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ABSTRACT

A graph is supereulerian if it has a spanning closed trail. Catlin in 1990 raised the problem of determining the reduced nonsupereulerian graphs with small orders, as such results are of particular importance in the study of Eulerian subgraphs and Hamiltonian line graphs. We determine all reduced graphs with order at most 14 and with few vertices of degree 2, extending former results of Chen and Chen in 2016. In 1985, Bauer proposed the problems of determining best possible sufficient conditions on minimum degree of a simple graph (or a simple bipartite graph, respectively) G to ensure that its line graph $L(G)$ is Hamiltonian. These problems have been settled by Catlin and Lai in 1988, respectively. As an application of our main results, we prove the following for a connected simple graph G on n vertices:

i. If $\delta(G) \geq \frac{n}{10}$, then for sufficiently large n, $L(G)$ is Hamilton-connected if and only if both $\kappa(G) \geq 3$ and G is not nontrivially contractible to the Wagner graph.
ii. If G is bipartite and $\delta(G) > \frac{n}{20}$, then for sufficiently large n, $L(G)$ is Hamilton-connected if and only if both $\kappa(G) \geq 3$ and G is not nontrivially contractible to the Wagner graph.

KEYWORDS
Eulerian graphs; collapsible graphs; reduced graphs

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1. Introduction

We generally follow the notation and terminology of Bondy and Murty [3], except as otherwise stated. Graphs considered in this paper are finite and loopless, but multiple edges are allowed. A cycle on n vertices, denoted by C_n, is called an n-cycle. A graph G is Hamiltonian if it has a spanning cycle, and is Hamiltonian-connected if for any distinct vertices u and v, G contains a spanning (u, v)-path. As in [3], $\kappa(G)$ and $\kappa'(G)$ denote connectivity and the edge-connectivity of a graph G, respectively. If G has a cycle, the *girth* of G, denoted by $girth(G)$, is the length of a shortest cycle in G. For a connected graph G and $u, v \in V(G)$, $dist(u, v)$ denotes the distance between u and v. For a vertex $v \in V(G)$, define $N_G(v) = \{u \in V(G) : vu \in E(G)\}$, $E_G(v) = \{e \in E(G) : e \text{ is incident with } v \text{ in } G\}$, and $d_G(v) = |E_G(v)|$. For an integer $i \geq 0$, define $D_i(G) = \{v \in V(G) : d_G(v) = i\}$ and $d_i(G) = |D_i(G)|$. For any $v \in D_1(G)$, the edge $e \in E_G(v)$ is called a *pendant edge* of G. Let $O(G)$ be the set of vertices of odd degree in G. A connected graph G is *Eulerian* if $O(G) = \emptyset$. An Eulerian subgraph H in G is a spanning Eulerian subgraph if $V(H) = V(G)$. A graph is *supereulerian* if it has a spanning Eulerian subgraph, which is equivalent to the statement that G has a spanning closed trail. Throughout this paper, $\mathcal{S}_n$ denotes the family of all supereulerian graphs on n vertices, and $\mathcal{S} = \cup_{n\geq 1}\mathcal{S}_n$ is the family of all supereulerian graphs. We use $P(10)$ for the Petersen graph and let $P^-(10), P(11), P^1(12), P^2(12), P^3(12), P^1(13), P^2(13)$, $P^1(14)$, and $P^2(14)$ be the graphs shown in Figure 1, respectively.

Let G be a graph and $X \subseteq E(G)$ be an edge subset. The *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. We define $G/\emptyset = G$. If $H \subseteq G$, then we write G/H for $G/E(H)$. If H is a connected subgraph of G, and if v_H is the vertex in G/H onto which H is contracted, then H is the *preimage* of v_H, and is denoted by $PI(v_H)$. As an edgeless graph is viewed as trivial, if G is contracted to a graph G' in such a way that every vertex of G' has nontrivial preimage in G, we say that G' is a *nontrivial contraction* of G.

To study supereulerian graphs, Catlin [5] introduced collapsible graphs in his investigation on graphs H with the property that for any graph G containing H as a subgraph, G is supereulerian if and only if the contraction G/H is supereulerian. A graph G is *collapsible* if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph G_R such that $O(G_R) = R$. Catlin indicated in [5] that for any graph G, every vertex of G lies in a unique maximal collapsible subgraph of G. The *reduction* of G, denoted by G', is obtained from G by contracting all maximal collapsible subgraphs of G. A graph is *reduced* if it is the reduction of some graph. Catlin [5] proved that a graph

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In honor of Lowell Beineke's 80th birthday.

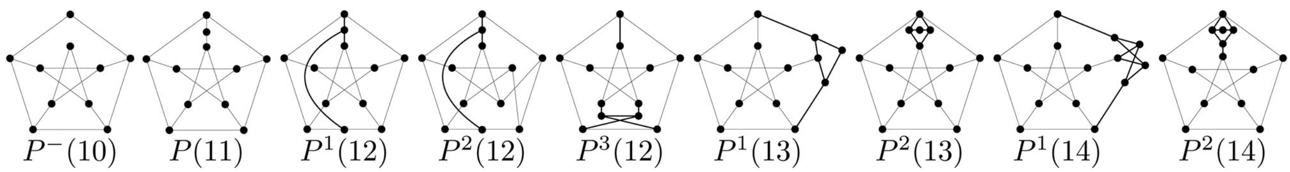

Figure 1. The graphs $P^-(10), P(11), P^1(12), P^2(12), P^3(12), P^1(13), P^2(13), P^1(14), P^2(14)$.

G is supereulerian if and only if the reduction of G is supereulerian, thereby developing a reduction method in [5] to study supereulerian graphs.

In order to apply Catlin's reduction method by contracting collapsible subgraphs, identifying small reduced graphs is of particular importance [7,14,18]. Catlin first raised the problem of determining all reduced graphs with small orders and proposed the following conjecture.

Conjecture 1.1. (Catlin [6]). *Any 3-edge-connected simple graph of order at most 17 is either supereulerian or is contractible to the Petersen graph.*

Conjecture 1.1 has several extended versions, as seen in [6,8,21]. The following theorem shows some progresses toward Conjecture 1.1. By Catlin's reduction method, it is common to reduce the generic study on eulerian subgraphs into the study of reduced graphs with small orders. Because of this, results on reduced graph with small orders play important roles in applications of Catlin's reduction method, and have been applied to study eulerian subgraphs and Hamiltonian line graphs by many authors, as seen in [5,6,9,10,13,17,19–25,28–31], among others. Theorem 1.2 presents some of the frequently applied such results.

Theorem 1.2. *Let G be a connected graph of order n and with G' as defined above.*

(i) (Chen and Lai [13]) If $n \le 11$ and $\delta(G) \ge 3$, then $G' \in \{K_1, K_2, P(10)\}$.
(ii) (Chen [11]) If $\kappa'(G) \ge 3$ and $n \le 11$, then $G' \in \{K_1, P(10)\}$.
(iii) (Chen and Chen [12]) If $\kappa'(G) \ge 3$ and $n \le 13$, then either $G \in \mathcal{S}$ or $G' = P(10)$.
(iv) (Chen and Chen [12]) If $\kappa'(G) \ge 3$ and $n \le 14$, then either $G \in \mathcal{S}$ or $G' \in \{P(10), P^1(14)\}$.

Let s_1, s_2, s_3, m, l, t be the nonnegative integers with $t \ge 2$ and $m, l \ge 1$. Let $M \cong K_{1,3}$ with center a and ends a_1, a_2, a_3. Define $K_{1,3}(s_1, s_2, s_3)$ to be the graph obtained from M by adding s_i vertices with neighbors $\{a_i, a_{i+1}\}$, where $i \equiv 1, 2, 3 (\text{mod } 3)$. Let $K_{2,t}(u, u')$ be a $K_{2,t}$ with u, u' being the nonadjacent vertices of degree t. Let $K'_{2,t}(u, u', u'')$ be the graph obtained from a $K_{2,t}(u, u')$ by adding a new vertex u'' that joins to only u'. Hence u'' has degree 1 and u has degree t in $K'_{2,t}(u, u', u'')$. Let $K^+_{2,t}(u, u', u'')$ be the graph obtained from a $K_{2,t}(u, u')$ by adding a new vertex u'' that joins to a vertex of degree 2 of $K_{2,t}$. Hence u'' has degree 1 and both u and u' have degree t in $K_{2,t}''(u, u', u'')$. We shall use $K'_{2,t}$ and $K_{2,t}''$ for a $K'_{2,t}(u, u', u'')$ and a $K_{2,t}''(u, u', u'')$, respectively. Let $S(m, l)$ be the graph obtained from a $K_{2,m}(u, u')$ and $K'_{2,l}(w, w')$ by identifying u with w, and connecting $u'w'$; let $J(m, l)$ denote the graph obtained from a $K_{2,m+1}$ and a $K'_{2,l}(w, w', w'')$ by identifying w, w'' with the two ends of an edge in $K_{2,m+1}$, respectively; let $T(m, l)$ denote the graph obtained from a $K_{2,m+2}$ and a $K'_{2,l}(w, w', w'')$ by identifying w, w'' with two vertices of degree 2 in $K_{2,m+2}$, respectively. See Figure 2 for examples of these graphs. Let

$$\mathcal{EG} = \{K_1, K_2, K_{2,t}, K'_{2,t}, K^+_{2,t}, K_{1,3}(s, s', s''), S(m, l), J(m, l), T(m, l), P\},$$

where t, s, s', s'', m, l are nonnegative integers.

Theorem 1.3 (Chen and Chen [12]). *Let G be a connected graph of order n and with G' as defined above.*

(i) Let $\delta(G) \ge 2$ and $d_2(G) \le 2$. If $n \le 6$, then $G' = K_1$, and if $n \le 7$, then $G' \in \{K_1, K_2\}$.
(ii) If $G \ne K_1$ is reduced, $n \le 7, \kappa'(G) \ge 2$ and $d_2(G) = 3$, then $G \in \{K_{2,3}, K_{1,3}(1,1,1), T(1,1)\}$.
(iii) If $n \le 9, d_1(G) = 0$ and $d_2(G) \le 1$, then $G' \in \{K_1, K_2, K_{1,2}\}$.
(iv) If $n \le 9, \kappa'(G) \ge 2$ and $d_2(G) \le 2$, then $G' \in \{K_1, K_{2,3}\}$. Furthermore, if G is K_3-free, $G' = K_1$.
(v) If $n \le 10, \kappa'(G) \ge 2$, and $d_2(G) \le 1$, then $G' \in \{K_1, P(10)\}$.

Theorem 1.4 (Li et al. [20]). *Let G be a connected graph of order n and with G' as defined above. If $n \le 8, d_1(G) = 0$ and $d_2(G) \le 2$, then $G' \in \{K_1, K_2, K_{2,3}\}$. Furthermore, if $G' = K_{2,3}$ with $D_2(G') = \{v_1, v_2, v_3\}$ and $D_3(G') = \{u_1, u_2\}$, then $PI(v_1)$ is either K_4 or K_4 minus an edge, and other vertices in G' are trivial.*

Following Catlin [5], let $F(G)$ be the minimum number of additional edges that must be added to a graph G to result in a graph with two edge-disjoint spanning trees.

Theorem 1.5 (Chen and Lai [13]). *Let G be a connected reduced graph with $|V(G)| \le 11$ and $F(G) \le 3$. Then $G \in \mathcal{EG}$. In particular, if $d_1(G) = 0$ and $d_2(G) \le 2$, then $G \in \{K_1, P(10)\}$.*

Theorem 1.6 (Chen [10,13]). *Let G be a connected simple graph of order n. Let G' be the reduction of G. If $n \le 13$ and $\delta(G) \ge 3$, then either $G \in \mathcal{S}_{12}$, or $G' \in \{K_1, K_2, K_{1,2}, K_{1,3}, P(10)\}$.*

To present our results in this paper, more graphs need to be introduced. Let $T_1^+(1,1), T_2^+(1,1), K^2_{2,3}, P^2(11), P^4(12)$, and $P^5(12)$ be the graphs as shown in Figure 3. We use $K^+_{1,3}(1,1,1)$, $C_4^+, (P^-(10))^+, (K^2_{2,3})^+$ to denote the graphs obtained from $K_{1,3}(1,1,1), C_4, P^-(10), K^2_{2,3}$, respectively, by attaching a pendant edge to a vertex of degree two, use $(P(10))^+$ to denote the graph obtained from $P(10)$ by adding a pendant edge, and use $K^+_{1,3}$ to denote the graph

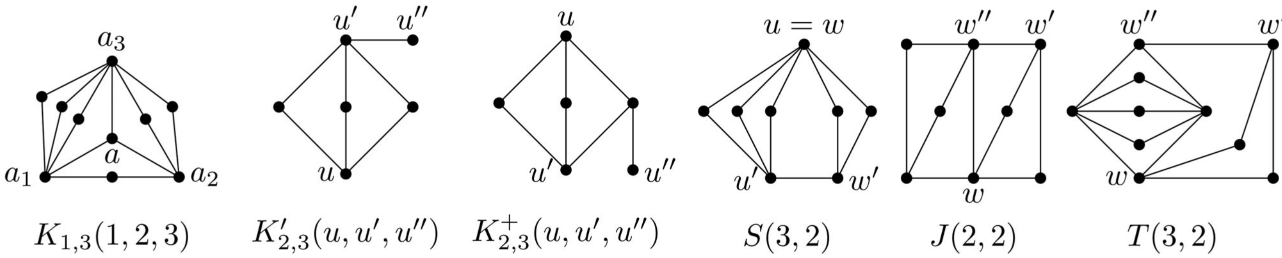

Figure 2. Some graphs in $\mathcal{EG}$ with small parameters.

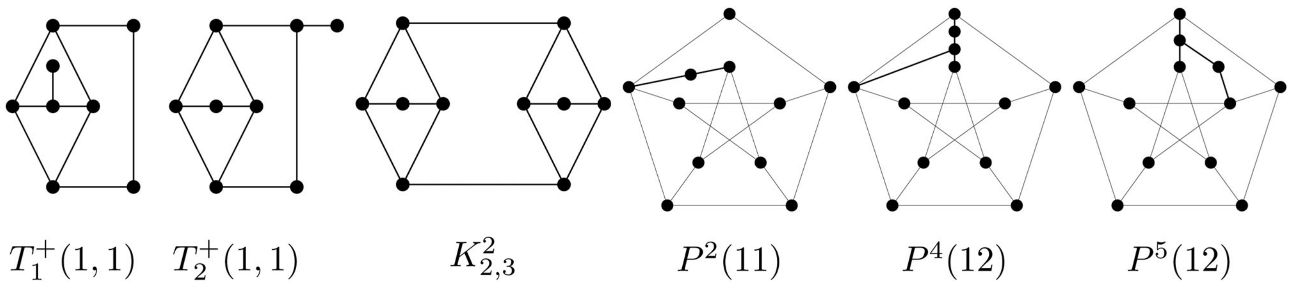

Figure 3. Some graphs used in Theorem 1.7.

obtained from $K_{1,3}$ by adding a pendant edge to a vertex of degree one. Denote $K^{++}_{2,3}$ be the graph obtained from $K_{2,3}$ by adding two pendant edges to two vertices of degree two, respectively. Define $\mathcal{F}_{11} = \{K_1, K_2, K_{1,2}, K_{2,3}, P(10), P(11)\}$, $\mathcal{F}_{12} = \{K^+_{2,3}\}$, $\mathcal{F}_{13} = \{K_{1,3}, P_4, K^{++}_{2,3}, K_{1,3}(1,1,1), P^-(10), P^1(13), P^2(13)\}$, and $\mathcal{F}_{14} = \{K^+_{1,3}, C_4^+, (K^2_{2,3})^+, K^+_{1,3}(1,1,1), T_1^+(1,1), T_2^+(1,1), (P^-(10))^+, (P(10))^+, P^1(14), P^2(14)\}$. For application purposes, relaxations of the above Theorems are often needed. This motivates our current research.

Theorem 1.7. *Let G' be the reduction of a connected simple graph G of order n. If $n \le 11, d_1(G) = 0$ and $d_2(G) \le 2$, then $G' \in \mathcal{F}_{11} \cup \{P_4, C_4, K^+_{2,3}, K^2_{2,3}, K_{1,3}(1,1,1), T(1,1), T(1,2), P^-(10), K^+_{1,3}(1,1,1), T_1^+(1,1), T_2^+(1,1)\}$. Furthermore, if $d_2(G) \le 1$, then $G' \in \mathcal{F}_{11}$.*

Theorem 1.8. *Let G' be the reduction of a connected simple graph G of order n. Suppose that $d_1(G) = 0$ and $d_2(G) \le 1$. Then the following statements hold:*

(i) If $n \le 12$, then $G' \in \mathcal{F}_{11} \cup \mathcal{F}_{12} \cup \{P^1(12), P^2(12), P^3(12)\}$. Therefore, either $G \in \mathcal{S}_{12}$ or $G' \in \mathcal{F}_{11} \cup \mathcal{F}_{12}$.
(ii) If $n \le 13$, then either $G \in \mathcal{S}_{12} \cup \mathcal{S}_{13}$, or $G' \in \mathcal{F}_{11} \cup \mathcal{F}_{12} \cup \mathcal{F}_{13}$.
(iii) If $n \le 14$, then either $G \in \mathcal{S}_{12} \cup \mathcal{S}_{13} \cup \mathcal{S}_{14}$, or $G' \in \mathcal{F}_{11} \cup \mathcal{F}_{12} \cup \mathcal{F}_{13} \cup \mathcal{F}_{14}$.

The paper is organized as follows: In Section 2, we present the needed tools to facilitate our proofs for the main results. In Section 3, we will prove Theorems 1.7 and 1.8. Applications of Theorems 1.7 and 1.8 will be given in Section 4.

2. Collapsible graphs

We will present basic properties of collapsible graphs in this section. The next theorem summarizes some basic properties needed in our arguments in the proofs.

Theorem 2.1. *Let G be a connected graph, H a collapsible subgraph of G, and G' the reduction graph of G. Then each of the following holds:*

(i) (Caltin [5]) G is collapsible if and only if G/H is collapsible. In particular, G is collapsible if and only if the reduction $G' = K_1$.
(ii) (Caltin [5]) G is reduced if and only if G has no nontrivial collapsible subgraphs.
(iii) (Caltin [5]) G' is simple, girth$(G') \ge 4$ and $\delta(G') \le 3$.
(iv) (Caltin [5]) G is supereulerian if and only if G' is supereulerian.
(v) (Caltin [5]) K_3 is the smallest nontrivial collapsible simple graph and the nontrivial reduced graphs with at most 5 vertices are either a tree, a 4-cycle, $K_{2,3}$, or $K_{2,3}$ minus an edge.
(vi) (Caltin, Han, and Lai [9]) If G is connected and if $F(G) \le 2$, then $G' \in \{K_1, K_2\} \cup \{K_{2,t} : t \ge 1\}$.
(vii) (Caltin, Han, and Lai [9]) If G is reduced, then $F(G) = 2|V(G)| - |E(G)| - 2 = \frac{1}{2}(3d_1(G) + 2d_2(G) + d_3(G) - \sum_{i \ge 4}(i-4)d_i(G)) - 2$.
(viii) If G is a reduced connected graph with $n \le 8$ and $d_1(G) = 0$, then either $\kappa'(G) \ge 2$, or G is the graph obtained from two 4-cycles C_4 and C'_4 by adding an edge xx', where $x \in V(C_4)$ and $x' \in V(C'_4)$.

Proof. We only present a proof of (viii) and refer the reader to the references cited for others. Let e be a cut edge of G and let H_1 and H_2 be the components of $G - e$. As $d_1(G) = 0$, for $i = 1, 2$, H_i can not be a tree. By Theorem 2.1(v), $H_1 = H_2 = C_4$ Thus, G is the graph obtained from two 4-cycles C_4 and C'_4 by adding an edge xx', where $x \in V(C_4)$ and $x' \in V(C'_4)$. □

Definition 2.2. Let $H = C_4 = v_1v_2v_3v_4v_1$ be a 4-cycle, or let $H = \Gamma_8$ denote the graph obtained from a 8-cycle *avdxbwcua* by adding two edges *cd* and *ab*. Consider a partition $\pi = (V_1, V_2) = (\{v_1, v_3\}, \{v_2, v_4\})$ of $V(C_4)$, or a partition $\pi = (V_1, V_2) = (\{a, b, c, d\}, \{x, w, u, v\})$ of $V(\Gamma_8)$. Following [4], if

H is a subgraph of a graph G, we define G/π to be the graph obtained from $G - E(H)$ by identifying all vertices of V_1 to form a single vertex v', by identifying all vertices of V_2 to form a single vertex v'', and by adding an edge $e_\pi = v'v''$.

Theorem 2.3 (Caltin [4]). *Let $H = C_4 = v_1v_2v_3v_4v_1$ be a 4-cycle in G, or let $H = \Gamma_8$ be a subgraph of G obtained from a 8-cycle avdxbwcua by adding two edges cd and ab. Let G/π be defined as in Definition 2.2. Then the following hold:*

(i) If G/π is collapsible, then G is collapsible.
(ii) If G/π has a spanning eulerian subgraph, then G has a spanning eulerian subgraph.
(iii) If G is a reduced graph with a 4-cycle C_4, then $F(G/\pi) \le F(G) - 1$.

Lemma 2.4. *$P^2(11), P^4(12), P^5(12)$ are collapsible.*

Proof. By definition, each of $P^2(11), P^4(12)$, and $P^5(12)$ contains a subgraph isomorphic to Γ_8. Let G/π be the graph defined as in Definition 2.2. Then G/π contains K_3 as a subgraph. As cycles of length at most 3 are collapsible, it is a routine matter to verify that contracting all cycles of length at most 3 in G/π results in a collapsible graph. By Theorem 2.3(i), $P^2(11), P^4(12), P^5(12)$ are collapsible. □

Lemma 2.5. *Let G be a connected reduced graph with $n \le 11$. If $d_1(G)=0$ and $d_2(G) \le 1$, then $G \in \{K_1, P(10), P(11)\}$.*

Proof. By Theorem 1.5, if $F(G) \le 3$, then $G \in \{K_1, P(10)\}$, and so we assume that $F(G) \ge 4$. By Theorem 2.1(vii), $d_3(G) \ge 10$. As $n \le 11$, we have $n = 11$, $d_2(G) = 1$, $d_3(G) = 10$, and $V(G) = D_2(G) \cup D_3(G)$. Let $D_2(G) = \{y\}$.

Assume that G has a cut edge e. Let H_1 and H_2 be the components of $G - e$. As $d_1(G) = 0$ and $d_2(G) = 1$, by Theorem 2.1(v), we have $|V(H_i)| \notin \{1, 2, 3, 4\}$ for $i = 1, 2$. Thus $|V(H_i)| \in \{5, 6\}$. Since $d_2(G) = 1$, we may assume that $y \in V(H_2)$. Then $d_1(H_1) = 0$ and $d_2(H_1) \le 1$. By Theorem 1.3(iii), $H_1 \in \{K_1, K_2, K_{1,2}\}$, a contradiction. Hence, G must be 2-edge-connected.

Next, we claim that $girth(G) \ge 5$. Otherwise, let $C_4 = v_1v_2v_3v_4v_1$ be a 4-cycle of G. Let $\pi = (\{v_1, v_3\}, \{v_2, v_4\})$ be a partition of $V(C_4)$. Form the graph G/π with the new edge e_π defined as in Definition 2.2. Then $|V(G/\pi)| = 11 - 2 = 9$. As $d_1(G) = 0$ and $d_2(G) = 1$, we have $|D_2(G) \cap \{v_1, v_2, v_3, v_4\}| \le 1$, $d_1(G/\pi) = 0$, and $d_2(G/\pi) \le 1$. By Theorems 1.3(iii) and 2.3(i), G/π is not 2-edge-connected. As G is 2-edge-connected, e_π is the cut edge of G/π, and so $\{v_1, v_2, v_3, v_4\}$ is a vertex-cut of G. Let L_1 and L_2 be the components of $G - \{v_1, v_2, v_3, v_4\}$ with $|V(L_1)| \le |V(L_2)|$ and $N_G(v_1) \cap V(L_2) = N_G(v_3) \cap V(L_2) = \emptyset$ and $N_G(v_2) \cap V(L_1) = N_G(v_4) \cap V(L_1) = \emptyset$ (see Figure 4). As $n = 11$, $|V(L_1)| \in \{1, 2, 3\}$. If $|V(L_1)| \in \{2, 3\}$, then L_1 is either P_2 or P_3. As $d_2(G) = 1$, the number of edges between $V(L_1)$ and $\{v_1, v_3\}$ is at least 3. Thus, either $d_G(v_1) \ge 4$ or $d_G(v_3) \ge 4$, contrary the fact that $V(G) = D_2(G) \cup D_3(G)$. Thus, $V(L_1) = \{y\}$ and $yv_1, yv_3 \in E(G)$. Let $L_3 =$

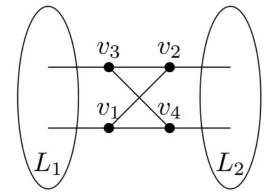

Figure 4. An illustration for the proof of Lemma 2.5.

$G - \{y, v_1, v_2, v_3, v_4\}$. Then $|V(L_3)| = 6$, and either $d_1(L_3) = 1$ and $d_3(L_3) = 5$, or $d_2(L_3) = 2$ and $d_3(L_3) = 4$. Thus, $F(L_3) \le 2$. As L_3 is reduced, by Theorem 2.1(vi), $L_3 = K_{2,4}$, a contradiction. Thus, $girth(G) \ge 5$.

As $V(G) = D_2(G) \cup D_3(G)$, there is a vertex $w \in D_3(G)$ such that the distance between y and w is 3. For an integer $i \ge 0$, define $T_i = \{x \in V(G) : dist(x, w) = i\}$. Then $|T_0| = 1, |T_1| = 3, |T_2| = 6, |T_3| = 1$ and $y \in T_3$. Let H be the subgraph in G induced by $T_2 \cup T_3$. Then H is a 7-cycle $a_1a_2 \cdots a_7a_1$. Let $T_1 = \{u_1, u_2, u_3\}$. Then for $i = 1, 2, 3$, $|N_G(u_i) \cap V(H)| = 2$. As $girth(G) \ge 5$, without loss of generality, we assume that $u_1a_1, u_1a_4 \in E(G)$. By symmetry, we assume that $a_7 \ne y$ and $a_7u_2 \in E(G)$. As $girth(G) \ge 5$, $u_2a_3 \in E(G)$. Thus, $u_3a_2 \in E(G)$ and $|N_G(u_3) \cap \{a_5, a_6\}| = 1$. Therefore, $G = P(11)$. □

Lemma 2.6. *Let G be a connected simple graph with $n \le 13$ and let G' be the reduction of G. If $\delta(G) \ge 3$, then $G' \in \{K_1, K_2, K_{1,2}, K_{1,3}, P(10), P^1(12), P^2(12), P^3(12)\}$.*

Proof. Let $n' = |V(G')|$. As $\delta(G) \ge 3$, by Theorem 2.1(v), we have $13 \ge \sum_{v \in D_1(G')} |PI(v)| + \sum_{v \in D_2(G')} |PI(v)| \ge 4d_1(G') + 4d_2(G')$. Thus, $d_1(G') + d_2(G') \le 3$.

Assume that G has a cut edge e, and L_1 and L_2 are the components of $G - e$. As $\delta(G) \ge 3$, we have $|V(L_i)| \notin \{1, 2, 3\}$ for $i = 1, 2$. Thus, $|V(L_i)| \in \{4, 5, ..., 9\}$. As $d_1(L_i) = 0$ and $d_2(L_i) \le 1$, by Theorem 1.3(iii), the reduction of L_i is in $\{K_1, K_2, K_{1,2}\}$. Thus, $G' \in \{K_2, K_{1,2}, K_{1,3}\}$. Next we assume that G is 2-edge-connected. Then $d_1(G') = 0$ and $d_2(G') \le 3$, and so $n' \le 13 - 3d_2(G')$. By Theorem 1.3(ii), (iv), (v), we have $d_2(G') = 0$. Thus, $\delta(G') \ge 3$. If $n' \ne 12$, then, by Theorem 1.6, we have $G = P(10)$. Next we assume that $n' = 12$. Then $G = G'$.

Case 1. $girth(G) \ge 5$.

Assume that $w \in V(G)$ such that $d_G(w) = \Delta(G) \ge 4$. For an integer $i \ge 0$, let $T_i = \{x \in V(G) : dist(x, w) = i\}$. Then $|T_0| = 1, |T_1| = 4, |T_2| \ge 8$, and so $n \ge 13$, a contradiction. So G is cubic. By Theorem 1.6, G is Hamiltonian. Let $v_0v_1 \cdots v_{11}v_0$ be a Hamiltonian cycle of G.

If $girth(G) \ge 7$, then $N_G(v_0) = \{v_1, v_{11}, v_6\}$ and $N_G(v_1) = \{v_0, v_2, v_7\}$. This results in a 4-cycle $v_0v_1v_7v_6v_0$, a contradiction. If $girth(G) = 6$, then $v_0v_6 \notin E(G)$ and so $N_G(v_0) \cap \{v_5, v_7\} \ne \emptyset$. Without loss of generality, we assume that $v_0v_5 \in E(G)$. Then $N_G(v_1) = \{v_0, v_2, v_8\}$. Thus, $N_G(v_2) = \{v_1, v_3\}$, a contradiction, and so $girth(G) = 5$. Without loss of generality, we assume that $v_0v_4 \in E(G)$. Then $N_G(v_5) \subseteq \{v_4, v_6, v_9, v_{10}\}$. Similarly, $N_G(v_{11}) \subseteq \{v_0, v_{10}, v_6, v_7\}$.

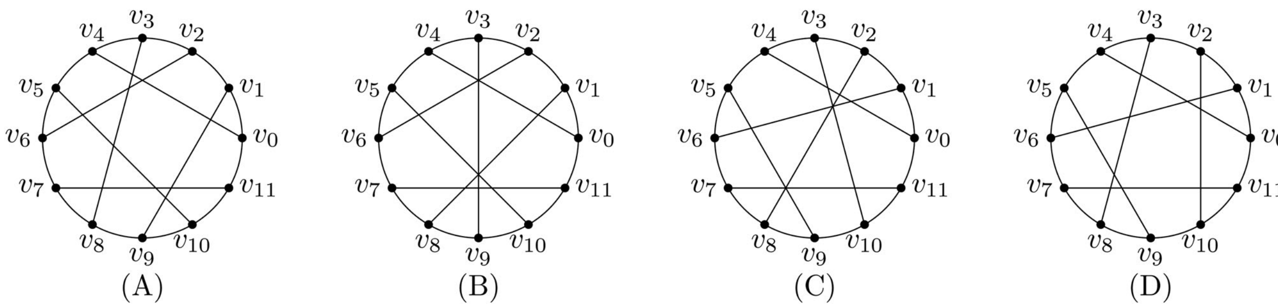

Figure 5. Illustrations for the proof of Lemma 2.6.

Case 1.1. Either $v_5v_{10} \in E(G)$ or $v_6v_{11} \in E(G)$.

We assume that $v_5v_{10} \in E(G)$. As $N_G(v_{11}) \subseteq \{v_0, v_{10}, v_6, v_7\}$ and $girth(G) \geq 5$, we have $v_{11}v_7 \in E(G)$. Then $N_G(v_2) \cap \{v_8, v_9\} = \emptyset$ (otherwise, $N_G(v_3) = \{v_2, v_4\}$, a contradiction). So $v_2v_6 \in E(G)$. Therefore, we have either $v_1v_9, v_3v_8 \in E(G)$, or $v_1v_8, v_3v_9 \in E(G)$ (Graphs A and B in Figure 5). In the former case, $G = P^1(12)$, and in the latter case, $G = P^2(12)$.

Case 1.2. $v_5v_{10}, v_6v_{11} \notin E(G)$.

Then $v_5v_9, v_{11}v_7 \in E(G)$. Thus, we have either $v_2v_8, v_3v_{10}, v_1v_6 \in E(G)$, or $v_2v_{10}, v_3v_8, v_1v_6 \in E(G)$ (Graphs C and D in Figure 5, respectively). In the former case, $G = P^2(12)$, and in the latter case, $G = P^1(12)$.

Case 2. $girth(G) = 4$.

Let $C_4 = v_1v_2v_3v_4v_1$ be a 4-cycle in G. Let $H = G/\pi$ be defined as in Definition 2.2. Then $\delta(H) \geq 3$ and $|V(H)| = 10$. Assume that H is not 2-edge-connected. Then e_π is a cut edge of H. Thus, $\{v_1, v_2, v_3, v_4\}$ is the vertex-cut of G. Let L_1 and L_2 be the components of $G - \{v_1, v_2, v_3, v_4\}$ such that $N_G(x) \cap V(L_2) = \emptyset$ for $x \in \{v_1, v_3\}$, and let $Q_i (i = 1, 2)$ be the subgraph induced by $V(L_i) \cup \{v_1, v_2, v_3, v_4\}$. As $\delta(G) \geq 3$, by Theorem 1.4, $|V(Q_i)| \geq 9$. Thus, $|V(L_i)| \geq 5$ and so $|V(G)| \geq 5 + 5 + 4 = 14$, a contradiction. Hence $\kappa'(H) \geq 2$. By Theorem 1.6, $H = P(10)$. Thus, $G = P^3(12)$. □

Lemma 2.7. *Let G be a 2-edge-connected reduced graph with $n = 13$. If $d_1(G) = 0, d_2(G) = 1, d_3(G) = 12$, and $girth(G) \geq 5$, then G is supereulerian.*

Proof. Let $D_2(G) = \{v\}$. As $girth(G) \geq 5$ and $V(G) = D_2(G) \cup D_3(G)$, we have $|\{x \in V(G) : dist(x, v) \leq 2\}| = 7$. Thus, there exists a vertex w such that the distance between w and v is at least three. For an integer $i \geq 0$, let $T_i = \{x \in V(G) : dist(x, w) = i\}$. Then $|T_0| = 1, |T_1| = 3, |T_2| = 6, |T_3 \cup T_4| = 3$, and $v \in T_3 \cup T_4$. Let $T_1 = \{u_1, u_2, u_3\}$ and let H be the subgraph in G induced by $T_2 \cup T_3 \cup T_4$. Then $|V(H)| = 9$ with $d_3(H) = 2$ and $d_2(H) = 7$. As $girth(G) \geq 5$, H is a 9-cycle $C_9 = v_1v_2 \cdots v_9v_1$ by adding a chord v_1v_5. Thus, for $i = 1, 2, 3$, $|N_G(u_i) \cap \{v_2, v_3, v_4, v_6, ..., v_9\}| = 2$.

First, we claim that $v \in \{v_2, v_3, v_4\}$. Otherwise, we may assume that $u_1v_2, u_2v_3, u_3v_4 \in E(G)$. If $u_3v_9 \in E(G)$, then $u_3v_9v_8v_7v_6v_5v_1v_2u_1wu_2v_3v_4u_3$ is a Hamiltonian cycle of G, a contradiction. So $u_3v_9 \notin E(G)$. Similarly, $u_1v_6 \notin E(G)$. As $girth(G) \geq 5$, $u_3v_6, u_1v_9 \notin E(G)$. Thus, $N_G(u_2) \cap \{v_6, v_9\} \neq \emptyset$. Without loss of generality, we assume that $u_2v_9 \in E(G)$. Then $v_9u_2v_3\ v_4u_3wu_1v_2v_1v_5v_6v_7v_8v_9$ is a Hamiltonian cycle, a contradiction. So $v \in \{v_2, v_3, v_4\}$.

As $girth(G) \geq 5$, we may assume that $u_3v_6, u_2v_7, u_1v_8 \in E(G)$. As $N_G(v_9) \cap \{u_1, u_2, u_3\} \neq \emptyset, u_3v_9 \in E(G)$. If $u_1v_4 \in E(G)$, then $v_4u_1v_8v_9u_3wu_2v_7v_6v_5v_1v_2v_3v_4$ is a Hamiltonian cycle of G, a contradiction. So $u_1v_4 \notin E(G)$. Similarly, $u_2v_2 \notin E(G)$. If $v = v_3$, then $u_1v_2, u_2v_4 \in E(G)$. Thus, $wu_1v_2v_3v_4u_2v_7v_8v_9v_1v_5v_6u_3w$ is a Hamiltonian cycle of G, a contradiction. So $v \in \{v_2, v_4\}$. Without loss of generality, we assume that $v = v_2$. As $v_4u_1 \notin E(G)$, we have $v_4u_2 \in E(G)$. Thus, $v_3u_1 \in E(G)$. So $v_3v_2v_1v_9v_8v_7u_2v_4v_5v_6u_3wu_1v_3$ is a Hamiltonian cycle of G, a contradiction. □

Lemma 2.8. *Let G be a 2-edge-connected reduced graph with $n = 14$. If $d_1(G) = 0, d_2(G) = 1, d_3(G) = 12, d_4(G) = 1$, and $girth(G) \geq 5$, then G is supereulerian.*

Proof. By contradiction, we assume that G is not supereulerian. Let $D_2(G) = \{v\}$ and $D_4(G) = \{w\}$. For an integer $i \geq 0$, define $T_i = \{x \in V(G) : dist(x, w) = i\}$. Then $|T_0| = 1$ and $|T_1| = 4$. Let $T_1 = \{u_1, u_2, u_3, u_4\}$ and H be the subgraph in G induced by $V(G) - (T_0 \cup T_1)$. Then $|V(H)| = 9$.

Claim 1. $vw \in E(G)$.

Assume that $vw \notin E(G)$. Then $|T_2| = 8, |T_3| = 1, v \in T_2 \cup T_3$, $|N_G(u_i) \cap V(H)| = 2$ for $i = 1, 2, 3, 4$, and $|N_G(x) \cap T_1| = 1$ for each $x \in T_2$.

If $v \in T_3$, then $d_H(x) = 2$ for each $x \in V(H)$. As $girth(G) \geq 5$, $H = C_9$. Assume that $C_9 = a_1a_2 \cdots a_9a_1$, where $a_9 = v$. As $girth(G) \geq 5$, we may assume that $u_1a_1, u_2a_2, u_3a_3 \in E(G)$. If $u_1a_4 \in E(G)$, then $a_5u_4, a_8u_4 \in E(G)$. Thus, $wu_4a_5a_6a_7a_8a_9a_1a_2u_2wu_3a_3a_4u_1w$ is a spanning eulerian subgraph of G, a contradiction. So $u_1a_4 \notin E(G)$. Therefore, $u_4a_4 \in E(G)$ and $wu_2a_2a_3u_3wu_4a_4a_5a_6a_7a_8a_9a_1u_1w$ is a spanning Eulerian subgraph of G, a contradiction. So $v \in T_2$.

Then $d_1(H) = 1, d_2(H) = 7, d_3(H) = 1$. As $girth(G) \geq 5$, H is connected. Thus, H is a cycle $C_k = a_1a_2 \cdots a_ka_1$ by attaching a path $a_ka_{k+1} \cdots v_9$, where $a_9 = v$ and $a_k \in T_3$. As $girth(G) \geq 5, k \in \{5, 6, 7, 8\}$. If $k = 8$, as $girth(G) \geq 5$, we assume that $u_1a_9, u_2a_1, u_3a_7 \in E(G)$. As G is not supereulerian, $N_G(u_4) \cap \{a_2, a_6\} = \emptyset$. Thus, $N_G(u_4) \subseteq \{w, a_3, a_4, a_5\}$. This implies that $girth(G) \leq 4$, a contradiction. If $k = 7$, then we assume that $a_9u_4, a_8u_3 \in E(G)$. Thus, $(N_G(a_1) \cup N_G(a_6)) \cap \{u_1, u_2\} \neq \emptyset$. Without loss of generality, we assume that $u_1a_2 \notin E(G)$. As G is not supereulerian, $u_1a_2 \notin E(G)$. Thus, $u_1a_3, u_1a_6 \in E(G)$. As $girth(G) \geq 5$, $N_G(u_2) \cap \{a_4, a_5\} \neq \emptyset$. This would result in a spanning subgraph of G, a contradiction. If $k = 6$, by symmetry, we assume that $u_2a_7, u_3a_8, u_4a_9 \in$

$E(G)$. Thus, $N_G(u_1) \cap \{a_1, a_5\} \neq \emptyset$. Without loss of generality, we assume that $u_1a_1 \in E(G)$. Thus, $wu_1a_1a_2a_3a_4a_5a_6a_7u_2wu_3a_8a_9u_4w$ is a spanning eulerian subgraph of G, a contradiction. So $k=5$. As $girth(G) \geq 5$, we assume that $u_1a_1, u_2a_2, u_3a_3 \in E(G)$. Then $u_4a_9 \notin E(G)$ (otherwise, $wu_3a_3a_4a_5a_6a_7a_8a_9u_4wu_2a_2a_1u_1w$ is a spanning eulerian subgraph of G, a contradiction). Thus, $u_4a_4 \in E(G)$. Notice that $|N_G(a_9) \cap \{u_1, u_2, u_3\}| = 1$. If $a_9u_3 \in E(G)$, then $wu_4a_4a_3a_2u_2wu_1a_1a_5a_6a_7a_8a_9u_3w$ is a spanning eulerian subgraph of G; if $a_9u_2 \in E(G)$, then $wu_2a_9a_8a_7a_6a_5a_4u_4w\ u_1a_1a_2a_3u_3w$ is a spanning eulerian subgraph of G; if $a_9u_1 \in E(G)$, then $wu_4a_4a_3u_3wu_2\ a_2a_1a_5a_6a_7a_8a_9u_1w$ is a spanning eulerian subgraph of G. We finish the proof of Claim 1.

By Claim 1, $vw \in E(G)$. Then $|T_2| = 7$ and $|T_3| = 2$. Thus, $d_2(H) = 7$ and $d_3(H) = 2$. As $girth(G) \geq 5$, H is 2-connected. Thus, H is the 9-cycle $C_9 = a_1a_2\cdots a_9a_1$ by adding the chord a_1a_5. Since $girth(G) \geq 5$, we assume that $u_1a_2, u_2a_3, u_3a_4 \in E(G)$. Then $N_G(u_4) \cap \{a_6, a_9\} = \emptyset$. Thus, $N_G(u_4) \cap \{a_7, a_8\} \neq \emptyset$. Without loss of generality, we assume that $u_4a_7 \in E(G)$. Then $u_4 = v$. Thus, $u_2a_6 \notin E(G)$ (otherwise, $wu_2a_6a_5a_1a_9a_8a_7u_4wu_1a_2a_3a_4u_3w$ is a spanning eulerian subgraph of G, a contradiction), Similarly, $u_2a_8 \notin E(G)$ Thus, $u_2a_9 \in E(G)$. As $girth(G) \geq 5, u_3a_8 \in E(G)$ and so $u_1a_6 \in E(G)$. Hence $wu_3a_8a_7u_4wu_1a_6a_5a_4a_3a_2a_1a_9u_2w$ is a spanning eulerian subgraph of G, a contradiction. □

Lemma 2.9. *Let G be a 2-edge-connected reduced graph with $n=14$. If $d_1(G)=0, d_2(G)=1$, $\Delta(G)=4$, and $girth(G) \geq 5$, then G is supereulerian.*

Proof. By Theorem 2.1(vi), $F(G) \geq 3$. By Theorem 2.1(vii), $2d_2(G) + d_3(G) \geq 10$. Thus, $d_3(G) \geq 8$. By Lemma 2.8, it suffices to consider the cases when $(d_2(G), d_3(G), d_4(G)) \in \{(1, 8, 5), (1, 10, 3)\}$. Let $D_2(G) = \{v\}$.

Claim 1. If $d_2(G) = 1, d_3(G) = 10$ and $d_4(G) = 3$, then G is supereulerian.

By Lemma 2.8, $D_4(G)$ is independant. As $d_4(G) = 3$, there is a vertex $w \in D_4(G)$ such that $v \notin N_G(w)$. Choose such the vertex w such that the distance between w and v is longest. Thus for any $x \in N_G(w), x \in D_3(G)$. For an integer $i \geq 0$, define $T_i = \{x \in V(G) : dist(x, w) = i\}$. Then $|T_0| = 1, |T_1| = 4, |T_2| = 8, |T_3| = 1$, and $v \in T_2 \cup T_3$. Let $T_1 = \{u_1, u_2, u_3, u_4\}$, $T_3 = \{z\}$ and let H be the subgraph in G induced by $T_2 \cup T_3$. Then $d_G(z) \in \{2, 3, 4\}$.

If $d_G(z) = 4$, then $v \in T_2$. Thus, $d_1(H) = 1, d_2(G) = 6$, $d_3(H) = 1$ and $d_4(H) = 1$. As $girth(G) \geq 5$, H is formed from the 8-cycle $a_1a_2\cdots a_8a_1$ by adding the chord a_1a_5 and the pendant edge $a_1v \in E(G)$, where $a_1 = x \in D_4(G)$. As $D_4(G)$ is independant, $N_G(a_5) \cap \{u_1, u_2, u_3, u_4\} = \emptyset$. Since the number of edges between $\{u_1, u_2, u_3, u_4\}$ and $\{v, a_2, a_3, a_4, a_6, a_7, a_8\}$ is 8 and $v \in D_2(G)$, there is a vertex $y \in \{a_2, a_3, a_4, a_6, a_7, a_8\}$ such that $|N_G(y) \cap \{u_1, u_2, u_3, u_4\}| \geq 2$. This results in a 4-cycle in G, a contradiction. If $d_G(z) = 2$, then $d_2(H) = 7$ and $d_3(H) = 2$. As $girth(G) \geq 5$, H is a 9-cycle $a_1a_2\cdots a_9a_1$ by adding a chord, say a_1a_5. As $girth(G) \geq 5$, $|N_G(y) \cap \{u_1, u_2, u_3, u_4\}| \leq 1$ for $y \in \{a_1, a_2, ..., a_9\}$. As $D_4(G) = 3, a_1, a_5 \in D_4(G)$, contrary to the fact that $D_4(G)$ is independant. Thus, $d_G(z) = 3$ and the distance between w and v is 2.

Therefore, $d_1(H) = 1, d_2(H) = 5$ and $d_3(H) = 3$. As $girth(G) \geq 5$, H is formed from the 8-cycle $a_1a_2\cdots a_8a_1$ by adding the chord a_1a_5 and the pendant edge $va_{i_0} \in E(G)$, where $i_0 \notin \{1, 5\}$. By symmetry, we assume that $va_{i_0} \in E(G)$, where $i_0 \in \{2, 3\}$. As $D_4(G)$ is independant, $a_{i_0} \in D_4(G)$ and $|\{a_1, a_5\} \cap D_4(G)| = 1$. Without loss of generality, we assume that $a_5 \in D_4(G)$. Thus, $a_1 = z$, and the distance between a_5 and v is 3. This contradicts the choose of w. Hence Claim 1 follows.

By Claim 1, we assume that $d_2(G) = 1, d_3(G) = 8$ and $d_4(G) = 5$. By Claim 1, $D_4(G)$ is independant. As $d_4(G) = 5$, choose $w \in D_4(G)$ such that $v \notin N_G(w)$. Thus, for any $x \in N_G(w), x \in D_3(G)$. For an integer $i \geq 0$, define $T_i = \{u \in V(G) : dist(u, w) = i\}$. Then $|T_0| = 1, |T_1| = 4, |T_2| = 8$, $|T_3| = 1$, and $v \in T_2 \cup T_3$. Let $T_1 = \{u_1, u_2, u_3, u_4\}$, and let $T_2 = \{x_1, x_2, ..., x_8\}$ such that $x_{2i-1}, x_{2i} \in N_G(u_i)$ for $i = 1, 2, 3, 4$. Let $T_3 = \{x\}$.

We claim that $x \notin D_4(G)$. Otherwise, as $girth(G) \geq 5$, we may assume that $N_G(x) = \{x_1, x_3, x_5, x_7\}$. Then $|\{x_2, x_4, x_6, x_8\} \cap D_4(G)| = 3$. Without loss of generality, we assume that $x_4, x_6, x_8 \in D_4(G)$. Thus, $|\{x_1, x_2\} \cap N_G(x_4)| = 1$ and $x_5, x_7 \in N_G(x_4)$. It implies that $x_4x_5xx_7x_4$ is a 4-cycle, a contradiction. So $x \in D_2(G) \cup D_3(G)$. Thus, $|T_2 \cap D_4(G)| = 4$.

Assume that $x_1 \in D_4(G)$ such that $xx_1 \notin E(G)$. As $girth(G) \geq 5$, we assume that $N_G(x_1) = \{u_1, x_3, x_5, x_7\}$. Then $|\{x_2, x_4, x_6, x_8\} \cap D_4(G)| = 3$ and $N_G(x_2) \cap \{x_1, x_3, x_5, x_7\} = \emptyset$. Thus, $N_G(x_2) \subseteq \{u_1, x, x_4, x_6, x_8\}$. As $D_4(G)$ is independant, $x_2 \notin D_4(G)$. Thus, $x_4, x_6, x_8 \in D_4(G)$. Therefore, $N_G(x_4) \subseteq \{x, u_2, x_2, x_5, x_7\}$. Notice that $x_4x_5x_1x_7x_4$ would be a 4-cycle if $x_4x_5, x_4x_7 \in E(G)$. We have $xx_4, x_2x_4 \in E(G)$. Similarly, $xx_6, x_2x_6 \in E(G)$. This results in a 4-cycle $x_2x_4xx_6x_2$, a contradiction. □

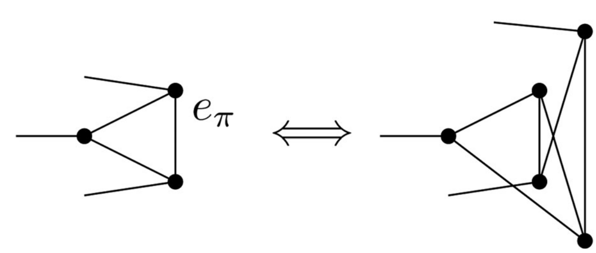

Figure 6. An illustration for Claim 8 in the proof of Theorem 1.8.

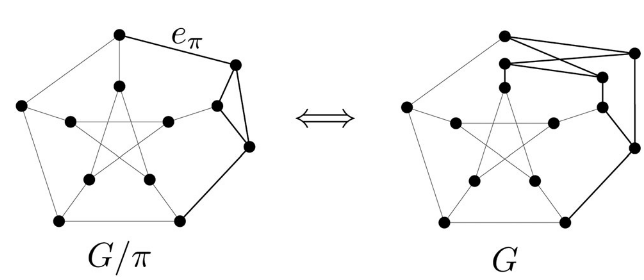

Figure 7. An illustration for Claim 8 in the proof of Theorem 1.8.

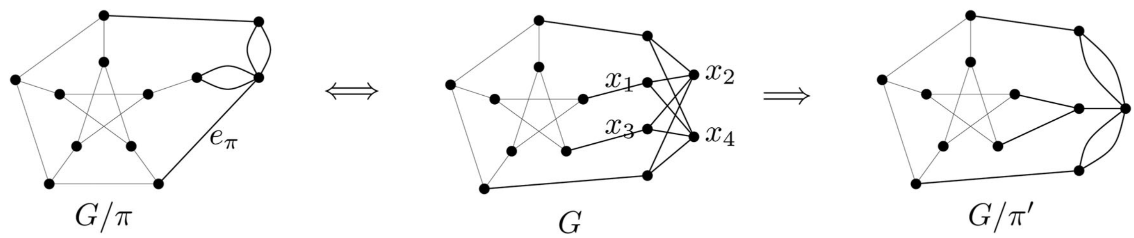

Figure 8. An illustration for Claim 8 in the proof of Theorem 1.8.

3. Proof of Theorems 1.7 and 1.8

In this section, we will justify both Theorems 1.7 and 1.8. Some of the parts (or graphs) in Figures 6–8 are adapted from [12].

Proof of Theorem 1.7. Let G' be the reduction of G. Assume that the conclusion of Theorem 1.7 is false, and in particular, $G' \neq K_1$.

$$\tag{1}$$

By assumption, it is known that

$$n \leq 11, d_1(G) = 0, \text{ and } d_2(G) \leq 2. \tag{2}$$

By Theorem 2.1(v), we have $|\cup_{v\in D_1(G')} PI_G(v)| \geq 4d_1(G')$, and if $d_2(G') \geq 2$, then $|\cup_{v\in D_2(G')} PI_G(v)| \geq 4(d_2(G') - 2) + 2$.

$$\tag{3}$$

By (2) and (3), we must have $d_1(G') \leq 2$ and $d_1(G') + d_2(G') \leq 4$. In particular, if $d_1(G') + d_2(G') = 4$, then $n \in \{10, 11\}, |V(G')| = 4$ with $d_2(G) = 2$ and $G' \in \{P_4, C_4\}$, contrary to (1). Therefore, we assume that $d_1(G') + d_2(G') \leq 3$. Let $m' = |E(G')|$ and $n' = |V(G')|$.

Claim 1. $d_1(G') = 0$.

Otherwise, $d_1(G') \in \{1, 2\}$. If $d_1(G') = 2$, then $n' \leq 11 - 8 + 2 = 5$ and $d_2(G') \leq 3 - 2 = 1$. By Theorem 2.1(vii), $F(G') \leq 3$. By Theorem 1.5, $G' \in \{K_2, K_{1,2}\}$, contrary to (1). If $d_1(G') = 1$, then $d_2(G') \leq 3 - 1 = 2$. Assume that $D_1(G') = \{a_1\}$. Let $a_1a_2 \cdots a_k (k \geq 2)$ be a path in G' such that $d_{G'}(a_i) = 2 (i = 2, ..., k-1)$ and $d_{G'}(a_k) \geq 3$ and $H = G' - \{a_1, ..., a_{k-1}\}$. Then $d_1(H) = 0, d_2(H) \leq 3$ and $|V(H)| \leq 11 - 4 = 7$, implying that $\kappa'(H) \geq 2$. By Theorem 1.3(ii) and (iv), $H \in \{K_{2,3}, K_{1,3}(1,1,1), T(1,1)\}$ and so $G' \in \{K^+_{2,3}, K^+_{1,3}(1,1,1), T^+_1(1,1), T^+_2(1,1)\}$. If $G' \in \{K^+_{1,3}(1,1,1), T^+_1(1,1), T^+_2(1,1)\}$, then $n = 11$ and $d_2(G) = 2$; and if $G' = K^+_{2,3}$, then $d_2(G) = 2$, contrary to (1) in either case. Hence, Claim 1 must hold.

Claim 2. $d_2(G') = 2$, and $D_2(G')$ is independant.

Assume that $d_2(G') = 3$. Then $n' \leq 11 - 6 + 3 = 8$. By Claim 1 and Theorem 2.1(viii), G′ is 2-edge connected. If $n' \leq 7$, Theorem 1.3(ii) $G' \in \{K_{2,3}, K_{1,3}(1,1,1), T(1,1)\}$ and when $G' \in \{K_{1,3}(1,1,1), T(1,1)\}, d_2(G) = 2$, contrary to (1). If $n' = 8$, then, by Theorem 2.1(vii), $m' \geq 11$ and $F(G') \leq 3$. By Theorem 1.5, $G' = T(1,2)$ and $d_2(G) = 2$, contrary to (1). Hence $d_2(G') \leq 2$. By Lemma 2.5, $d_2(G') = 2$.

Let $D_2(G') = \{a_1, a_2\}$. If $a_1a_2 \in E(G')$, then setting $L_4 = G' - \{a_1, a_2\}$, we have $|V(L_4)| \leq 9$. As $d_2(G') = 2$, we have $d_1(L_4) = 0$ and $d_2(L_4) \leq 2$. By Theorem 1.4, $|V(L_4)| = 9$. By Theorem 1.3(iv), L_4 has a cut edge e. Assume that Y_1 and Y_2 are components of $L_4 - e$ with $|V(Y_1)| \leq |V(Y_2)|$. As $|V(L_4)| = 9, |V(Y_1)| \in \{1, 2, 3, 4\}$. Since G' is reduced, $Y_1 \in \{K_1, K_2, K_{1,2}, K_{1,3}, P_4, C_4\}$. For each of these four cases, we have either $d_1(G') \neq 0$ or $d_2(G') \geq 3$, a contradiction occurring in any case. Hence $D_2(G')$ must be independant. This proves Claim 2.

Claim 3. G' is 2-edge-connected.

If G' has a cut edge e, then we assume that H_1 and H_2 are components of $G' - e$. Notice that G' is reduced. For $i = 1$, 2, by Theorem 2.1(v) and by Claims 1 and 2, $|V(H_i)| \notin \{1, 2, 3, 4\}$, and if $|V(H_i)| = 5$, then $H_i = K_{2,3}$. By Claim 2, we assume that $|V(H_1)| = 5$ and $|V(H_2)| = 6$. As $H_1 = K_{2,3}$ and $d_2(G') = 2$, we have $d_1(H_2) = 0, d_2(H_2) \leq 1$. By Theorem 1.3(iii), $H_2 \in \{K_1, K_2, K_{1,2}\}$, contrary to Claim 1. This justifies Claim 3.

By Theorem 1.3(iv), $n' \in \{10, 11\}$. Thus, G is reduced. By Theorem 1.5, $F(G') \geq 4$. Thus, $d_3(G') = 8$ if $n' = 10$, or $d_3(G') = 8$ and $d_4(G') = 1$ if $n' = 11$.

Claim 4. $girth(G') = 4$.

Assume that $girth(G') \geq 5$. First of all, we assume that $n' = 11$. Let $D_4(G') = \{w_1\}$. For an integer $i \geq 0$, define $T_i = \{u \in V(G') : dist(u, w_1) = i\}$. Then $|T_0| = 1, |T_1| = 4, |T_2| = 6$, and $D_2(G') \subseteq T_1$. Let L_1 be the subgraph in G' induced by T_2. Then L_1 is a 6-cycle $a_1a_2 \cdots a_6a_1$. Let $T_1 = \{u_1, u_2, u_3, u_4\}$, where $u_1, u_4 \in D_2(G')$ and $u_2, u_3 \in D_3(G')$. Then for $i = 2$, 3, $|N_{G'}(u_i) \cap T_2| = 2$, and for $i = 1$, 4, $|N_{G'}(u_i) \cap T_2| = 1$. By symmetry and $girth(G') \geq 5$, we may assume that $u_2a_1, u_2a_4 \in E(G')$, and $u_3a_2, u_3a_5 \in E(G')$. We also assume that $u_1a_3, u_4a_6 \in E(G')$. So $G' = P^2(11)$. By Lemma 2.4, G' is collapsible, a contradiction. Next we assume that $n' = 10$ and so $V(G') = D_2(G') \cup D_3(G')$.

As $d_3(G') = 8$, there is a vertex $w_2 \in D_3(G')$ such that $N_{G'}(w_2) \cap D_2(G') = \emptyset$. Define $T_i = \{u \in V(G) : dist(u, w_2) = i\}$. Then $|T_0| = 1, |T_1| = 3, |T_2| = 6$, and $D_2(G') \subseteq T_2$. Let $T_1 = \{u_1, u_2, u_3\}$ and let L_2 be the subgraph in G' induced by T_2. Then L_2 is a 6-path $a_1a_2a_3a_4a_5a_6$. Notice that for $i = 1$, 2, 3, $|N_{G'}(u_i) \cap T_2| = 2$. As $girth(G') \geq 5$, by symmetry, we may assume that $u_1a_2, u_1a_5 \in E(G), u_2a_1, u_2a_4 \in E(G)$ and $u_3a_3, u_3a_6 \in E(G)$. Then $G' = P^-(10)$ and $d_2(G) = 2$, contrary to (1). This proves Claim 4.

Let $C_4 = v_1v_2v_3v_4v_1$ be a 4-cycle in G'. Let $\pi = (V_1, V_2) = (\{v_1, v_3\}, \{v_2, v_4\})$ be a partition of $V(C_4)$. Form the graph G'/π with the new edge e_π as in Definition 2.2. Then $|V(G'/\pi)| \in \{8, 9\}$.

Claim 5. G'/π is not 2-edge-connected.

Assume that G'/π is 2-edge-connected. As $d_1(G') = 0$ and $d_2(G') = 2$, we have $d_2(G'/\pi) \leq 2$. If G'/π is K_3-free, then by Theorem 1.3(iv), the reduction of G'/π is K_1, Thus, $G' = K_1$, contrary to (1). Hence G'/π must contains a K_3. Let $uvxu$ be a K_3 in G'/π. As G' is reduced, either $x \in \{v_1, v_3\}$ or $x \in \{v_2, v_4\}$. Without loss of generality, we assume that $x \in \{v_1, v_3\}$, and $uv_1, vv_3 \in E(G')$. By Claim 2, either $d_{G'}(u) \geq 3$ or $d_{G'}(v) \geq 3$. Without loss of generality, we assume that $d_{G'}(u) \geq 3$. Let H be the graph obtained from G'/π by contracting $uvxu$ and let z be the vertex on which $uvxu$ is contracted. Then $|V(H)| \in \{6,7\}$ and H is 2-edge-connected. If $d_2(H) \leq 2$, then by Theorem 1.3(i), H is collapsible, forcing that G' is collapsible, a contradiction. Thus, $d_2(H) \geq 3$. As $d_2(G'/\pi) \leq 2, d_H(z) = 2$. It implies that $d_{G'/\pi}(v) = 2$, and so $d_2(G'/\pi) = 3$, contrary to the fact that $d_2(G'/\pi) \leq 2$. Claim 5 follows.

Claim 6. $d_{G'}(v_1) + d_{G'}(v_3) \geq 5$ and $d_{G'}(v_2) + d_{G'}(v_4) \geq 5$.

Assume that $d_{G'}(v_1) + d_{G'}(v_3) \leq 4$. Then $v_1, v_3 \in D_2(G')$. Thus, $d_{G'}(v_2) \geq 3$ and $d_{G'}(v_4) \geq 3$. If $d_{G'}(v_2) = 4$, then we set $Q_1 = G' - \{v_1, v_3, v_4\}$. Thus, Q_1 is connected, $|V(Q_1)| \in \{7,8\}, d_1(Q_1) = 0$ and $d_2(Q_1) \leq 2$. By Theorem 1.4, $Q_2 \in \{K_1, K_2, K_{2,3}\}$, a contradiction. So $d_{G'}(v_2) = d_{G'}(v_4) = 3$. Let $Q_2 = G' - \{v_1, v_3, v_2, v_4\}$. Then Q_2 is connected, $|V(Q_2)| \in \{6,7\}$, $d_1(Q_2) = 0$ and $d_2(Q_2) \leq 2$. By Theorem 1.4, $Q_2 \in \{K_1, K_2, K_{2,3}\}$, a contradiction. Claim 6 is justified.

As G' is 2-edge-connected, by Claims 5 and 6, $\{v_1, v_2, v_3, v_4\}$ is a vertex-cut of G'. Let L_1 and L_2 be the components of $G' - \{v_1, v_2, v_3, v_4\}$ such that $N_{G'}(v_1) \cap V(L_2) = N_{G'}(v_3) \cap V(L_2) = \emptyset$ and $N_{G'}(v_2) \cap V(L_1) = N_{G'}(v_4) \cap V(L_1) = \emptyset$. Also we assume that $|V(L_1)| \leq |V(L_2)|$. As $n' \in \{10, 11\}, |V(L_1)| \in \{1,2,3\}$. By Claims 2 and 3, $|V(L_1)| \neq 3$. If $|V(L_1)| = 2$, as G' has no triangles, we have $D_2(G') = V(L_2)$, contrary to Claim 2. So $|V(L_1)| = 1$.

Let $V(L_1) = \{v\}$. Then $vv_1, vv_3 \in E(G')$. If $d_{G'}(v_2) = 4$, then we set $Q_3 = G' - \{v, v_1, v_3, v_4\}$. Thus, Q_3 is connected, $|V(Q_3| \in \{6,7\}, d_1(Q_3) = 0$ and $d_2(Q_3) \leq 2$. By Theorem 1.4, $Q_3 \in \{K_1, K_2, K_{2,3}\}$, a contradiction. Hence $d_{G'}(v_2) = d_{G'}(v_4) = 3$. Let $Q_4 = G' - \{v, v_1, v_3, v_2, v_4\}$. Then Q_4 is connected, $|V(Q_4)| \in \{5,6\}, d_1(Q_4) = 0$ and $d_2(Q_4) \leq 2$. By Theorem 1.4, $Q_4 = K_{2,3}$. Therefore, $n' = 10$ and $G = K_{2,3}^2$, a contradiction. □

Proof of Theorem 1.8. Let G' be the reduction of G. By Theorem 1.7, if $n \leq 11$, then $G' \in \mathcal{F}_{11}$. Hence we assume that $n \in \{12, 13, 14\}$. Arguing by contradiction to prove Theorem 1.8, we assume that

$$G' \neq K_1, \text{and none of (i), (ii), and (iii) holds.} \tag{4}$$

By assumption, it is known that

$$d_1(G) = 0, \text{ and } d_2(G) \leq 1. \tag{5}$$

By Theorem 2.1(v), we have

$$|\cup_{v \in D_1(G')} PI_G(v)| \geq 4d_1(G'), \text{ and if } d_2(G') \geq 1, \text{ then } |\cup_{v \in D_2(G')} PI_G(v)| \geq 4(d_2(G') - 1) + 1. \tag{6}$$

By (5) and (6), we must have $d_1(G') \leq 3$ and $d_1(G') + d_2(G') \leq 5$. Let $n' = |V(G')|$.

Claim 1. $d_1(G') + d_2(G') \leq 3$.

Otherwise, $d_1(G') + d_2(G') = 4$. Thus, $(d_1(G'), d_2(G')) \in \{(3,1), (2,2), (1,3), (0,4)\}$. If $d_1(G') = 3$ and $d_2(G') = 1$, then $n' = 5$, and so $G' = K_{1,3}^+$ and $n = 14$; if $d_1(G') = 2$ and $d_2(G') = 2$, then $n' = 4, n \in \{13, 14\}$, and $G' = P_4$; if $d_1(G') = 1$ and $d_2(G') = 3$, then $n' = 5, G' = C_4^+$ and $n = 14$; if $d_1(G') = 0$ and $d_2(G') = 4$, then $n \in \{13, 14\}$ and $G' = C_4$, and so $G \in \mathcal{S}_{13} \cup \mathcal{S}_{14}$, contrary to (4).

Claim 2. $d_1(G') = 0$.

Otherwise, $d_1(G') \in \{1,2,3\}$. Assume that $d_1(G') = 3$. By Claim 1, $d_2(G') = 0$. Thus, $n' = 4, n \in \{13, 14\}$ and $G' = K_{1,3}$, contrary to (4).

Assume that $d_1(G') = 2$ and $D_1(G') = \{a_1, b_1\}$ with $N_{G'}(a_1) = a_2$ and $N_{G'}(b_1) = b_2$. By Claim 1, $d_2(G') \leq 1$. If $a_2 = b_2$ and $d_{G'}(a_2) = 2$, then $G' = K_{1,2}$ and $n \leq 14$; if $a_2 = b_2$ and $d_{G'}(a_2) \geq 3$, then $|V(G') - \{a_1, b_1\}| \leq 14 - 8 = 6, d_1(G' - \{a_1, b_1\}) \leq 1$ and $d_1(G' - \{a_1, b_1\}) + d_2(G' - \{a_1, b_1\}) \leq 2$. By Theorem 1.7, $d_1(G' - \{a_1, b_1\}) = 1$. As the number of odd degree vertices in a graph is even, by Theorem 2.1(vii), $F(G' - \{a_1, b_1\}) \leq 2$. Thus, $G' - \{a_1, b_1\} \in \{K_2, K_{2,t}\}, t \in \{1,2,3,4\}$, contrary to Claim 1 and the hypothesis that $d_1(G') = 2$. So $a_2 \neq b_2$ and either $d_{G'}(a_2) \geq 3$ or $d_{G'}(b_2) \geq 3$. Hence, $|V(G') - \{a_1, b_1\}| \leq 14 - 8 = 6$. If $d_{G'}(a_2) \geq 3$ and $d_{G'}(b_2) \geq 3$, then $d_1(G' - \{a_1, b_1\}) = 0$ and $d_2(G'_1 - \{a_1, b_1\}) \leq 3$. By Theorem 2.1(vii), $F(G' - \{a_1, b_1\}) \leq 2$, and so $G' - \{a_1, b_1\} = K_{2,3}, G' = K_{2,3}^{++}$ and $n \in \{13, 14\}$. If $d_{G'}(a_2) = 2$ and $d_{G'}(b_2) \geq 3$, then by Claim 1, $d_1(G' - \{a_1, b_1\}) = 1$ and $d_2(G'_1 - \{a_1, b_1\}) = 1$. By Theorem 2.1(vii), $F(G' - \{a_1, b_1\}) \leq 2$, and so $G' - \{a_1, b_1\} = K_2$, a contradiction.

Assume that $d_1(G') = 1$ with $D_1(G') = \{a_1\}$. Let $a_1 a_2 \cdots a_k (k \geq 2)$ be a path in G' such that $d_{G'}(a_i) = 2 (i = 2, \ldots, k-1)$ and $d_{G'}(a_k) \geq 3$ and $H = G' - \{a_1, \ldots, a_{k-1}\}$. Then $d_1(H) = 0$, and $d_2(H) \leq 3$ and $|V(H)| \leq 14 - 4 = 10$. If $d_2(H) \leq 2$, by Theorem 1.7, $H \in \{K_{2,3}^2, P^-(10), P(10)\}$. So $n = 14$, and $G' \in \{(K_{2,3}^2)^+, (P^-(10))^+, (P(10))^+\}$. If $d_2(H) = 3$, then $|V(H)| \leq 14 - 4 - 3 = 7$. By Theorem 2.1(vii), $F(H) \leq 3$. By Theorem 1.5, $H \in \{K_{2,3}, K_{1,3}(1,1,1), T(1,1)\}$. Thus, $G' \in \{K_{2,3}^+, K_{1,3}^+(1,1,1), T_1^+(1,1), T_2^+(1,1)\}$, and if $G' = K_{2,3}^+$, then $n \in \{12, 13, 14\}$, and if $G' \in \{T_1^+(1,1), T_2^+(1,1), K_{1,3}^+(1,1,1)\}$, then $n = 14$. Claim 2 holds.

Claim 3. $d_2(G') = 1$.

Otherwise, $d_2(G') \in \{0,2,3\}$. If $d_2(G') = 3$, then $n' \leq 14 - 8 + 2 = 8$. By Theorem 2.1(vii), $F(G') \leq 3$. By Theorem 1.5, $G' \in \{K_{1,3}(1,1,1), K_{2,3}, T(1,1)\}$. If $G' = K_{1,3}(1,1,1)$, then $n \in \{13, 14\}$, and if $G' = T(1,1)$, then $G \in \mathcal{S}_{13} \cup \mathcal{S}_{14}$, contrary to (4). If $d_2(G') = 2$, then $n' \leq 14 - 4 + 1 = 11$. By Theorem 1.7 and Claim 2, $G' \in \{K_{2,3}^2, P^-(10), P^2(11)\}$. If $G' = K_{2,3}^2$, then $G \in \mathcal{S}_{13} \cup \mathcal{S}_{14}$; if $G' = P^2(11)$, then $G' \in \mathcal{S}_{14}$; if $G' = P^-(10)$, then $n \in \{13, 14\}$, contrary to (4).

Next we assume that $d_2(G') = 0$. Then $\delta(G') \geq 3$. By Theorem 1.6 and Lemma 2.5, $n' \in \{12, 14\}$. If $n' = 12$, by Lemma 2.6, $G' \in \{P^1(12), P^2(12), P^3(12)\}$, contrary to (4). If $n' = 14$, then $G = G'$. If G has an edge-cut X with $|X| \leq 2$, then we set Z_1 and Z_2 are the components of $G - X$ with

$|Z_1| \le |Z_2|$. Thus, $|Z_1| \in \{6, 7\}$ and $F(Z_1) \le 2$. Therefore, $Z_1 = K_{2,t}(t = 4, 5)$, a contradiction. So G is 3-edge-connected. By Theorem 1.2(iv), either $G \in \mathcal{S}_{14}$ or $G = P^1(14)$, contrary to (4). Claim 3 holds.

By Claim 3, we denote $D_2(G') = \{v\}$.

Claim 4. G' is 2-edge-connected.

Let e be an edge-cut of G' and let H_1 and H_2 be the components of $G' - e$ such that $V(H_1) \cap D_2(G') = \emptyset$. By Claims 2 and 3 and by Theorem 2.1(v), $|V(H_i)| \ge 6(i = 1, 2)$. Thus, $|V(H_1)| \in \{6, 7, 8\}$. Since $d_1(H_1) = 0$ and $d_2(H_1) \le 1$ and since H_1 is reduced, by Theorem 1.4, $H_1 = K_{2,3}$, a contradiction. Claim 4 holds.

Claim 5. $n' \in \{13, 14\}$. Therefore, $G' = G$ and Theorem 1.8(i) holds.

Otherwise, by Lemma 2.5, $n' = 12$, Let $H = G' - v$ and $N_{G'}(v) = \{u_1, u_2\}$. As H is reduced and $|V(H)| = 11$, by Theorem 1.2(i), either $u_1 \in D_3(G')$ or $u_2 \in D_3(G')$. Thus, $d_2(H) \in \{1, 2\}$. By Theorem 1.7, $H = P(11)$. Thus, $G' = P^5(12)$. By Lemma 2.4, G' is collapsible, a contradiction. So Claim 5 holds.

Claim 6. (i) If $n' = 13$, then $d_2(G') = 1, d_3(G') \in \{8, 9, 10, 11, 12\}$, and $\Delta(G') \le 7$. Furthermore, if $\Delta(G') = 3$, then $d_3(G') = 12$ and $girth(G') = 4$.

(ii) If $n' = 14$, then $d_2(G') = 1$ and $d_3(G') \in \{8, 9, ..., 12\}$ and $\Delta(G') \in \{4, 5, ..., 8\}$. Furthermore, if $\Delta(G') \ge 5$, then $girth(G') = 4$.

(i) Assume that $n' = 13$. Then $F(G') \ge 3$. By Claims 2 and 3, we have $d_3(G') \ge 8$ and $\Delta(G') \le 7$. If $\Delta(G') = 3$, then $d_3(G') = 12$. By Lemma 2.7, $girth(G') = 4$.

(ii) Assume that $n' = 14$. As $F(G') \ge 3$, we have $d_3(G') \ge 8$ and $\Delta(G') \le 8$. Assume that $v \in V(G')$ such that $d_{G'}(v) = \Delta(G') \ge 5$ and $girth(G') \ge 5$. Let $T_i = \{x \in V(G') : dist_{G'}(x, v) = i\}$. Then $|T_0| = 1, |T_1| = 5$ and $|T_2| \ge 9$. Thus, $n' \ge 1 + 5 + 9 \ge 15$, a contradiction. So, if $\Delta(G') \ge 5$, then $girth(G') = 4$.

Claim 7. $girth(G') = 4$.

Assume that $girth(G') \ge 5$. If $n' = 14$, by Claim 5(ii), $\Delta(G) = 4$. By Lemma 2.9, G is supereulerian, contrary to (4). Next we assume that $n' = 13$. Notice that $G = G'$. As $girth(G) \ge 5, \Delta(G) = 4$. So we have $(d_2(G), d_3(G), d_4(G)) \in \{(1, 8, 4), (1, 10, 2)\}$. If $|N_G(v) \cap D_4(G)| \ge 1$, then $|V(G - v)| = 12, d_1(G - v) = 0$ and $d_2(G - v) \le 1$. By Theorem 1.8(i), $G - v \in \{P^1(12), P^2(12), P^3(12)\}$. Thus, $G \in \mathcal{S}_{13}$, contrary to (4). So $|N_G(v) \cap D_4(G)| = 0$. Choose $w \in D_4(G)$. For an integer $i \ge 0$, define $T_i = \{u \in V(G) | dist(u, w) = i\}$ Then $|T_0| = 1, |T_1| = 4, |T_2| \ge 8$ and $v \in T_2$. As $n = 13$, $|T_2| = 8$. Thus, for any $x \in T_1, x \in D_3(G)$. Let $T_1 = \{u_1, u_2, u_3, u_4\}$. As $v \in T_2$, we assume that $vu_1 \in E(G)$.

Consider $H = G - \{u_1, v\}$. Then $|V(H)| = 11, d_1(H) = 0$ and $d_2(H) \le 2$. By Claim 4, H is connected. By Theorem 1.7, $H = P(11)$. Let $D_2(H) = \{z\}$. Then either $vz \in E(G)$ or $u_1z \in E(G)$. If $zv \in E(G)$, then $N_G(v) = \{z, u_1\}$. As $|N_G(u_1) \cap V(H)| = 2$, G must have a 4-cycle, a contradiction. So $u_1z \in E(G)$. Therefore, $|N_G(u) \cap (V(H) - \{z\})| = 1$ and $N_G(v) \cap (V(H) - \{z\})| = 1$. As $girth(G) \ge 5$, the subgraph induced by $V(H) \cup \{u_1\}$ is $P^5(12)$. By Lemma 2.4, G is collapsible, a contradiction. So Claim 7 holds.

By Claim 7, we assume that G has a 4-cycle $C_4 = v_1v_2v_3v_4v_1$. Let $\pi = (V_1, V_2) = (\{v_1, v_3\}, \{v_2, v_4\})$ be a partition of $V(C_4)$. Form the graph G/π with the new edge e_π as in Definition 2.2.

Claim 8. $\kappa'(G'/\pi) \ge 2$.

By Claim 4, $\kappa'(G') \ge 2$. If G'/π has a cut edge, then it must be e_π. Thus, $\{v_1, v_2, v_3, v_4\}$ is a vertex-cut of G'. Let H_1 and H_2 be the components of $G' - \{v_1, v_2, v_3, v_4\}$ such that $|V(H_1)| \le |V(H_2)|$. Also we assume that $N_{G'}(v_1) \cap V(H_2) = N_{G'}(v_3) \cap V(H_2) = \emptyset$ and $N_{G'}(v_2) \cap V(H_1) = N_{G'}(v_4) \cap V(H_1) = \emptyset$. As $n' \in \{13, 14\}, |V(H_1)| \in \{1, 2, 3, 4, 5\}$. For $i = 1$, 2, let L_i induced by $V(H_i) \cup \{v_1, v_2, v_3, v_4\}$. By Claims 2, 3, and 4, $\kappa'(L_i) \ge 2$ and $d_2(L_i) \le 3$. If $|V(H_1)| \le 3$, by Theorem 1.3(ii) and (iv), we have $|V(L_1)| = 7$ and $L_1 \in \{K_{1,3}(1, 1, 1), T(1, 1)\}$. It contradicts the fact that $N_{L_1}(v_2) = N_{L_1}(v_4)$. So $|V(H_1)| \in \{4, 5\}$ and $|V(L_1)| \in \{8, 9\}$. By Theorem 1.3(iv), $d_2(L_1) = 3$. As $d_2(G) = 1$, we have $d_2(L_2) = 2$. By Theorem 1.3(iv), $|V(L_2)| = 10$. By Theorem 1.7, $L_2 = P^-(10)$. It contradicts the hypothesis that L_2 contains the 4-cycle $v_1v_2v_3v_4v_1$. Claim 8 holds.

Consider the reduction $(G/\pi)'$ of G/π. Let $x \in V(G/\pi)'$. with $d_{(G/\pi)'}(x) = 2$, either $x \in D_2(G)$ or $PI(x)$ contains either u_1 or u_2. So $d_2((G/\pi)') \le 3$. In particular, if $d_2((G\pi)') = 3$ with $x, y \in D_2((G/\pi)')$ such that $u_1 \in PI(x)$ and $u_2 \in PI(y)$, then $xy \in E((G/\pi)')$. Next we will use $(G/\pi)'$ to find the graph G. Figures 6–8 are originally from [12].

Assume that $n' = 13$. Then $|V(G/\pi)| = 11, d_1(G/\pi) = 0$ and $d_2(G/\pi) \le 1$. By Claim 8 and Theorem 1.7, $(G/\pi)' \in \{P(10), P(11)\}$. If $(G/\pi)' = P(10)$, then G/π contains the parallel edges. Thus, $G = P^2(13)$ as shown in Figure 9.

If $(G/\pi)' = P_{11}$, then $G/\pi = P(11)$. If e_π is incident to the degree two vertex in G/π, then $G = P^1(13)$ as shown in Figure 10. If e_π is not incident to the degree two vertex, then $G = P^3(12)(e)$ meaning subdividing an edge e in

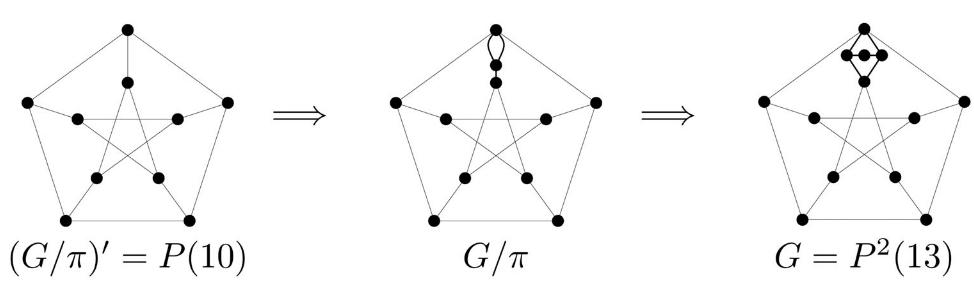

Figure 9. An illustration for Claim 8 in the proof of Theorem 1.8.

$P^3(12)$ as shown in Figure 1, where $e \in E(P^3(12)) - E(C_4)$. Thus, $G \in \mathcal{S}_{13}$. So Theorem 1.8(ii) holds.

Next we assume that $n' = 14$. Then $|V(G/\pi)| = 12$, $d_1(G/\pi) = 0$ and $d_2(G/\pi) \leq 1$. By Theorem 1.8(ii) and Claim 8, $(G/\pi)' \in \{P(10), P(11), P^1(12), P^2(12), P^3(12)\}$. If $(G/\pi)' \in \{P^1(12), P^2(12), P^3(12)\}$, then is supereulerian. Thus, $G \in \mathcal{S}_{14}$.

Assume that $(G/\pi)' = P(10)$. Then G/π either contains a K_3 or two C_2 such that $(G/\pi)/K_3 = P_{10}$ or $(G/\pi)/(C_2 \cup C_2) = P(10)$. Assume that $(G/\pi)/K_3 = P(10)$. If $e_\pi \in E(K_3)$, since G is K_3-free, G is the a graph with the structure as shown in Figure 6. Thus, G contains a collapsible subgraph K_3, contrary to the fact that $G = G'$ is reduced. If $e_\pi \notin E(K_3)$, then G/π and G are graphs as shown in Figure 7. Thus, $G \in \mathcal{S}_{14}$, contrary to (4).

Assume that $(G/\pi)/(C_2 \cup C_2) = P(10)$. Then two C_2 cycles must be incident with the edge e_π in G/π. Thus, G/π and G are shown in Figure 8. Let $\pi' = (\{x_1, x_3\}, \{x_2, x_4\})$ be a partition of a 4-cycle in G as shown in Figure 8. Then G/π' contains two C_2. Let $J = (G/\pi')/(C_2 \cup C_2)$. Then $|V(J)| = 10$, $\delta(J) \geq 3$ and $\kappa'(J) \geq 3$. By Theorem 1.2(i), J is collapsible. By Theorem 2.3(i), G is collapsible, a contradiction.

Next, we assume that $(G/\pi)' = P(11)$. As $d_2(G) = 1, G/\pi$ and G are the graphs as shown in Figure 11. So Theorem 1.8(iii) holds. The proof of Theorem 1.8 is now complete. □

4. Applications

Spanning trailable graphs are a special class of supereulerian graphs. Let $e, e' \in E(G)$. A trail from e to e' is called an (e, e')-trail. A graph is *spanning trailable* if for any pair of edges $e, e' \in E(G)$, G has a spanning (e, e')-trail. As $e = e'$ is possible, spanning trailable graphs are supereulerian. Luo et al. [23] first studied spanning trailable graphs (called Eulerian-connected graphs in [23]). They showed that every 4-edge-connected graph is spanning trailable, improving the former result of Caltin [5] and Jaeger [16] that every 4-edge-connected graph is supereulerian. Thus it is natural to study which 3-edge-connected graphs are spanning trailable.

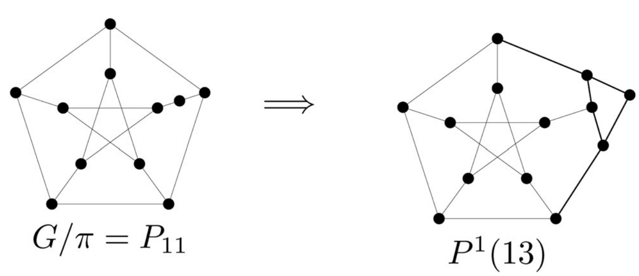

Figure 10. An illustration for Claim 8 in the proof of Theorem 1.8.

Suppose that $e = u_1v_1, e' = u_2v_2 \in E(G)$ denote two edges of G. If $e \neq e'$, then the graph $G(e, e')$ is obtained from G by replacing $e = u_1v_1$ by a path $u_1v_ev_1$ and by replacing $e' = u_2v_2$ by a path $u_2v_{e'}v_2$, where $v_e, v_{e'}$ are two new vertices not in $V(G)$. If $e = e'$, then $G(e, e')$ is also denoted by $G(e)$ and is obtained from G by replacing $e = u_1v_1$ by a path $u_1v_ev_1$. Let $u, v \in V(G)$, a (u, v)-trail is a trail from u to v. A graph G is *strongly spanning trailable* if for any $e, e' \in E(G), G(e, e')$ has a spanning $(v_e, v_{e'})$-trail. By definition,

$$\text{every strongly spanning trailable graphis also spanning trailable.} \tag{7}$$

Let $\mathbb{Z}_8$ denote the set of integers modulo 8, and V_8 denote the *Wagner graph*, which has vertex set $.V(V_8) = \{v_i : i \in \mathbb{Z}_8\}$ and edge set $E(V_8) = \{v_iv_{i+1} : i \in \mathbb{Z}_8\} \cup \{v_1v_5, v_2v_6, v_3v_7, v_4v_8\}$. As the Wagner graph V_8 is spanning trailable but not strongly spanning trailable [27], strongly spanning trailable graphs and spanning trailable graphs are not equivalent.

Theorem 4.1. *Let G be a 3-edge-connected non-strongly spanning spanning trailable simple graph. If $|V(G)| \leq 11$, then $G \in \{V_8, P(10)\}$.*

Proof. Let G be a non-strongly spanning trailable graph with $\kappa'(G) \geq 3$. Then there exist edges $e, e' \in E(G)$ such that $G(e, e')$ does not have a spanning $(v_e, v_{e'})$-trail. Let H be the graph obtained from $G(e', e'')$ by adding a new vertex z_0 and new edges $z_0v_{e'}, z_0v_{e''}$. Then H is not supereulerian. As $|V(G)| \leq 11$, we have $|V(H)| \leq 14$. Let H' be the reduction of H. As G is 3-edge-connected, $H' \neq K_1$ is 2-edge-connected, and $d_2(H') \leq 1$. In addition, if $d_2(H') = 0$, then $|V(H')| \leq 11 - 4 = 7$, and if $d_2(H') = 1$, then $D_2(H') = \{z_0\}$. By Theorem 1.8(iii), $H' \in \{P(11), P^1(13), P^2(13), P^2(14)\}$. Since G is a simple graph, $H' \notin \{P^2(13), P^2(14)\}$. If $H' = P(11)$, then $G = V_8$. If $H' = P^1(13)$, then $G = P(10)$. □

Harary and Nash-Williams showed that there is a close relationship between a graph and its line graph concerning Hamilton cycles.

Theorem 4.2 (Harary and Nash-Williams [15]). *Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is hamiltonian if and only if G has an eulerian subgraph H with $E(G - V(H)) = \emptyset$.*

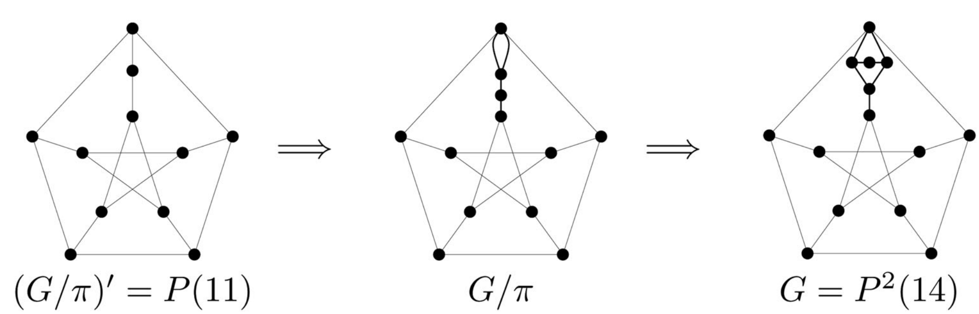

Figure 11. An illustration for Claim 8 in the proof of Theorem 1.8.

Let G be a graph such that $\kappa(L(G)) \geq 3$ and $G \neq K_{1,n-1}$. The *core* of this graph G, denoted by G_0, is obtained from $G - D_1(G)$ by contracting exactly one edge xy or yz for each path xyz in G with $d_G(y) = 2$.

Lemma 4.3 (Shao [26]). *Let G be a connected nontrivial graph such that $\kappa(L(G)) \geq 3$, and let G_0 denote the core of G.*

(i) G_0 is uniquely determined by G with $\kappa'(G_0) \geq 3$.
(ii) (see also Lemma 2.9 of [19]) If for any $e, e' \in E(G_0)$, $G_0(e, e')$ has a spanning $(v_e, v_{e'})$-trail, then $L(G)$ is Hamilton-connected.

In [1] and [2], Bauer proposed the problems of determining best possible sufficient conditions on the vertex degrees of a simple graph (or a simple bipartite graph, or a simple triangle-free graph, respectively) G to ensure that its line graph $L(G)$ is Hamiltonian. These problems have been settled by Catlin [5] and Lai [17], respectively. Similar problems are considered in this paper. We seek best possible sufficient degree conditions of a simple graph G to assure that $L(G)$ is Hamilton-connected. In [22], Liu et al. proved several results which imply that for a simple graph G with sufficiently large $n = |V(G)|$, if either $\delta(G) \geq \frac{n}{8} - 1$, or G is bipartite and $\delta(G) \geq \frac{n}{16} - 1$, then $L(G)$ is Hamilton-connected if and only if $\kappa(G) \geq 3$ and V_8 is not a nontrivial contraction of G. As an application of our main result, we prove the following.

Theorem 4.4. *Let G be a connected simple graph on n vertices. Each of the following holds:*

(i) If $\delta(G) \geq \frac{n}{10}$, then for sufficiently large n, $L(G)$ is Hamilton-connected if and only if both $\kappa(G) \geq 3$ and G are not nontrivially contractible to V_8.
(ii) If G is bipartite and $\delta(G) > \frac{n}{20}$, then for sufficiently large n, $L(G)$ is Hamilton-connected if and only if both $\kappa(G) \geq 3$ and G are not nontrivially contractible to V_8.

Proof. As the proof for (ii) is similar to that for (i), we only present the proof for (ii). Let G be a graph satisfying the hypotheses of Theorem with $\kappa(L(G)) \geq 3$ and $n \geq 141$. Then $\delta(G) \geq 8$, and so $D_i(G) = \emptyset$ for $i \in \{1, 2, ..., 7\}$. As G is essentially 3-edge-connected, G is 3-edge-connected. Thus, $G = G_0$. Let $e_1, e_2 \in E(G)$ and G' be the reduction of $G(e_1, e_2)$. Then $D_2(G') \subseteq \{v_{e_1}, v_{e_2}\}$. Let $v \in V(G') - \{v_{e_1}, v_{e_2}\}$ such that $d_{G'}(v) \leq 7$. Then $PI(v)$ is nontivial and there is a vertex $x \in V(PI(v))$ such that $N_G(x) \subseteq V(PI(v))$. As G is bipartite, $PI(v)$ is also bipartite. Assume that the vertex partition of $PI(v)$ is (A, B) and $x \in A$. Then $N_G(x) \subseteq B$. Thus, $|B| \geq d_G(x) \geq 8$. As $d_{G'}(v) \leq 7$, there is a vertex $y \in B$ such that $N_G(y) \subseteq V(PI(v))$. Thus, $|V(PI(v))| \geq d_G(x) + d_G(y) > \frac{n}{10}$. So $d_3(G') + \cdots + d_7(G') \leq 9$. As $F(G') \leq 2$, we have $2d_2(G') + d_3(G') \geq 10 + \sum_{i \geq 5}(i-4)d_i(G')$. As $d_2(G') \leq 2$, we have $d_i(G') = 0$ for $i \geq 8$. So

$$|V(G')| = d_2(G') + d_3(G') + \cdots + d_7(G') \leq 2 + 9 = 11.$$

In addition, if $|V(G')| = 11$, then $d_2(G') = 2$. By Theorem 1.7, $G' \in \{K^2_{2,3}, P^-(10)\}$. If $G' = K^2_{2,3}$, then G' has a spanning (v_{e_1}, v_{e_2})-trail. Thus, $L(G)$ is Hamilton-connected. If $G' = P^-(10)$, then $D_2(P^-(10)) = \{v_{e_1}, v_{e_2}\}$ and G is contractible to V_8. □

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No potential conflict of interest was reported by the authors.

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