\pi\} \cap E(C)|$ is maximized. By contradiction, assume that $|E(C)| \ge 7$. If $e_\pi \in E(C)$, then we may assume that $G[E(C - e_\pi)]$ is an x_1x_2-path in G. It follows that $G[E(C - e_\pi) \cup \{x_1y_2, y_2y_1, y_1x_2\}]$ is a cycle of G with length at least 9, contrary to $c(G) \le 8$. Hence e_π is not on any longest cycle of H, and so $|\{v_1, v_2\} \cap V(C)| \le 1$.

Suppose $|\{v_1, v_2\} \cap V(C)| = 0$. As $\kappa(H) \ge 2$, for $j \in \{1, 2\}$, H contains disjoint $v_iu_{i_j}$-path P_j' such that u_{i_1}, u_{i_2} are distinct vertices of C and $V(P_j') \cap V(C) = u_{i_j}$. By symmetry, we may assume $G[E(P_j')]$ is an $x_ju_{i_j}$-path P_j. Since $|E(C)| \ge 7$, C contains an $u_{i_1}u_{i_2}$-path P_3 such that $|E(P_3)| \ge 4$. Therefore $u_{i_1}P_3[u_{i_1}, u_{i_2}]u_{i_2}P_2^{-1}[x_2, u_{i_2}]x_2y_1y_2x_1P_1[x_1, u_{i_1}]u_{i_1}$ is a cycle of G with length at least 9, contrary to $c(G) \le 8$. Hence $|\{v_1, v_2\} \cap V(C)| = 1$. By (2D)(ii), we have $\{v_1, v_2\} \cap V(C) = \{v_1\}$. By $\kappa(H) \ge 2$, $H - v_1$ contains a v_2u_k-path P_4 such that $V(P_4) \cap V(C - v_1) = \{u_k\}$. By definition of L_1, $u_k \in V(L_1)$. By (2D)(iii), $|V(P_4)| \ge 3$. As e_π is not on any longest cycle of H, replacing edges in a v_1u_k-path on C by $E(P_4) \cup \{e_\pi\}$ will not result in a longest cycle of H, and so $|E(C)| = 8$. If $x_1, y_1 \in V(G[E(C)])$, then $G[E(C) \cup \{x_1x_2, x_2y_1\}]$ is a cycle of length at least 9, a contradiction. Hence we may assume that $V(C') \cap V(G[E(C)]) = \{x_1\}$. By symmetry, we assume that P_4 is a y_2u_k-path in G. Let P_5 be a longest x_1u_k-path on C with $|E(P_5)| \ge 4$. It follows that $x_1x_2y_1y_2P_4[y_2, u_k]u_kP_5^{-1}[x_1, u_k]x_1$ is a cycle of G with length at least 9. This proves (2E).

(**2F**) $|V(L_2) - \{x_2, y_2, w_2\}| \le 1$.

Suppose $V(L_2) - \{x_2, y_2, w_2\}$ contains two vertices a_1, a_2 with $d_G(a_1) \ge d_G(v)$ for any $v \in V(L_2) - \{x_2, y_2, w_2\}$. If $V(L_2) - \{x_2, y_2, w_2\} \subseteq D_2(G)$, then $V(L_2) - \{x_2, y_2, w_2\} = \{a_1, a_2\}$ and $\{a_1, a_2\} \subseteq N_G(w_2)$. Assume $a_1x_2 \in E(G)$ by symmetry. Suppose first that $\{x_1, y_1\}$ is a vertex-cut of G and let S' be the (x_1, y_1)-component contained in L_1. Then $G[E(S') \cup E(C')]$ satisfies each hypotheses of Theorem 3.2, whence by (16), $G[E(S') \cup E(C')]$ is collapsible, contrary to the assumption that G is reduced. Hence $\{x_1, y_1\}$ is not a vertex-cut of G, and so $G - y_1$ has two internally disjoint $z_1'a_1$-paths. Thus $L_1 - \{y_1\}$ contains internally disjoint $z_1'x_1$-path Q and $z_2'w_1$-path Q'. It follows that $z_1'z_2'Q'[z_2', w_1]w_1w_2a_1x_2y_1y_2x_1Q^{-1}[z_1', x_1]z_1'$ is a cycle of length at least 9, contrary to $c(G) \le 8$.

Hence $d_G(a_1) \ge 3$ and $\{a_1x_2, a_1y_2, a_1w_2\} \subseteq E(G)$. Since $\kappa(G) \ge 2$, G contains a cycle C'' with $z_1'z_2', a_1w_2 \in E(C'')$. As $\{z_1', z_2'\} \cap \{x_1, y_1, w_1\} = \emptyset$, C'' must use at least 3 edges in $E(L_1)$ and two edges incident with a_1, and so $|E(C'')| \ge 7$. By $c(G) \le 8$, it follows that $|V(C'') \cap V(C')| = 2$ and $|E(C'') \cap E(C')| = 1$. Hence $C'' \triangle C'$ is a cycle of length at least 9, a contradiction to the assumption $c(G) \le 8$. This proves (2F).

By (2F), we use $\bar{a}$ to denote the possible vertex in $V(L_2) - \{x_2, y_2, w_2\}$. By (2D), (2F) and by $|D_2(G)| \le 2$, we conclude that L_2 must contain one of the following graphs H_i, $(1 \le i \le 5)$, depicted in Fig. 2, as a subgraph.

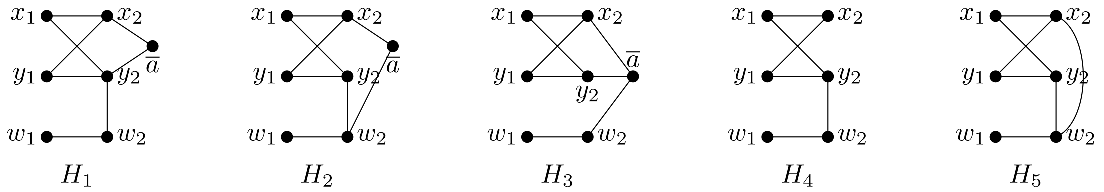

Fig. 2. The possible subgraphs in L_2.

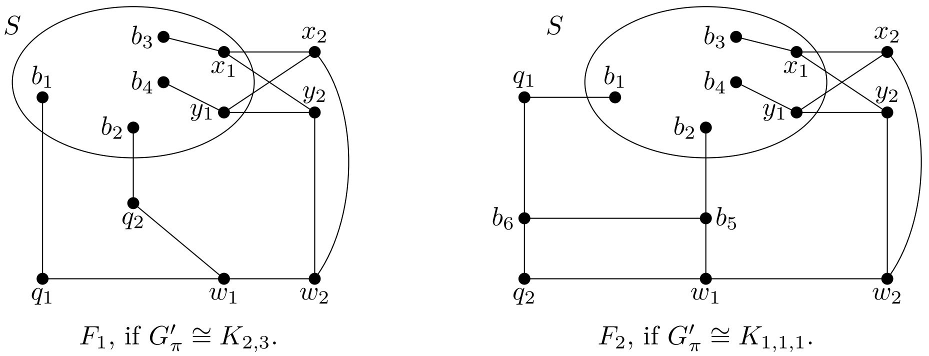

Fig. 3. The two possible structures of G.

(**2G**) None of H_1, H_2, H_4 can be a subgraph of L_2.

By contradiction, suppose that L_2 contains $H' \in \{H_1, H_2, H_4\}$ as a subgraph. Then we have $|D_2(B')| \leq 3$ and $d_{B'}(v'_2) = 2$. By (2E), $c(B') \leq 6$. If $|D_2(B')| \leq 2$, then by Lemma 4.4 and Observation 5.1(i), B' is collapsible, contrary to (B)(i). Hence we may assume $|D_2(G'_\pi)| = 3$ and $H' \neq H_2$, and so by (2C)(i) $d_{B'}(v'_1) = d_{B'}(v'_2) = 2$, contrary to (2C) (ii).

By (2G), either H_3 or H_5 is a subgraph of L_2. If $|D_2(B')| = 4$, then by (2C)(i), $d_{B'}(v'_1) = d_{B'}(v'_2) = 2$, violating (2C)(ii). This implies that $|D_2(B')| = 3$, $d_{B'}(v'_2) = 2$ and $d_{B'}(v'_1) \geq 3$. Let G_1 be a graph contains H_3 as a subgraph, and $G_2 = G_1 - \bar{a} + x_2w_2 + y_2w_2$. Then G_2 is obtained from G_1 by replacing H_3 by H_5 and so $c(G_1) \geq c(G_2)$. It is suffice to show $c(G_2) \geq 9$ to complete the proof of Claim 4. Hence we may assume that G contains H_5 as a subgraph.

(**2H**) $B' = G'_\pi$.

Assume that $G'_\pi - v'_2$ has a block $B'' \neq B'$. Then by (2D), $V(B') \cap V(B'') = \{v'_1\}$. If follows that $H'' = G[E(B'') \cup E(C')]$ is also a 2-edge-connected subgraph of G. As $|D_2(B')| = 3$, $D_2(G) \cap V(H'') = \emptyset$, and so $D_2(H'') = \{x_2, y_2\}$. Furthermore, any edge-cut of B'' not intersecting $E(C')$ is also an edge-cut of G, and any edge-cut of B'' intersecting C must be either the two edges incident with x_2 or y_2, or of size at least 3. Hence $ess'(B'') \geq 3$. Since $c(H'') \leq c(G) \leq 8$, it follows by (16) that H'' is collapsible, contrary to the assumption that G is reduced. This proves (2H).

By (2E) and (2H), $c(G'_\pi) = c(B') \leq 6$. It follows by Lemma 4.4, Observation 5.1 and $|D_2(B')| \leq 3$ that $G'_\pi \in \{K_{2,3}, K_{1,3}(1, 1, 1)\}$.

By $|D_2(G)| \leq 2$, (2C) and the structures of $K_{2,3}$ and $K_{1,3}(1, 1, 1)$, G_π contains only one maximal nontrivial collapsible subgraph S_1 with $v_1 \in V(S_1)$ in each of these two cases. Hence G must have one of the following structures.

In the following, we adopt the notation in Fig. 3, and so S denotes the preimage of S_1, $D_2(G) = \{q_1, q_2\}$ and $b_3 \in N_{L_1}(x_1)$, $b_4 \in N_{L_1}(y_1)$ in both of F_1, F_2, $N_G(q_1) = \{b_1, w_1\}$ in F_1 and $N_{L_1}(q_1) = \{b_1, b_6\}$ in F_2, $N_{L_1}(w_1) = \{q_1, q_2\}$ in F_1 and $N_{L_1}(w_1) = \{q_2, b_5\}$ in F_2.

Suppose that G has structure F_2. By symmetry we may assume that $d_G(b_1, x_1) \leq d_G(b_1, y_1)$. Let P_8 be a shortest b_1x_1-path in S. Then $x_1x_2y_1y_2w_2w_1b_5b_6q_1b_1P_8[b_1, x_1]x_1$ is a cycle of G with length at least 10, contrary to $c(G) \leq 8$. Hence G must have structure F_1.

(**2I**) $b_3 \neq b_4$ and $\{b_1, b_2\} \cap \{x_1, y_1\} = \emptyset$.

If $b_3 = b_4$, then $G[\{b_3x_1, b_3y_1, w_2x_2, w_2y_2\} \cup E(C')] \cong K^-_{3,3}$ is collapsible, contrary to the assumption that G is reduced. Thus $b_3 \neq b_4$.

Assume that $\{b_1, b_2\} \cap \{x_1, y_1\} \neq \emptyset$. Then by symmetry we assume that $b_1 = x_1$. Let $C_1 = y_1x_2w_2y_2y_1$. Using the notation in Definition 5.4, we let e'_π be the new edge in $G_{\pi(C_1)}$, the $\pi(C_1)$-reduction of G, and B'' be the block of $G'_{\pi(C_1)}$ containing e'_π. As $|D_2(G)| \leq 2$ and by applying (2C) and (2E) to $G_{\pi(C_1)}$ with B'' replacing B', we observe that $|D_2(B'')| \leq 3$ and $c(B'') \leq 6$.

Let $G'_{\pi(C_1)}$ be the reduction of $G_{\pi(C_1)}$, v_0 be the vertex onto which the collapsible subgraph of $G_{\pi(C_1)}$ containing x_1 is contracted. If $d_{G'_{\pi(C_1)}}(v) = 2$, then $q_1v \in E(G'_{\pi(C_1)})$ with $d_{G'_{\pi(C_1)}}(q_1) = d_{G'_{\pi(C_1)}}(v) = 2$. As $c(B'') \leq 6$, by Lemma 4.4 and

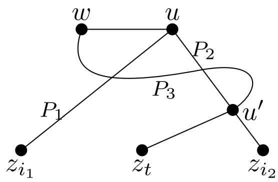

Fig. 4. Proof of Claim 7.

Observation 5.1, B'' is collapsible, and so is $G'_{\pi(C_1)}$. Hence by Theorems 2.1(iii) and 5.5(i), G is collapsible, contrary to (16). Hence $d_{G'_{\pi(C_1)}}(v) \geq 3$, and so by (2C), $|D_2(B'')| \leq 2$. As $c(B'') \leq 6$, by Lemma 4.4 and Observation 5.1(i), B'' is collapsible, which implies that G is collapsible, contrary to (16). This proves that $\{b_1, b_2\} \cap \{x_1, y_1\} = \emptyset$ and so (2I) is justified.

Let P_9 be a longest b_1v_1-path contained in S_1. By (2I), $\{b_1, b_2\} \cap \{x_1, y_1\} = \emptyset$, and so $|E(P_9)| \geq 1$. Suppose $|E(P_9)| = 1$. Let B_1 be the block of S_1 which contains P_9 and e_a, e_b be two edges incident with b_1 in B_1. Since any longest b_1v_1-path in S_1 has length 1, by $g(G) \geq 4$, we may assume that $e_a = b_1x_1$ and $e_b = b_1y_1$ in G. Then $G[\{b_1, x_1, y_1, x_2y_2, w_2\}] \cong K^-_{3,3}$ is collapsible, contrary to the assumption that G is reduced. Hence $|E(P_9)| \geq 2$. By symmetry, we may assume $G[E(P_9)]$ is a b_1, x_1-path P'_9. Thus $x_1x_2y_1y_2w_2w_1\, q_1b_1P'_9[b_1, x_1]x_1$ is a cycle of G of length at least 9, contrary to $c(G) \leq 8$. This completes the proof of Claim 4.

Let $c = c(G)$ and $C = z_1z_2...z_cz_1$ be a longest cycle of G. As C is longest, for $z_i, z_j \in V(C)$ with $1 \leq i < j \leq c$, we have:

any (z_i, z_j)-path in G internally disjoint from $V(C)$ has length at most $d_C(z_i, z_j)$. (18)

Claim 5. *$|E(G[V(C)]) - E(C)| \leq 2$ and $V(G) - V(C) \neq \emptyset$.*

Suppose there exist three edges $e_1, e_2, e_3 \in E(G[V(C)]) - E(C)$. If $c(G) = 7$, then as $g(G) \geq 4$, $G[E(C) \cup \{e_1\}]$ contains a reducible 4-cycle, contrary to Claim 4(iii). Hence $c(G) = 8$. By Claim 4(iii), we must have $\{e_1, e_2, e_3\} \subset \{z_1z_5, z_2z_6, z_3z_7, z_4Z_8\}$. This forces that $E(C) \cup \{e_1, e_2, e_3\}$ contains a reducible 4-cycle, which is also contrary to Claim 4(iii). Hence $E(G[V(C)]) - E(C)| \leq 2$. Since $|D_2(G)| \leq 2$, we must have $V(G) - V(C) \neq \emptyset$. This proves Claim 5.

Claim 6. *There exists $v \in V(G) - V(C)$ such that $d_G(v) \geq 3$.*

Suppose $d_G(v) = 2$ for any $v \in V(G) - V(C)$. Then $|V(G) - V(C)| \leq |D_2(G)| \leq 2$, and so there exists $z_iz_j \in E[G(V(C))] - E(C)$ where $1 \leq i < j \leq c$. As $|V(C)| = 7$ would imply that $G[E(C) \cup \{z_iz_j\}]$ contains a reducible 4-cycle, violating Claim 4(iii), we must have $c = 8$, and so by $|D_2(G)| \leq 2$ and Claim 5, $|E(G[V(C)]) - E(C)| = 2$. It follows by $V(G) - V(C) \subseteq D_2(G)$ that $|V(G) - V(C)| = 2$. Suppose that $E[G(V(C))] - E(C) = \{z_{i_1}z_{i_2}, z_{i_3}z_{i_4}\}$, $V(G) - V(C) = D_2(G) = \{u, v\}$ and $uz_{i_5}, uz_{i_6}, vz_{i_7}, vz_{i_8} \in E(G)$. Since G is reduced and by Claim 4(iii), each pair of vertices in $\{z_{i_1}, z_{i_2}, z_{i_3}, z_{i_4}\}$ must have distance 2 on C, and so we may assume $i_1 = 1, i_3 = 3, i_2 = 5, i_4 = 7$. Then by Claim 4 (*iii*), we must have $\{\{i_5, i_6\}, \{i_7, i_8\}\} = \{\{2, 6\}, \{4, 8\}\}$. It follows that $G = P(10)^-$. This implies that $c(G) = 9$, contrary to $c(G) = 8$, and so Claim 6 follows (see Fig. 4).

Claim 7. *There exists a vertex $v \in V(G) - V(C)$ such that there are three internally disjoint vz_{i_j}-path P_j where $z_{i_j} \in V(C)$ and $V(P_j) \cap V(C) = \{z_{i_j}\}$ for $j \in \{1, 2, 3\}$.*

By Claim 6, there exists a vertex $u \in V(G) - V(C)$ with $d_G(u) \geq 3$. As $\kappa(G) \geq 2$, G contains a uz_{i_1}-path Q_1 and a uz_{i_2}-path Q_2 with $z_{i_1} \neq z_{i_2}$, $V(Q_1) \cap V(Q_2) = \{u\}$ and for $j \in \{1, 2\}$, $V(Q_j) \cap V(C) = \{z_{i_j}\}$. Let f_j be the edge in Q_j incident with z_{i_j}. Since $d_G(u) \geq 3$, there exists an edge $uw \in E(G) - E(Q_1) \cup E(Q_2)$. If $w \in V(C)$, then done. Assume that $w \notin V(C)$. As $ess'(G) \geq 3$, $G - \{f_1, f_2\}$ has a uz_t-path Q_3 with $V(Q_3) \cap V(C) = \{z_t\}$. We may assume that $V(Q_3) \cap V(Q_1) \cup V(Q_2) - (V(C) \cup \{u\}) \neq \emptyset$, as otherwise the claim holds. Let $v \in V(Q_3) \cap V(Q_1) \cup V(Q_2) - (V(C) \cup \{u\})$ such that for $j \in \{1, 2\}$, if $v \in V(Q_j)$, then $V(Q_j[v, z_{i_j}]) \cap V(Q_3) \subseteq V(C) \cup \{v\}$. Assume that $v \in V(Q_1)$. Let $P_1 = Q_1[v, z_{i_1}]$, $P_2 = Q_1^{-1}[v, u]Q_2[u, z_{i_2}]$ and $P_3 = Q_3[v, z_t]$. Then P_1, P_2, P_3 are the paths satisfying the claim. This justifies Claim 7.

Let $v \in V(G) - V(C)$ and P_j be vz_{i_j}-path where $z_{i_j} \in V(G)$ and $V(P_j) \cap V(C) = \{z_{i_j}\}$ for $j \in \{1, 2, 3\}$. By Claim 7, there exists a vertex $v \in V(G) - V(C)$ and three internally disjoint vz_{i_j}-paths $P_j, j \in \{1, 2, 3\}$, with $V(P_j) \cap V(C) = \{z_{i_j}\}$. We label the P_i's so that $|E(P_1)| \leq |E(P_2)| \leq |E(P_3)|$.

Claim 8. *Each of the following holds.*
(i) $|\{z_{i_1}, z_{i_2}, z_{i_3}\}| < 3$, $|E(P_2)| \geq 2$ and $|E(P_3)| \leq 3$.
(ii) If $|E(P_1)| \geq 2$, then $|E(P_1)| = |E(P_2)| = |E(P_3)| = 2$.
(iii) If $|E(P_1)| = 1$, then $z_{i_2} = z_{i_3}$.

Assume by contradiction that $|\{z_{i_1}, z_{i_2}, z_{i_3}\}| = 3$. If $|E(P_2)| \geq 2$, then by (18), we have $8 \geq c(G) \geq d_C(z_{i_3}, z_{i_1}) + d_C(z_{i_3}, z_{i_2}) + d_C(z_{i_1}, z_{i_2}) = 3 + 4 + 3 = 10$, a contradiction. Thus we may assume that $|E(P_2)| = 1$. Then no matter whether

$|E(P_3)| \geq 2$ or $|E(P_3)| = 1$, as G is reduced and $c(G) \leq 8$ and by (18), either P_1 or P_2 is always in a reducible 4-cycle of G, contrary to Claim 4(iii). Hence $|\{z_{i_1}, z_{i_2}, z_{i_3}\}| < 3$. Next we assume that $|E(P_2)| = |E(P_1)| = 1$. As $g(G) \geq 4$, we cannot have $z_{i_1} = z_{i_2}$, and so by the symmetry between P_1 and P_2, we may assume that $z_{i_1} = z_{i_3}$. By $g(G) \geq 4$, we have $|E(P_3)| \geq 3$. If $E(P_3)| = 3$, then $E(P_1) \cup E(P_3)$ induces a reducible 4-cycle of G, contrary to Claim 4(iii). Hence $|E(P_3)| \geq 4$. But then $E(P_3) \cup E(P_2)$ induces a path of length at least 5 with both ends on $V(C)$, contrary to (18). This proves that $|E(P_2)| \geq 2$. If $|E(P_3)| \geq 4$, then for some $j \in \{1, 2\}$, $E(P_3) \cup E(P_j)$ induces a path of length at least 5 with end vertices on $V(C)$, contrary to (18), and so (i) is justified.

Now assume that $|E(P_1)| \geq 2$. Then for $j \in \{2, 3\}$, $E(P_1) \cup E(P_j)$ is either a cycle or a path. If $E(P_1) \cup E(P_j)$ is a path, then both ends of this path are on $V(C)$. As $c(G) \leq 8$, $4 \leq 2|E(P_1)| \leq |E(P_1) \cup E(P_j)| \leq 4$, implying $|E(P_1)| = |E(P_j)| = 2$, and so we may assume that $j = 2$ and $E(P_1) \cup E(P_j)$ is a cycle. But then, $E(P_2) \cup E(P_3)$ is a path with both ends of it on $V(C)$. As $c(G) \leq 8$, we also have $|E(P_2)| = |E(P_3)| = 2$, and so (ii) follows.

Assume that $|E(P_1)| = 1$. If $z_{i_1} = z_{i_j}$ for some $j \in \{2, 3\}$. Then as $g(G) \geq 4$, we have $|E(P_3)| \geq |E(P_j)| \geq 3$. It follows by Claim 8(i) that $E(P_3) \cup E(P_2)$ induces a path of length at least 5 with both ends on $V(C)$, contrary to (18). Hence we must have $z_{i_2} = z_{i_3}$. This completes the proof of the claim.

By Claim 8, we may assume $z_{i_2} = z_{i_3}$, and there exist vertices $u^2 \in V(P_3) - \{v, z_{i_2}\}$ and $u^3 \in V(P_2) - \{v, z_{i_3}\}$ such that $\{vu^2, vu^3\} \subseteq E(G)$. By (18), $|V(P_1)| \leq 3$. Let p be the possible vertex of P_1 such that $p \notin \{v, z_{i_1}\}$.

Claim 9. *Each of the following holds.*
(i) For $j \in \{2, 3\}$, if $|E(P_j)| = 2$, then $u^j \in D_2(G)$.
(ii) $D_2(G) = \{u^2, u^3\}$.

Let $j \in \{j, j'\} = \{2, 3\}$ and $d_G(u^j) \geq 3$. By $\kappa(G) \geq 2$, u^j is adjacent to a vertex $q \in V(G) - V(P_j)$ such that u^jq is on a u^jz-path P_4 with $V(P_4 - u^j) \cap (V(C) \cup V(P_1) \cup V(P_2) \cup V(P_3)) = \{z\}$. By Claim 8 with v being replaced by u^j, we conclude that $z \in V(P_1) \cup V(P_2) \cup V(P_3)$.

Assume that $|E(P_j)| = 2$. As $g(G) \geq 4$ and $c(G) \leq 8$, either $z = z_{i_1}$, $|E(P_4)| = 1$ and $|E(P_1)| = 2$, whence $vpzu^jv$ is a reducible 4-cycle of G, contrary to Claim 4(iii); or $z = z_{i_1}$, $|E(P_4)| \geq 2$, whence $P_{j'}[z_{i_2}, v]vu^jP_4[u_j, z]$ is a path of length at least 5 with end vertices on $V(C)$, contrary to (18); or $z = z_{i_2}$ and $|E(P_4)| \geq 3$ whence $P_4[z_{i_2}, u^j]u^jvP_1[v, z_{i_1}]$ is a path of length at least 5 with end vertices on $V(C)$, contrary to (18). This proves (i).

If $|E(P_3)| = 2$, then by Claim 8, $|E(P_2)| = |E(P_3)| = 2$, and so Claim 9(i) implies (ii). Thus we assume that $|E(P_3)| = 3$ to show that $u^3 \in D_2(G)$. By (18), $|E(P_1)| = 1$. If $z = z_{i_1}$, then by $g(G) \geq 4$ and by Claim 4(iii), $|E(P_4)| \geq 3$. It follows that $P_3[z_{i_2}, u^3]P_4[u^3, z_{i_1}]$ is a path of length at least 5 with end vertices on $V(C)$, contrary to (18). This proves (ii), as well as Claim 9.

We now complete the proof of Theorem 3.2 by finding a contradiction. If there exists a vertex $v' \in V(G) - V(C) \cup \{v, u, w\}$, then by Claim 9, $d_G(v') \geq 3$. Applying Claims 6–9 to the case when v is replaced by v', we are led to the conclusion that v' must be adjacent to both vertices in $D_2(G)$. It follows that for $j \in \{2, 3\}$, the vertex u^j must be adjacent to distinct vertices v, v' and a vertex in $V(P_j) - \{v\}$, contrary to the fact that $u^j \in D_2(G)$. Hence we must have $V(G) = V(C) \cup \{v, u^2, u^3\}$. As $D_2(G) = \{u^2, u^3\}$, we must have $|E(G[V(C)]) - E(C)| \geq 3$, contrary to Claim 5. This completes the proof of the theorem. ■

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory, Springer, New York, 2008.
[2] H.J. Broersma, On some intriguing problems in hamiltonian graph theory-A survey, Discrete Math. 251 (2002) 47–69.
[3] H.J. Broersma, H.J. Veldman, 3-connected line graphs of triangular graphs are panconnected and 1-hamiltonian, J. Graph Theory 11 (1987) 399–407.
[4] J. Brousek, Z. Ryjáček, O. Favaron, Forbidden subgraphs hamiltonianicity and closure in claw-free graph, Discrete Math. 196 (1999) 29–50.
[5] P.A. Catlin, Super-eulerian graphs, collapsible graphs, and four-cycles, Congr. Numer. 58 (1987) 233–246.
[6] P.A. Catlin, A reduction method to find spanning eulerian subgraph, J. Graph Theory 12 (1988) 29–45.
[7] P.A. Catlin, Super-eulerian graphs, a survey, J. Graph Theory 16 (1992) 177–196.
[8] Z.-H. Chen, H.-J. Lai, Reduction techniques for super-eulerian graphs and related topics-A survey, in: Combinatorics and Graph Theory 95, Vol. 1, World Sci. Publishing, Hefei, River Edge, NJ, 1995, pp. 53–69.
[9] Z.-H. Chen, H.-J. Lai, Supereulerian graph and the petersen graph, II, Ars Combin. 48 (1998) 271–282.
[10] Z.H. Chen, H.-J. Lai, D.Y. Li, W. Shiu, An s-hamiltonian line graph problem, Graphs Combinatoircs 23 (2007) 241–248.
[11] R. Faudree, E. Flandrin, Z. Ryjáček, Claw-free graphs-A survey, Discrete Math. 164 (1977) 87–147.
[12] J. Fujisawa, Forbidden subgraphs for hamiltonicity of 3-connected cCaw-free graphs, J. Graph Theory 73 (2013) 146–160.

[13] R.J. Gould, Advances on the hamiltonian problem-A survey, Graphs Combin. 19 (2003) 7–52.
[14] R.J. Gould, Recent advances on the hamiltonian problem: Survey III, Graphs Combin. 30 (2014) 1–16.
[15] F. Harary, C. St. J. A. Nash-Williams, On eulerian and hamiltonian graphs and line graphs, Canad. Math. Bull. 8 (1965) 701–709.
[16] Z. Hu, S. Zhang, Every 3-connected $\{K_{1,3}, N_{1,2,3}\}$-free graph is hamiltonian-connected, Graphs Combin. 32 (2016) 685–705.
[17] T. Kaiser, Z. Ryjáček, P. Vrána, On 1-Hamilton-connected claw-free graphs, Discrte Math. 321 (2014) 1–11.
[18] T. Kaiser, P. Vrána, Hamilton cycles in 5-connected line graphs, European J. Combin. 33 (2012) 924–947.
[19] R. Kučzel, L. Xiong, Every 4-connected line graph is hamiltonian if and only if it is hamiltonian connected in: R. Kučzel (Ed.), Hamiltonian Properties of Graphs, (Ph.D. Thesis), U.W.B. Pilsen, 2004.
[20] H.-J. Lai, Reduced graphs of diameter 2, J. Graph Theory. 14 (1990) 77–87.
[21] H.-J. Lai, Y. Shao, On s-hamiltonian line graphs, J. Graph Theory 74 (2013) 344–358.
[22] H.-J. Lai, Y. Shao, H. Yan, An update on supereulerian graphs, Wseas Trans. Math. 12 (2013) 926–940.
[23] H.-J. Lai, Y. Shao, G. Yu, M. Zhan, Hamiltonian connectedness in 3-connected line graphs, Discrete Appl. Math. 157 (2009) 982–990.
[24] H.-J. Lai, L. Xiong, H. Yan, J. Yan, Every 3-connected claw-free Z_8-free graph is hamiltonian, J. Graph Theory 664 (2010) 1–11.
[25] H.-J. Lai, M. Zhan, T. Zhang, J. Zhou, On s-hamiltonian line graphs of claw-free graphs, Discrete Math. 342 (2019) 3006–3016.
[26] D. Li, H.-J. Lai, M. Zhan, Eulerian subgraphs and hamiltonian connected line graphs, Discrete Appl. Math. 145 (2005) 422–428.
[27] P. Li, H. Li, Y. Chen, H. Fleischner, H-.J. Lai, Supereulerian graphs with width s and s-collapsible graphs, Discrete Appl. Math. 200 (2016) 79–94.
[28] P. Li, K. Wang, M. Zhan, H.-J. Lai, Strongly spanning trailable graphs with short longest paths, Ars Combin. 137 (2018) 3–39.
[29] X. Ma, H.-J. Lai, W. Xiong, B. Wu, X. An, Supereulerian graphs with small circumference and 3-connected hamiltonian claw-free graphs, Discrete Appl. Math. 202 (2016) 111–130.
[30] M.M. Matthews, D.P. Sumner, Hamiltonian results in $K_{1,3}$-free graphs, J. Graph Theory 8 (1984) 139–146.
[31] J.G. Oxley, Matroid Theory, second ed., Oxford University Press, New York, 2011.
[32] Z. Ryjáček, On a closure concept in claw-free graphs, J. Combin. Theory Ser. B 70 (1997) 217–224.
[33] Z. Ryjáček, P. Vrána, Line graphs of multigraphs and hamilton-connectedness of claw-free graphs, J. Graph Theory 66 (2011) 152–173.
[34] Y. Shao, Claw-Free Graphs and Line Graphs (Ph. D. Dissertation), West Virginia University, 2005.
[35] C. Thomassen, Reflections on graph theory, J. Graph Theory 10 (1986) 309–324.
[36] W. Xiong, H.-J. Lai, X. Ma, K. Wang, M. Zhang, Hamilton cycles in 3-connected claw-free and net-free graphs, Discrete Math. 313 (2013) 784–795.
[37] S.M. Zhan, On hamiltonian line graphs and connectivity, Discrete Math 89 (1991) 89–95.