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Connectivity and eigenvalues of graphs with given girth or clique number

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ABSTRACT

Let $\kappa'(G)$, $\mu_{n-1}(G)$ and $\mu_1(G)$ denote the edge-connectivity, the algebraic connectivity and the Laplacian spectral radius of G, respectively. In this paper, we prove that for integers $k \geq 2$ and $r \geq 2$, and any simple graph G of order n with minimum degree $\delta \geq k$, girth $g \geq 3$ and clique number $\omega(G) \leq r$, the edge-connectivity $\kappa'(G) \geq k$ if $\mu_{n-1}(G) \geq \frac{(k-1)n}{N(\delta,g)(n-N(\delta,g))}$ or if $\mu_{n-1}(G) \geq \frac{(k-1)n}{\varphi(\delta,r)(n-\varphi(\delta,r))}$, where $N(\delta, g)$ is the Moore bound on the smallest possible number of vertices such that there exists a δ-regular simple graph with girth g, and $\varphi(\delta, r) = \max\{\delta+1, \lfloor \frac{r\delta}{r-1} \rfloor\}$. Analogue results involving $\mu_{n-1}(G)$ and $\frac{\mu_1(G)}{\mu_{n-1}(G)}$ to characterize vertex-connectivity of graphs with fixed girth and clique number are also presented. Former results in Liu et al. (2013) [22], Liu et al. (2019) [20], Hong et al. (2019) [15], Liu et al. (2019) [21] and Abiad et al. (2018) [1] are improved or extended.

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1. Introduction

We only consider finite and simple graphs in this paper. Undefined notation and terminologies will follow Bondy and Murty [3]. Let $G = (V, E)$ be a graph of order n. We use $\kappa(G)$, $\kappa'(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex-connectivity, the edge-connectivity, the minimum degree and the maximum degree of a graph G, respectively. The *girth* $g(G)$ of a graph G is the length of a shortest cycle in G if it contains at least one cycle, and $g(G) = \infty$ if G is acyclic. A *clique* of a graph is a set of mutually adjacent vertices, and that the maximum size of a clique of a graph G, the *clique number* of G, is denoted by $\omega(G)$. For a vertex subset $S \subseteq V(G)$, $G[S]$ is the subgraph of G induced by S.

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G)$. The *adjacency matrix* of G is defined to be a $(0, 1)$-matrix $A(G) = (a_{ij})_{n\times n}$, where $a_{ij} = 1$ if v_i and v_j are adjacent, $a_{ij} = 0$ otherwise. As G is simple and undirected, $A(G)$ is a symmetric $(0, 1)$-matrix. The *adjacency eigenvalues* of G are the eigenvalues of $A(G)$. Denoted by $D(G) = \mathrm{diag}\{d_G(v_1), d_G(v_2), \ldots, d_G(v_n)\}$, the *degree diagonal matrix* of G, where $d_G(v_i)$ denotes the degree of v_i. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are called the *Laplacian matrix* and the *signless Laplacian matrix* of G, respectively. We use $\lambda_i(G)$, $\mu_i(G)$ and $q_i(G)$ to denote the ith largest eigenvalue of $A(G)$, $L(G)$ and $Q(G)$, respectively.

The second smallest Laplacian eigenvalue $\mu_{n-1}(G)$ is called *algebraic connectivity* by Fiedler [9,10]. Fiedler [9] initiated the investigation on the relationship between graph connectivity and graph eigenvalues, and showed that $\mu_{n-1}(G) \le \kappa(G) \le \kappa'(G)$. Kirkland, Molitierno, Neumann and Shader [16] investigated the graphs with equal algebraic connectivity and vertex-connectivity. It is worth mentioning that Cioabă in [6] investigated the relationship between edge-connectivity and adjacency eigenvalues of regular graphs. From then on, the edge-connectivity problem has been intensively studied by many researchers, as found in [1,5–7,11,12,17–20,22,23], among others. For the vertex-connectivity of graphs, one can refer to [1,15,21,24]. In [1], Abiad, Brimkov, Martínez-Rivera, O and Zhang raised the following research problem.

Problem 1.1. (Abiad, Brimkov, Martínez-Rivera, O and Zhang [1]) For a d-regular simple graph or multigraph G and for $2 \le k \le d$, what is the best upper bound on $\lambda_2(G)$ which guarantees $\kappa'(G) \ge k$ or $\kappa(G) \ge k$?

A number of results are related to Problem 1.1, as shown in the following theorem.

Theorem 1.2. *Let d, k be integers with $d \ge k \ge 2$, and let G be a simple graph of order n with minimum degree $\delta \ge k$.*

(i) (Cioabă [6]) If G is d-regular and $\lambda_2(G) \le d - \frac{(k-1)n}{(d+1)(n-d-1)}$, then $\kappa'(G) \ge k$.

(ii) (Li and Shi [17], Liu, Hong and Lai [18]) If $\lambda_2(G) \le \delta - \frac{(k-1)n}{(\delta+1)(n-\delta-1)}$, then $\kappa'(G) \ge k$.

(iii) (Liu, Lu and Tian [22]) If $\mu_{n-1}(G) \geq \frac{(k-1)n}{(\delta+1)(n-\delta-1)}$ or $q_2(G) \leq 2\delta - \frac{(k-1)n}{(\delta+1)(n-\delta-1)}$, then $\kappa'(G) \geq k$.

(iv) (Abiad, Brimkov, Martínez-Rivera, O and Zhang [1]) Let G be a d-regular graph. If $k \geq 3$ and $\lambda_2(G) < d - \frac{(k-1)dn}{2(d-k+2)(n-d+k-2)}$, then $\kappa(G) \geq k$. If $\lambda_2(G) < d - \frac{dn}{2(d+1)(n-d-1)}$, then $\kappa(G) \geq 2$.

As can be seen in [15] or will be seen in Section 4, for any real number $p > 0$, if $q_2(G) \leq 2\delta(G) - p$ or $\lambda_2(G) \leq \delta(G) - p$, then $\mu_{n-1}(G) \geq p$. Moreover, it is known that if $\mu_{n-1}(G) > 0$, then $\kappa'(G) \geq \kappa(G) \geq 1$. Therefore, we focus on establishing the lower bounds on $\mu_{n-1}(G)$ which guarantee $\kappa'(G) \geq k$ or $\kappa(G) \geq k$. By Theorem 1.2, it is natural to discuss Problem 1.1 for bipartite graphs or triangle-free graphs and drop the graph regularity. Note that triangle-free graphs have girth at least 4, or equivalently clique number at most 2. Thus, to get better lower bounds on algebraic connectivity, we consider graphs with fixed girth or clique number. In this paper, we improve or extend some recent results. In order to state some known results, we need the following definition.

Definition 1.3. For integers δ, g with $\delta \geq 2$ and $g \geq 3$, let $t = \lfloor \frac{g-1}{2} \rfloor$. Define

$$N(\delta, g) = \begin{cases} 1 + \delta \sum_{i=0}^{t-1} (\delta - 1)^i, & \text{if } g = 2t + 1; \\ 2 \sum_{i=0}^{t} (\delta - 1)^i, & \text{if } g = 2t + 2. \end{cases}$$

Tutte [26] initiated the cage problem, which seeks, for any given integers d and g with $d \geq 2$ and $g \geq 3$, the smallest possible number of vertices $n(d, g)$ such that there exists a d-regular simple graph with girth g. $N(d, g)$ in Definition 1.3 is a tight lower bound (often called the Moore bound) on $n(d, g)$ which can be found in [8].

The results in Theorem 1.2 have been improved or extended in [22,20,21,15] as follows.

Theorem 1.4. *(Liu, Lu and Tian [22]) Let $k \geq 2$ be an integer, and G be a connected graph of order n with girth $g \geq 3$ and minimum degree $\delta \geq k$. If $\mu_{n-1}(G) \geq \frac{(k-1)n}{g(n-g)}$, then $\kappa'(G) \geq k$. Moreover, if $\delta \geq 3$ and $\mu_{n-1}(G) \geq \frac{(k-1)n}{\frac{4}{9}N(\delta,g)(n-\frac{4}{9}N(\delta,g))}$, then $\kappa'(G) \geq k$.*

Theorem 1.5. *(Liu, Lai and Tian [20]) Let $k \geq 2$ be an integer, and G be a connected graph of order n with girth $g \geq 3$ and minimum degree $\delta \geq k$. Let $f(2, g) = g$, $t = \lfloor \frac{g-1}{2} \rfloor$ and for $\delta \geq 3$ $f(\delta, g) = N(\delta, g) - \sum_{i=1}^{t-1} (\delta - 1)^i$. If $\mu_{n-1}(G) \geq \frac{(k-1)n}{f(\delta,g)(n-f(\delta,g))}$, then $\kappa'(G) \geq k$.*

Theorem 1.6. *(Liu, Lai, Tian and Wu [21]) Let $k \geq 2$ be an integer, and G be a connected graph of order n with maximum degree Δ, minimum degree $\delta \geq k$, girth $g \geq 3$. Let $t = \lfloor \frac{g-1}{2} \rfloor$ and*

$$\nu(\delta, g, k) = \begin{cases} N(\delta, g) - (k-1) \sum_{i=0}^{t-1} (\delta - 1)^i, & \text{if } g = 2t + 1, \text{ or } g = 2t + 2 \text{ and } \delta \geq 3; \\ 2t + 1, & \text{if } g = 2t + 2 \text{ and } \delta = 2. \end{cases}$$

If $\mu_{n-1}(G) \geq \frac{(k-1)n\Delta}{2\nu(\delta,g,k)(n-\nu(\delta,g,k))}$, *then* $\kappa(G) \geq k$.

Theorem 1.7. *(Hong, Xia and Lai [15]) Let k be an integer and G be a simple graph of order n with maximum degree Δ and minimum degree $\delta \geq k \geq 2$.*

(i) If $\mu_{n-1}(G) > \frac{(k-1)n\Delta}{(n-k+1)(k-1)+4(\delta-k+2)(n-\delta-1)}$, *then* $\kappa(G) \geq k$.

(ii) If G is triangle-free and $\mu_{n-1}(G) > \frac{(k-1)n\Delta}{(n-k+1)(k-1)+4(2\delta-k+1)(n-2\delta)}$, *then* $\kappa(G) \geq k$.

Although Theorem 1.5 and Theorem 1.6 improve some previous results, there still exist some graphs which do not satisfy the sufficient conditions of Theorem 1.5 and Theorem 1.6. We aim to find better bounds on $\mu_{n-1}(G)$ such that the connectivity of more graphs can be determined. As will be seen in the appendix, we list some graphs whose connectivity can be determined by Theorems 1.8, 1.9, 1.10, 1.12, 1.13 but can not be determined by Theorem 1.5 and Theorem 1.6. This is the motivation of the current research.

For edge-connectivity, in this paper we obtain the following two theorems, where Theorem 1.8 improves Theorems 1.4 and 1.5, and Theorem 1.9 extends Theorem 1.8 when $g(G) = 3$.

Theorem 1.8. *Let k be an integer and G be a connected graph of order n with minimum degree $\delta \geq k \geq 2$ and girth $g \geq 3$. If* $\mu_{n-1}(G) \geq \frac{(k-1)n}{N(\delta,g)(n-N(\delta,g))}$, *then* $\kappa'(G) \geq k$.

Theorem 1.9. *Let $r \geq 2$ and k be integers, and G be a connected graph of order n with minimum degree $\delta \geq k \geq 2$ and clique number $\omega(G) \leq r$. Let* $\varphi(\delta, r) = \max\{\delta+1, \lfloor \frac{r\delta}{r-1} \rfloor\}$. *If* $\mu_{n-1}(G) \geq \frac{(k-1)n}{\varphi(\delta,r)(n-\varphi(\delta,r))}$, *then* $\kappa'(G) \geq k$.

For vertex-connectivity, we obtain the following three theorems, where Theorem 1.10 improves Theorem 1.6 and extends Theorem 1.7 when $g(G) \geq 5$, and Theorems 1.11 and 1.12 extend Theorem 1.10 when $g(G) = 3$.

Theorem 1.10. *Let g, k be integers and G be a connected graph of order n with maximum degree Δ, minimum degree $\delta \geq k \geq 2$ and girth $g \geq 3$. If*

$$\mu_{n-1}(G) > \frac{n(k-1)\Delta}{n(n-k+1) - (n - 2N(\delta,g) + k - 1)^2},$$

then $\kappa(G) \geq k$.

Theorem 1.11. *Let $r \geq 3$ and k be integers, and G be a connected graph of order n with maximum degree Δ, minimum degree $\delta \geq k \geq 2$ and clique number $\omega(G) \leq r$. Let* $\phi(\delta, k, r) = \max\{(n - \frac{2(r-1)}{r-2}\delta + \frac{r(k-1)}{r-2})^2, (n - \frac{2r\delta}{r-1} + k - 1)^2\}$. *If*

$$\mu_{n-1}(G) > \frac{n(k-1)\Delta}{n(n-k+1) - \phi(\delta,k,r)},$$

then $\kappa(G) \geq k$.

Theorem 1.12. *Let* $r \geq 2$ *and* $k \geq 2$ *be integers, and* G *be a connected graph of order* n *with maximum degree* Δ*, minimum degree* $\delta > (k-1)(r-1)$ *and clique number* $\omega(G) \leq r$. *If*

$$\mu_{n-1}(G) > \frac{n(k-1)\Delta}{n(n-k+1)-(n-\frac{2r\delta}{r-1}+k-1)^2},$$

then $\kappa(G) \geq k$.

Applying the result of Brouwer and Haemers [4], we get the following two results for vertex-connectivity with respect to $\mu_1(G)$ and $\mu_{n-1}(G)$.

Theorem 1.13. *Let* g, k *be integers and* G *be a connected graph of order* n *with minimum degree* $\delta \geq k \geq 2$ *and girth* $g \geq 3$. *If*

$$\frac{\mu_1(G)}{\mu_{n-1}(G)} < s+\sqrt{s^2-1} \text{ or equivalently } \frac{\mu_{n-1}(G)}{\mu_1(G)} > s-\sqrt{s^2-1},$$

then $\kappa(G) \geq k$*, where* $s = \frac{2(N(\delta,g)-k+1)(n-N(\delta,g))}{n(k-1)}+1$.

Theorem 1.14. *Let* $r \geq 2$ *and* $k \geq 2$ *be integers, and* G *be a connected graph of order* n *with minimum degree* $\delta > (k-1)(r-1)$ *and clique number* $\omega(G) \leq r$. *If*

$$\frac{\mu_1(G)}{\mu_{n-1}(G)} < s+\sqrt{s^2-1} \text{ or equivalently } \frac{\mu_{n-1}(G)}{\mu_1(G)} > s-\sqrt{s^2-1},$$

then $\kappa(G) \geq k$*, where* $s = \frac{2(\frac{r}{r-1}\delta-k+1)(n-\frac{r}{r-1}\delta)}{n(k-1)}+1$.

In Section 2, we display some preliminaries and mechanisms, including the bounds of Laplacian eigenvalues and the scale of the remained connected components when deleting vertex subset or edge subset in G. These will be applied in the proofs of the main results, to be presented in Section 3. As corollaries, adjacency and signless Laplacian eigenvalue conditions which guarantee that $\kappa'(G) \geq k$ or $\kappa(G) \geq k$ are presented in the last section.

2. Preliminaries

In this section, we present some of the preliminaries to be used in the proof of main results. For disjoint subsets X and Y of $V(G)$, let $E(X,Y)$ be the set of edges between X and Y. For $X \subseteq V(G)$, we use $d_G(X)$ or simply $d(X)$ to denote the number of edges between X and $V(G) \setminus X$, that is $d(X) = |E(X, V(G) \setminus X)|$. For a vertex $v \in V(G)$, we use $N_G(v)$ to denote the neighbor set of v in G. The following result is the famous theorem of Turán [27].

Lemma 2.1. *(Turán [27]) Let $r \geq 1$ be an integer, and G be a graph of order n. If the clique number $\omega(G) \leq r$, then $|E(G)| \leq \left\lfloor \frac{r-1}{2r} \cdot n^2 \right\rfloor$.*

Lemma 2.2. *Let $r \geq 2$ be an integer, and G be a graph with minimum degree δ and clique number $\omega(G) \leq r$. Let X be a nonempty proper subset of $V(G)$. If $d(X) < \delta$, then $|X| \geq \max\{\delta + 1, \left\lfloor \frac{r\delta}{r-1} \right\rfloor\}$.*

Proof. We first show that X contains at least $\delta + 1$ vertices. Since each vertex in X is adjacent to at most $|X| - 1$ vertices of X, we obtain

$$\delta|X| \leq \sum_{x \in X} d_G(x) \leq |X|(|X| - 1) + d(X) \leq |X|(|X| - 1) + \delta - 1,$$

and so $(|X| - 1)(|X| - \delta) \geq 1$, which means that $|X| \geq \delta + 1$.

Next we show that $|X| \geq \left\lfloor \frac{r\delta}{r-1} \right\rfloor$. By Lemma 2.1, we conclude that

$$|E(G[X])| \leq \frac{(r-1)|X|^2}{2r}. \tag{2.1}$$

Since $\sum_{x \in X} d_G(x) = 2|E(G[X])| + d(X)$, by (2.1)

$$|X|\delta \leq \sum_{x \in X} d_G(x) \leq 2\frac{(r-1)|X|^2}{2r} + d(X) \leq \frac{(r-1)|X|^2}{r} + \delta - 1$$

and so $|X|^2 - \frac{r\delta}{r-1}|X| + \frac{r(\delta-1)}{r-1} \geq 0$. It follows that

$$(|X| - 1)(|X| - \frac{r\delta}{r-1} + 1) \geq \frac{1}{r-1} > 0,$$

which means that $|X| > \frac{r\delta}{r-1} - 1$. Therefore we arrive at $|X| \geq \left\lfloor \frac{r\delta}{r-1} \right\rfloor$. $\square$

Lemma 2.3. *Let $r \geq 2$ be an integer, and G be a graph with minimum degree $\delta \geq 2$ and clique number $\omega(G) \leq r$. Let S be a vertex-cut of G and X be the vertex set of a component of $G - S$.*

(i) If $r \geq 3$ and $|S| < \delta$, then $|X| \geq \min\{\frac{r-1}{r-2}(\delta - |S|), \frac{r\delta}{r-1} - |S|\}$.
(ii) If $r \geq 3$ and $\frac{\delta}{r-1} \leq |S| < \delta$, then $|X| \geq \frac{r-1}{r-2}(\delta - |S|)$.
(iii) If $r \geq 2$ and $|S| < \frac{\delta}{r-1}$, then $|X| \geq \frac{r\delta}{r-1} - |S|$.

Proof. (i) If $\omega(G[X]) \leq r - 1$, then by Lemma 2.1, we have $2|E(G[X])| \leq \frac{r-2}{r-1}|X|^2$. Since $\delta > |S|$, each vertex in $G[X]$ has degree at least $\delta - |S|$ and so

$$|X|(\delta - |S|) \leq 2|E(G[X])| \leq \frac{r-2}{r-1}|X|^2.$$

Thus, in this case, we have $|X| \geq \frac{r-1}{r-2}(\delta - |S|)$.

If $\omega(G[X]) = r$, then there exists a complete subgraph K_r in $G[X]$. Consider the following two subcases. If $\delta \leq r-1$, then $|X| \geq r \geq \delta+1$. If $\delta > r-1$, then each vertex of K_r has at least $\delta - r + 1$ neighbors in $(X \cup S) \setminus V(K_r)$ and at most $r-1$ vertices of K_r have common neighbors in $(X \cup S) \setminus V(K_r)$. This leads to $|N(K_r)| \geq \frac{r(\delta-r+1)}{r-1}$ and so

$$|X| + |S| \geq |V(K_r)| + |N(K_r)| \geq r + \frac{r(\delta - r + 1)}{r-1} = \frac{r\delta}{r-1},$$

which implies $|X| \geq \frac{r\delta}{r-1} - |S|$.

By the discussions above, we conclude that

(A) if $\delta \leq r-1$, then $|X| \geq \min\{\frac{r-1}{r-2}(\delta - |S|), \delta + 1\} = \frac{r-1}{r-2}(\delta - |S|)$;
(B) if $\delta > r-1$, then $|X| \geq \min\{\frac{r-1}{r-2}(\delta - |S|), \frac{r\delta}{r-1} - |S|\}$.

Combining (A) with (B), (i) is proved.

(ii) If $r \geq 3$ and $|S| \geq \frac{\delta}{r-1}$, then

$$\frac{r\delta}{r-1} - |S| - \frac{r-1}{r-2}(\delta - |S|) = \frac{(r-1)|S| - \delta}{(r-1)(r-2)} \geq 0.$$

Therefore, by (i), $|X| \geq \min\{\frac{r-1}{r-2}(\delta - |S|), \frac{r\delta}{r-1} - |S|\} = \frac{r-1}{r-2}(\delta - |S|)$.

(iii) If $r \geq 3$ and $|S| < \frac{\delta}{r-1}$, then

$$\frac{r\delta}{r-1} - |S| - \frac{r-1}{r-2}(\delta - |S|) = \frac{(r-1)|S| - \delta}{(r-1)(r-2)} < 0.$$

Therefore, by (i), $|X| \geq \min\{\frac{r-1}{r-2}(\delta - |S|), \frac{r\delta}{r-1} - |S|\} = \frac{r\delta}{r-1} - |S|$.

If $r = 2$ and $|S| < \delta$, then X contains at least two vertices and there exists one edge xy in $G[X]$. As $r = 2$, G is triangle-free and so $N(x) \cap N(y) = \emptyset$. Since $N(x) \cup N(y) \subseteq X \cup S$, it follows that

$$|X| + |S| = |X \cup S| \geq |N(x) \cup N(y)| = |N(x)| + |N(y)| \geq 2\delta$$

and thus $|X| \geq 2\delta - |S| = \frac{r\delta}{r-1} - |S|$. The result follows. □

For any two vertices u, v in G, let $d(u, v)$ be the length of a shortest path between u and v in G. For any nonempty set $S \subseteq V$, let $d(v, S) = \min\{d(v, w), \forall w \in S\}$ for any vertex $v \in V(G)$. In particular, if $v \in S$, then $d(v, S) = 0$.

Lemma 2.4. *Let G be a simple connected graph with minimum degree $\delta \geq 2$ and girth $g \geq 3$. Let S be a vertex-cut of G and X be the vertex set of a component of $G - S$. If $|S| < \delta$, then $|X| \geq N(\delta, g) - |S|$.*

Proof. Claim 1. X contains at least $\delta + 1 - |S|$ vertices.

Since each vertex in X is adjacent to at most $|X| - 1$ vertices of X and at most $|S|$ vertices of S, we obtain

$$\delta|X| \leq \sum_{x \in X} d_G(x) \leq |X|(|X| - 1 + |S|),$$

and so $|X| \geq \delta + 1 - |S|$. Thus Claim 1 holds and implies that $|X| \geq 2$.

Claim 2. There exists a vertex $v \in X$ such that $d(v, S) \geq t$.

If $t = 1$, then Claim 2 holds obviously. So we only need to consider $t \geq 2$. Suppose to the contrary that each vertex $v \in X$ satisfies $d(v, S) \leq t - 1$. Let v_0 be an arbitrary vertex in X and $\{v_1, v_2, \dots, v_\delta\} \subseteq N(v_0)$ be the subset of the neighbors of v_0 in G. For each $i \in \{1, 2, \dots, \delta\}$, let P_i be a shortest path from v_i to S, then $|E(P_i)| \leq t - 1$. Note that v_i may be in S and P_i may be trivial. Since $|S| \leq \delta - 1$, there exist at least two paths P_j and P_k with $1 \leq j < k \leq \delta$ such that $V(P_j) \cap V(P_k) \neq \emptyset$. Thus, $P_j \cup P_k \cup \{v_0 v_j, v_0 v_k\}$ contains a cycle C of length

$$\ell(C) \leq |E(P_j)| + |E(P_k)| + 2 \leq 2t < g,$$

a contradiction to the girth of G is g. Claim 2 is proved.

(i) Assume that $g = 2t + 1$ is odd and $v \in X$ with $d(v, S) \geq t$. Then $N_i(v) \subseteq X \cup S$ for each $0 \leq i \leq t$, where $N_i(v) = \{u \in V(G) : d(u, v) = i\}$. Furthermore, for each $1 \leq i \leq t - 1$ and for any distinct vertices $x, y \in N_i(v)$, the neighbors of x and y in $N_{i+1}(v)$ are distinct as $G[X]$ contains no cycle of length less than g. Hence,

$$\begin{aligned} |X| + |S| = |X \cup S| &\geq |N_0(v)| + |N_1(v)| + |N_2(v)| + \cdots + |N_t(v)| \\ &\geq 1 + \delta + \delta(\delta - 1) + \cdots + \delta(\delta - 1)^{t-1} \\ &= 1 + \delta \sum_{i=0}^{t-1} (\delta - 1)^i = N(\delta, g). \end{aligned}$$

(ii) Assume that $g = 2t + 2$ is even and $v \in X$ with $d(v, S) \geq t$. Let $\{v_1, v_2, \dots, v_\delta\} \subseteq N(v)$ be the subset of the neighbors of v. Without loss of generality, assume that P is the shortest path from v to $v' \in S$ passing v_1 and P_1 is the subpath of P from v_1 to S. Let P_i be a shortest path from v_i to S for each $i \in \{2, 3, \dots, \delta\}$.

Claim 3. There exists a neighbor $u \in X$ of v such that $d(u, S) \geq t$.

Suppose that $d(v_i, S) \leq t - 1$ for each $2 \leq i \leq \delta$. If there exists some $i \geq 2$ such that $V(P_i) \cap V(P_1) \neq \emptyset$, then

$$t - 1 \geq d(v_i, S) = |E(P_i)| \geq |E(P_1)| = |E(P)| - 1 \geq t - 1$$

and so $P_i \cup P_1 \cup \{vv_1, vv_i\}$ contains a cycle C of length $\ell(C) \leq 2t$, a contradiction. In this case, if $\delta = 2$, then $|S| = 1$ and $V(P_2) \cap V(P_1) \neq \emptyset$, which yields a contradiction to $g > 2t$.

Hence, Claim 3 is true for $\delta = 2$. Next, it suffices to consider $\delta \geq 3$. If $V(P_i) \cap V(P_1) = \emptyset$ for each $2 \leq i \leq \delta$, then there exist at least two paths P_i and P_j with $2 \leq i < j \leq \delta$ such that $V(P_i) \cap V(P_j) \neq \emptyset$ as $|S \setminus \{v'\}| \leq \delta - 2$. Thus, $P_i \cup P_j \cup \{vv_i, vv_j\}$ contains a cycle C of length $\ell(C) \leq 2t$, a contradiction. This completes the proof of Claim 3.

By Claim 3, assume that u is a neighbor of v such that $d(u, S) \geq t$. Then $N_i(uv) \subseteq X \cup S$ for each $1 \leq i \leq t$, where $N_i(uv) = \{w \in V \setminus \{u, v\} : d(w, \{u, v\}) = i\}$. Furthermore, for each $1 \leq i \leq t-1$ and for any distinct vertices $x, y \in N_i(uv)$, the neighbors of x and y in $N_{i+1}(uv)$ are distinct and $N(u) \cap N(v) = \emptyset$ as $g(G[X \cup S]) \geq g = 2t + 2$. Hence,

$$\begin{aligned} |X| + |S| = |X \cup S| &\geq 2 + |N_1(uv)| + |N_2(uv)| + \cdots + |N_t(uv)| \\ &\geq 2 + 2(\delta - 1) + 2(\delta - 1)(\delta - 1) + \cdots + 2(\delta - 1)(\delta - 1)^{t-1} \\ &= 2 \sum_{i=0}^{t} (\delta - 1)^i = N(\delta, g). \end{aligned}$$

The result follows. □

Lemma 2.5. *Let G be a simple connected graph with minimum degree $\delta \geq 2$ and girth $g \geq 3$, X be a non-empty proper subset of $V(G)$. If $d(X) < \delta$, then $|X| \geq N(\delta, g)$.*

Proof. Let F be the set of edges between X and $V(G) \setminus X$, and $S = V(F) \cap X$ be the set of end-vertices of F in X. Since $d(X) < \delta$, by Lemma 2.2 we have $|X| \geq \delta + 1$. Thus, $X \setminus S \neq \emptyset$ and so S is a vertex cut of G with $|S| \leq d(X) < \delta$. Let $X_1, \ldots, X_k \subseteq X$ be the vertex sets of the components of $G - S$, where $k \geq 1$. By Lemma 2.4, $|X_1| \geq N(\delta, g) - |S|$ and so $|X| \geq |X_1| + |S| \geq N(\delta, g)$. □

Corollary 2.6. *Let G be a simple graph of order n with minimum degree $\delta \geq 2$ and girth $g \geq 3$.*

(i) If $n < 2N(\delta, g) - \kappa(G)$, then $\kappa(G) = \delta(G)$.

(ii) If $n < 2N(\delta, g)$, then $\kappa'(G) = \delta(G)$.

Proof. (i) Suppose to the contrary that $\kappa(G) < \delta(G)$. Assume that S is a minimum vertex-cut of G and X is the vertex set of a minimum component of $G - S$. Let $Y = V(G) \setminus (X \cup S)$. By Lemma 2.4, $|Y| \geq |X| \geq N(\delta, g) - \kappa(G)$ and so $n = |X| + |Y| + |S| \geq 2N(\delta, g) - \kappa(G)$, which is a contradiction.

(ii) Suppose to the contrary that $\kappa'(G) < \delta(G)$. Assume that $F = E(X, Y)$ is a minimum edge-cut of G and $|Y| \geq |X|$. By Lemma 2.5, $|Y| \geq |X| \geq N(\delta, g)$ and so $n = |X| + |Y| \geq 2N(\delta, g)$, which is a contradiction. □

Let $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, and let G be a graph with vertex set $V(G) = \{1, 2, \ldots, n\}$. Then x can be considered as a function defined on $V(G)$, that is, for any vertex i, we map it to $x_i = x(i)$. Fiedler [10] derived a very useful expression for algebraic connectivity as follows.

Lemma 2.7. *(Fiedler [10]) Let G be a connected graph with vertex set $V = \{1, 2, \dots, n\}$ and edge set E. Then the algebraic connectivity of G is positive and*

$$\mu_{n-1}(G) = \min_x f(x) = \min_x \frac{n \sum\limits_{ij \in E} (x_i - x_j)^2}{\sum\limits_{i,j \in V, i<j} (x_i - x_j)^2},$$

where the minimum is taken over all non-constant vectors $x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$. The characteristic vectors $y = (y_1, y_2, \dots, y_n)^T$ of G corresponding to $\mu_{n-1}(G)$ are then those non-constant vectors for which the minimum of $f(x)$ is attained and for which $\sum_{i=1}^n y_i = 0$.

The following lemma is one of the most important applications of algebraic connectivity and its proof could be found in [13]. By Lemma 2.7, we present an intuitive proof of this lemma as follows.

Lemma 2.8. *(Godsil and Royle, Lemma 13.7.1 in [13]) Let $G = (V, E)$ be a graph, and X be a nonempty proper subset of V and $Y = V \setminus X$. Then*

$$\mu_{n-1}(G) \le \frac{nd(X)}{|X||Y|}.$$

Proof. Let $x = (x_1, x_2, \dots, x_n)^T$ be a real vector. If $i \in X$, then set $x_i = 1$; if $i \in Y$, then set $x_i = -1$. By Lemma 2.7,

$$\mu_{n-1}(G) \le \frac{n \sum\limits_{ij \in E} (x_i - x_j)^2}{\sum\limits_{i,j \in V, i<j} (x_i - x_j)^2} \tag{2.2}$$

holds for the real vector x. Applying the values of the entries of x into the inequality (2.2), we obtain

$$\sum_{ij \in E} (x_i - x_j)^2 = \sum_{ij \in E(X,Y)} (1 - (-1))^2 = 4d(X),$$
$$\sum_{i,j \in V, i<j} (x_i - x_j)^2 = \sum_{i \in X, j \in Y} (1 - (-1))^2 = 4|X||Y|.$$

By (2.2), the result follows. □

In [25], in terms of the Laplacian eigenvalues, Pothen, Simon and Liou considered the lower bound on the size of vertex separator separating any pair of vertex disjoint sets A and B that are at a distance ρ from each other and generalized a result of Alon and Milman [2]. As a special case, we have the following result involving the upper bound on $\mu_{n-1}(G)$ by setting $\rho = 2$ and present the upper bound more precisely.

Lemma 2.9. *(Pothen, Simon and Liou [25]) Let $G = (V, E)$ be a graph of order n, and S be an arbitrary minimum vertex-cut of G and X be the vertex set of a component of $G - S$, and $Y = V \setminus (S \cup X)$. Then*

$$\mu_{n-1}(G) \leq \frac{nd(S)}{n(n-|S|) - (|X| - |Y|)^2}.$$

Proof. Let $x = (x_1, x_2, \ldots, x_n)^T$ be a real vector. If $i \in X$, then set $x_i = 1$; if $i \in Y$, then set $x_i = -1$; if $i \in S$, then set $x_i = 0$. By Lemma 2.7,

$$\mu_{n-1}(G) \leq \frac{n \sum\limits_{ij \in E} (x_i - x_j)^2}{\sum\limits_{i,j \in V, i<j} (x_i - x_j)^2} \tag{2.3}$$

holds for the real vector x. Applying the values of the entries of x into the inequality (2.3), we obtain

$$\sum_{ij \in E} (x_i - x_j)^2 = \sum_{ij \in E(S, X \cup Y)} (x_i - x_j)^2 = \sum_{ij \in E(S, X \cup Y)} 1 = d(S), \tag{2.4}$$

$$\begin{aligned} \sum_{i,j \in V, i<j} (x_i - x_j)^2 &= \sum_{i \in X, j \in S} (x_i - x_j)^2 + \sum_{i \in Y, j \in S} (x_i - x_j)^2 + \sum_{i \in X, j \in Y} (x_i - x_j)^2 \\ &= \sum_{i \in X, j \in S} (1-0)^2 + \sum_{i \in Y, j \in S} ((-1)-0)^2 + \sum_{i \in X, j \in Y} (1-(-1))^2 \\ &= |S||X| + |S||Y| + 4|X||Y| \\ &= (n - |X| - |Y|)(|X| + |Y|) + 4|X||Y| \\ &= n(n-|S|) - (|X| - |Y|)^2. \end{aligned} \tag{2.5}$$

Substituting (2.4) and (2.5) in (2.3), the result follows. □

The upper bounds in Lemma 2.8 and Lemma 2.9 are sharp. The hypercube Q_n is one of the graphs attaining the upper bound on $\mu_{n-1}(G)$ of Lemma 2.8. In fact, by setting $X = V(Q_{n-1})$, we have $\mu_{n-1}(Q_n) = 2 = \frac{2^n d(X)}{2^{n-1} 2^{n-1}}$. The graph G in Example 5.1 of Appendix is one of the graphs attaining the upper bound on $\mu_{n-1}(G)$ of Lemma 2.9. In fact, by setting $S = \{v_5, v_6, v_7, v_8\}$ and $X = \{v_1, v_2, v_3, v_4\}$, we get $\mu_{n-1}(G) = 2 = \frac{12 \cdot d(S)}{12 \cdot (12 - |S|) - (4-4)^2}$.

Lemma 2.10. *(Haemers [14]) Let G be a graph on n vertices, and let X and Y be disjoint sets of vertices, such that there is no edge between X and Y. Then*

$$\frac{|X||Y|}{(n-|X|)(n-|Y|)} \leq \left(\frac{\mu_1(G) - \mu_{n-1}(G)}{\mu_1(G) + \mu_{n-1}(G)} \right)^2.$$

For applications, a useful Lemma can be derived from Lemma 2.10 as follows.

Lemma 2.11. *(Brouwer and Haemers, Proposition 4.6.1 in [4]) Let G be a connected graph on n vertices, and let X and Y be disjoint sets of vertices, such that there is no edge between X and Y. Then*

$$\frac{|X||Y|}{n(n-|X|-|Y|)} \leq \frac{(\mu_1(G)-\mu_{n-1}(G))^2}{4\mu_1(G)\mu_{n-1}(G)}.$$

3. The proof of main results

Proof of Theorem 1.8. To the contrary, suppose that $1 \leq \kappa'(G) \leq k-1$. Let F be an arbitrary minimum edge-cut of G, and X, Y be the vertex sets of two components of $G-F$ with $|X| \leq |Y|$. Thus $d(X) = \kappa'(G) \leq k-1$. By Lemma 2.5 and $d(X) < \delta$, we obtain $|X| \geq N(\delta, g)$. Since $|Y| \geq |X|$ and $|X| + |Y| = n$,

$$|X| \cdot |Y| \geq N(\delta, g)(n - N(\delta, g)). \tag{3.1}$$

By Lemma 2.8 and (3.1), we have

$$\mu_{n-1}(G) \leq \frac{nd(X)}{|X||Y|} \leq \frac{(k-1)n}{N(\delta,g)(n-N(\delta,g))}.$$

According to the hypothesis, it follows that $\mu_{n-1}(G) = \frac{nd(X)}{|X||Y|} = \frac{(k-1)n}{N(\delta,g)(n-N(\delta,g))}$. By the proof of Lemma 2.8, $\mu_{n-1}(G) = \frac{n\sum_{ij\in E}(x_i-x_j)^2}{\sum_{i,j\in V, i<j}(x_i-x_j)^2}$, where $x_i = 1$ if $i \in X$ and $x_i = -1$ if $i \in Y$. By Lemma 2.7, x is a characteristic vector of G corresponding to $\mu_{n-1}(G)$. Since $d(X) < \delta$ and $|X| \geq N(\delta, g) \geq \delta + 1$, there exists one vertex j in X such that its neighbor set $N_G(j) \subset X$. Thus, by $\mu_{n-1}(G)x = (D-A)x$, we have $\mu_{n-1}(G)x_j = |N_G(j)|x_j - \sum_{\ell \in N_G(j)} x_\ell$. Since $x_j = x_\ell = 1$, it indicates $\mu_{n-1}(G) = 0$ and so $k-1 = 0$, which is a contradiction to $k \geq 2$. Hence, $\kappa'(G) \geq k$. □

Remark 3.1. The result in Theorem 1.8 improves the result of Theorem 1.4 when $\delta \geq 3$ and improves the result of Theorem 1.5 when $\delta \geq 3$ and $g \geq 5$. In fact, if $n < 2N(\delta, g)$, then by Corollary 2.6 we have $\kappa'(G) = \delta(G)$. Therefore, we only need to compare the bounds when $n \geq 2N(\delta, g)$. Note that $N(\delta, g) > N(\delta, g) - \sum_{i=1}^{t-1}(\delta-1)^i = f(\delta, g)$ when $\delta \geq 3$ and $g \geq 5$, and $N(\delta, g) > \frac{4}{9}N(\delta, g)$. As $N(\delta, g) \leq \frac{n}{2}$, it follows that $N(\delta, g)(n - N(\delta, g)) > \frac{4}{9}N(\delta, g)(n - \frac{4}{9}N(\delta, g))$ and $N(\delta, g)(n - N(\delta, g)) > f(\delta, g)(n - f(\delta, g))$, and so

$$\frac{(k-1)n}{N(\delta,g)(n-N(\delta,g))} < \frac{(k-1)n}{\frac{4}{9}N(\delta,g)(n-\frac{4}{9}N(\delta,g))},$$
$$\frac{(k-1)n}{N(\delta,g)(n-N(\delta,g))} < \frac{(k-1)n}{f(\delta,g)(n-f(\delta,g))}.$$

Proof of Theorem 1.9. To the contrary, suppose that $1 \leq \kappa'(G) \leq k-1$. Let F be an arbitrary minimum edge-cut of G, and X, Y be the vertex sets of two components of $G-F$ with $|X| \leq |Y|$. Thus $d(X) = \kappa'(G) \leq k-1$. By Lemma 2.2 and $d(X) < \delta$, we obtain $|X| \geq \varphi(\delta, r) = \max\{\delta+1, \lfloor \frac{r\delta}{r-1} \rfloor\}$. Since $|Y| \geq |X|$ and $|X| + |Y| = n$,

$$|X| \cdot |Y| \geq \varphi(\delta, r)(n - \varphi(\delta, r)). \tag{3.2}$$

By Lemma 2.8 and (3.2), we have

$$\mu_{n-1}(G) \leq \frac{nd(X)}{|X||Y|} \leq \frac{(k-1)n}{\varphi(\delta, r)(n-\varphi(\delta, r))}.$$

According to the hypothesis, it follows that $\mu_{n-1}(G) = \frac{nd(X)}{|X||Y|} = \frac{(k-1)n}{\varphi(\delta,r)(n-\varphi(\delta,r))}$. By the proof of Lemma 2.8, $\mu_{n-1}(G) = \frac{n\sum_{ij\in E}(x_i-x_j)^2}{\sum_{i,j\in V, i<j}(x_i-x_j)^2}$, where $x_i = 1$ if $i \in X$ and $x_i = -1$ if $i \in Y$. By Lemma 2.7, x is a characteristic vector of G corresponding to $\mu_{n-1}(G)$. Since $d(X) < \delta$ and $|X| \geq \varphi(\delta, r) \geq \delta+1$, there exists one vertex j in X such that $N_G(j) \subset X$. Thus, by $\mu_{n-1}(G)x = (D-A)x$, we have $\mu_{n-1}(G)x_j = |N_G(j)|x_j - \sum_{\ell \in N_G(j)} x_\ell$. Since $x_j = x_\ell = 1$, it indicates $\mu_{n-1}(G) = 0$ and so $k-1 = 0$, which is a contradiction to $k \geq 2$. Hence, $\kappa'(G) \geq k$. □

Proof of Theorem 1.10. To the contrary, suppose that $1 \leq \kappa = \kappa(G) \leq k-1$. Let S be an arbitrary minimum vertex-cut and X be the vertex set of a minimum component of $G-S$, and $Y = V \setminus (S \cup X)$. By Lemma 2.4 and $|S| = \kappa \leq k-1 < \delta$, we obtain $|X| \geq N(\delta, g) - |S|$. Thus

$$N(\delta, g) - k + 1 \leq |X| \leq |Y| \leq n - N(\delta, g), \tag{3.3}$$

and so $(|X| - |Y|)^2 \leq (n - 2N(\delta, g) + k - 1)^2$. Therefore,

$$n(n-|S|) - (|X|-|Y|)^2 \geq n(n-k+1) - (n - 2N(\delta, g) + k - 1)^2. \tag{3.4}$$

By $N(\delta, g) \geq \delta + 1 > k$ and (3.3), we have $n - k + 1 > n - 2N(\delta, g) + k - 1 \geq 0$, which implies $n(n-k+1) - (n-2N(\delta, g)+k-1)^2 > 0$. Combining Lemma 2.9 with the inequality (3.4), we have

$$\mu_{n-1}(G) \leq \frac{nd(S)}{n(n-|S|) - (|X|-|Y|)^2} \leq \frac{n(k-1)\Delta}{n(n-k+1) - (n-2N(\delta, g)+k-1)^2},$$

which is a contradiction to the hypothesis. Hence, $\kappa(G) \geq k$. □

Remark 3.2. Theorem 1.10 improves Theorem 1.6 and extends Theorem 1.7 when $g \geq 5$. In fact, if $n < 2N(\delta, g) - \kappa(G)$, then by Corollary 2.6 we have $\kappa(G) = \delta(G)$. Therefore, we only need to compare the bounds when $n \geq 2N(\delta, g) - \kappa(G)$.

(i) Theorem 1.10 improves Theorem 1.6. Denote $N := N(\delta, g)$, $\kappa := \kappa(G)$ and $\nu := \nu(\delta, g, k)$. Then $n \geq 2N - k + 1 > N$ and so $n - k + 1 \geq 2(N - k + 1)$. As $\nu = N - (k-1)\sum_{i=0}^{t-1}(\delta-1)^i \leq N - k + 1$, we get $n > 2(N - k + 1) \geq 2\nu$. Hence,

$$\begin{aligned} n(n-k+1) - (n-2N+k-1)^2 &= (n-k+1)(k-1) + 4(N-k+1)(n-N) \\ &\geq 2(N-k+1)(k-1) + 4(N-k+1)(n-N) \\ &= 2(N-k+1)(n-(N-k+1)+(n-N)) \\ &> 2(N-k+1)(n-(N-k+1)) \geq 2\nu(n-\nu), \end{aligned}$$

and we arrive at $\frac{n(k-1)\Delta}{n(n-k+1)-(n-2N(\delta,g)+k-1)^2} < \frac{n(k-1)\Delta}{2\nu(n-\nu)}$.

(ii) Theorem 1.10 extends Theorem 1.7. Suppose $n \geq 2N(\delta, g) - k + 1$ and $\delta \geq 2$. If $g \geq 3$, then $N(\delta, g) \geq N(\delta, 3) = \delta + 1$ and so

$$\begin{aligned} n(n-k+1) - (n-2N(\delta,g)+k-1)^2 &\geq n(n-k+1) - (n-2(\delta+1)+k-1)^2 \\ &= (n-k+1)(k-1) + 4(\delta-k+2)(n-\delta-1). \end{aligned}$$

If G is triangle-free, then $g \geq 4$ and $N(\delta, g) \geq N(\delta, 4) = 2\delta$, and thus

$$\begin{aligned} n(n-k+1) - (n-2N(\delta,g)+k-1)^2 &\geq n(n-k+1) - (n-4\delta+k-1)^2 \\ &= (n-k+1)(k-1) + 4(2\delta-k+1)(n-2\delta). \end{aligned}$$

Therefore, the lower bound on $\mu_{n-1}(G)$ in Theorem 1.10 is less than or equal to the one in Theorem 1.7, and Theorem 1.10 extends Theorem 1.7 when $g(G) \geq 5$.

Proof of Theorem 1.11. To the contrary, suppose that $1 \leq \kappa = \kappa(G) \leq k - 1$. Let S be an arbitrary minimum vertex-cut and X be the vertex set of a minimum component of $G - S$, and $Y = V \setminus (S \cup X)$. Consider the following two cases.

(i) $\frac{\delta}{r-1} \leq |S| = \kappa < \delta$. By Lemma 2.3 (ii),

$$\frac{r-1}{r-2}(\delta-\kappa) \leq |X| \leq |Y| \leq n - \kappa - \frac{r-1}{r-2}(\delta-\kappa), \tag{3.5}$$

and so $n - \frac{2(r-1)}{r-2}\delta + \frac{r\kappa}{r-2} \geq 0$ and $(|X| - |Y|)^2 \leq (n - \frac{2(r-1)}{r-2}\delta + \frac{r\kappa}{r-2})^2$. Therefore,

$$\begin{aligned} n(n-|S|) - (|X|-|Y|)^2 &\geq n(n-\kappa) - (n - \frac{2(r-1)}{r-2}\delta + \frac{r\kappa}{r-2})^2 \\ &\geq n(n-k+1) - (n - \frac{2(r-1)}{r-2}\delta + \frac{r(k-1)}{r-2})^2. \end{aligned} \tag{3.6}$$

By $\delta > k - 1$ and (3.5), we have $n - k + 1 > n - \frac{2(r-1)}{r-2}\delta + \frac{r(k-1)}{r-2} \geq 0$, which implies $n(n-k+1) - (n - \frac{2(r-1)}{r-2}\delta + \frac{r(k-1)}{r-2})^2 > 0$. Combining (3.6) with $d(S) \leq (k-1)\Delta$, by Lemma 2.9, we have

$$\mu_{n-1}(G) \leq \frac{n(k-1)\Delta}{n(n-k+1)-(n-\frac{2(r-1)}{r-2}\delta+\frac{r(k-1)}{r-2})^2}. \tag{3.7}$$

(ii) $|S| = \kappa < \frac{\delta}{r-1}$. By Lemma 2.3 (iii), we get

$$\frac{r\delta}{r-1} - k + 1 \leq \frac{r\delta}{r-1} - \kappa \leq |X| \leq |Y| \leq n - \frac{r\delta}{r-1}, \tag{3.8}$$

and so $(|X| - |Y|)^2 \leq (n - \frac{2r\delta}{r-2} + k - 1)^2$. Therefore,

$$n(n-|S|) - (|X|-|Y|)^2 \geq n(n-k+1) - (n - \frac{2r\delta}{r-1} + k - 1)^2. \tag{3.9}$$

By $\delta > k - 1$ and (3.8), we have $n - k + 1 > n - \frac{2r\delta}{r-1} + k - 1 \geq 0$, which implies $n(n-k+1) - (n - \frac{2r\delta}{r-1} + k - 1)^2 > 0$. Combining (3.9) with $d(S) \leq (k-1)\Delta$, by Lemma 2.9, we have

$$\mu_{n-1}(G) \leq \frac{n(k-1)\Delta}{n(n-k+1)-(n-\frac{2r\delta}{r-1}+k-1)^2}. \tag{3.10}$$

Now, let $\phi(\delta, k, r) = \max\{(n - \frac{2(r-1)}{r-2}\delta + \frac{r(k-1)}{r-2})^2, (n - \frac{2r\delta}{r-1} + k - 1)^2\}$. By (3.7) and (3.10), we have

$$\mu_{n-1}(G) \leq \frac{n(k-1)\Delta}{n(n-k+1)-\phi(\delta,k,r)},$$

which is a contradiction to the hypothesis. Hence, $\kappa(G) \geq k$. □

Proof of Theorem 1.12. If $r = 2$, then $g(G) \geq 4$ and so $N(\delta, g) \geq N(\delta, 4) = 2\delta = \frac{r}{r-1}\delta$. Thus, by Theorem 1.10, the theorem holds when $r = 2$. Next we consider $r \geq 3$. To the contrary, suppose that $1 \leq \kappa = \kappa(G) \leq k-1$. Let S be an arbitrary minimum vertex-cut and X be the vertex set of a minimum component of $G - S$, and $Y = V \setminus (S \cup X)$. By Lemma 2.3 and $|S| = \kappa \leq k - 1 < \frac{\delta}{r-1}$, we obtain $|X| \geq \frac{r\delta}{r-1} - \kappa$. Thus,

$$\frac{r\delta}{r-1} - k + 1 \leq \frac{r\delta}{r-1} - \kappa \leq |X| \leq |Y| \leq n - \frac{r\delta}{r-1}.$$

Using a similar argument as in the proof of Theorem 1.11, we have

$$\mu_{n-1}(G) \leq \frac{n(k-1)\Delta}{n(n-k+1)-(n-\frac{2r\delta}{r-1}+k-1)^2},$$

which is a contradiction to the hypothesis. Hence, $\kappa(G) \geq k$ and the result follows. □

Remark 3.3. If $\omega(G) \geq 3$, then $g(G) = 3$. In this case, since $\frac{r\delta}{r-1} \geq \delta+1$ for $2 \leq r \leq \delta+1$ and $\frac{r\delta}{r-1} > \delta+1$ for $3 \leq r \leq \delta$, it follows that Theorem 1.9 extends Theorem 1.8, and Theorems 1.11 and 1.12 extends Theorem 1.10 when $g(G) = 3$.

Proof of Theorem 1.13. To the contrary, suppose that $1 \leq \kappa = \kappa(G) \leq k-1$. Let S be an arbitrary minimum vertex-cut and X be the vertex set of a minimum component of $G - S$, and $Y = V \setminus (S \cup X)$. By Lemma 2.4 and $1 \leq \kappa \leq k-1 < \delta$, we obtain

$$N(\delta, g) - \kappa \leq |X| \leq |Y| \leq n - N(\delta, g),$$

and so

$$|X| \cdot |Y| \geq (N(\delta, g) - \kappa)(n - N(\delta, g)) \geq (N(\delta, g) - k + 1)(n - N(\delta, g)).$$

Combining this with $n - |X| - |Y| = \kappa \leq k-1$, by Lemma 2.11,

$$\frac{(\mu_1(G) - \mu_{n-1}(G))^2}{4\mu_1(G)\mu_{n-1}(G)} \geq \frac{|X||Y|}{n(n - |X| - |Y|)} \geq \frac{(N(\delta, g) - k + 1)(n - N(\delta, g))}{n(k-1)}. \tag{3.11}$$

Set $t = \frac{\mu_1(G)}{\mu_{n-1}(G)}$ and $s = \frac{2(N(\delta,g)-k+1)(n-N(\delta,g))}{n(k-1)} + 1$. Substituting t and s in (3.11), we obtain $t + t^{-1} \geq 2s$. Since $t \geq 1$ and $s \geq 1$, $t \geq s + \sqrt{s^2 - 1}$ is necessary. This contradicts to the hypothesis. Therefore, $\kappa(G) \geq k$. □

Proof of Theorem 1.14. To the contrary, suppose that $1 \leq \kappa = \kappa(G) \leq k-1$. Let S be an arbitrary minimum vertex-cut and X be the vertex set of a minimum component of $G - S$, and $Y = V \setminus (S \cup X)$. By Lemma 2.3 (iii) and $|S| \leq k-1 < \frac{\delta}{r-1}$, we obtain $|X| \geq \frac{r\delta}{r-1} - |S|$. Thus,

$$\frac{r\delta}{r-1} - \kappa \leq |X| \leq |Y| \leq n - \frac{r\delta}{r-1},$$

and so

$$|X| \cdot |Y| \geq (\frac{r\delta}{r-1} - \kappa)(n - \frac{r\delta}{r-1}) \geq (\frac{r\delta}{r-1} - k + 1)(n - \frac{r\delta}{r-1}).$$

Combining this with $n - |X| - |Y| = \kappa \leq k-1$, by Lemma 2.11,

$$\frac{(\mu_1(G) - \mu_{n-1}(G))^2}{4\mu_1(G)\mu_{n-1}(G)} \geq \frac{|X||Y|}{n(n - |X| - |Y|)} \geq \frac{(\frac{r\delta}{r-1} - k + 1)(n - \frac{r\delta}{r-1})}{n(k-1)}. \tag{3.12}$$

Set $t = \frac{\mu_1(G)}{\mu_{n-1}(G)}$ and $s = \frac{2(\frac{r\delta}{r-1}-k+1)(n-\frac{r\delta}{r-1})}{n(k-1)} + 1$. Substituting t and s in (3.12), we obtain $t + t^{-1} \geq 2s$. Since $t \geq 1$ and $s \geq 1$, $t \geq s + \sqrt{s^2 - 1}$ is necessary. This contradicts to the hypothesis. Therefore, $\kappa(G) \geq k$. □

4. Connectivity and adjacency or signless Laplacian eigenvalues

In this section, we present the relationship between (edge-)connectivity and the second largest adjacency eigenvalue or the second largest signless Laplacian eigenvalue.

Theorem 4.1. *(Weyl, Theorem 2.6.1 in [4]) Let A and B be Hermitian matrices of order n, and let $1 \leq i, j \leq n$. If $i + j \leq n + 1$, then $\lambda_i(A) + \lambda_j(B) \geq \lambda_{i+j-1}(A + B)$.*

For real numbers a, b with $b > 0$ and $a \geq -b$, let $\lambda_i(G, a, b)$ be the ith largest eigenvalue of the matrix $aD + bA$.

Corollary 4.2. *Let $p \geq 0$, $b > 0$ and $a \geq -b$ be real numbers and G be a graph of order n with minimum degree δ.*
(i) If $\lambda_2(G, a, b) < (a + b)\delta - bp$, then $\mu_{n-1}(G) > p$. In particular, if $q_2(G) < 2\delta - p$ or $\lambda_2(G) < \delta - p$, then $\mu_{n-1}(G) > p$.
(ii) If $\lambda_2(G, a, b) \leq (a + b)\delta - bp$, then $\mu_{n-1}(G) \geq p$. In particular, if $q_2(G) \leq 2\delta - p$ or $\lambda_2(G) \leq \delta - p$, then $\mu_{n-1}(G) \geq p$.

Proof. Let A and D be the adjacency matrix and degree diagonal matrix of G, respectively. Since $b(D-A)+(aD+bA) = (a+b)D$, by Theorem 4.1, $\lambda_{n-1}(b(D-A))+\lambda_2(aD+bA) \geq \lambda_n((a+b)D)$. As $b > 0$ and $a+b \geq 0$, $b\mu_{n-1}(G)+\lambda_2(G, a, b) \geq (a+b)\delta$. Therefore, if $\lambda_2(G, a, b) < (a + b)\delta - bp$, then $\mu_{n-1}(G) > p$. In particular, $\lambda_2(G, 1, 1) = q_2(G)$ and $\lambda_2(G, 0, 1) = \lambda_2(G)$. Thus, (i) is proved and (ii) can be proved similarly. $\square$

By Corollary 4.2, from the sufficient conditions on $\mu_{n-1}(G)$ in Theorems 1.8-1.12, we can obtain sufficient conditions on $\lambda_2(G, a, b)$, especially on $\lambda_2(G)$ and $q_2(G)$. For example, by Corollary 4.2 and Theorem 1.8, we have the following corollary.

Corollary 4.3. *Let k be an integer and G be a connected graph of order n with minimum degree $\delta \geq k \geq 2$ and girth $g \geq 3$. If $\lambda_2(G) \leq \delta - \frac{(k-1)n}{N(\delta,g)(n-N(\delta,g))}$ or $q_2(G) \leq 2\delta - \frac{(k-1)n}{N(\delta,g)(n-N(\delta,g))}$, then $\kappa'(G) \geq k$.*

5. Appendix

Example 5.1. Let G be the 4-regular graph in Fig. 1, where $n = |V(G)| = 12, \Delta(G) = \delta(G) = \kappa(G) = 4$, $g(G) = 3$, $N(\delta, g) = 5$ and $\mu_{n-1}(G) = 2$. The following table illustrates the lower bounds on $\mu_{n-1}(G)$ of Theorem 1.6 and Theorem 1.10 and the upper bound on $\frac{\mu_1(G)}{\mu_{n-1}(G)}$ of Theorem 1.13 for $k = 4$.

Theorem 1.6		$\mu_{n-1}(G)$		Theorem 1.10	$\frac{\mu_1(G)}{\mu_{n-1}(G)}$		Theorem 1.13
3.6	>	2	>	1.7349	3	<	3.2476

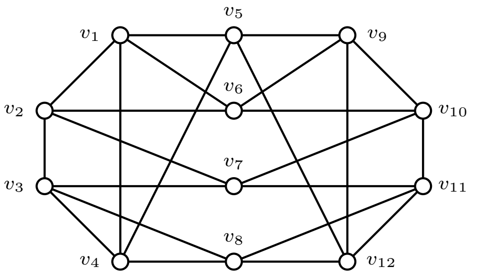

Fig. 1. The graph G in Example 5.1.

Example 5.2. Let G be the bipartite graph in Fig. 2, where $n = |V(G)| = 14$, $\Delta(G) = 5$, $\delta(G) = \kappa(G) = 4$, $g(G) = 4$, $N(\delta, g) = 8$ and $\mu_{n-1}(G) = 2$. The following table illustrates the lower bounds on $\mu_{n-1}(G)$ of Theorem 1.6 and Theorem 1.10 and the upper bound on $\frac{\mu_1(G)}{\mu_{n-1}(G)}$ of Theorem 1.13 for $k = 4$.

Theorem 1.6		$\mu_{n-1}(G)$		Theorem 1.10	$\frac{\mu_1(G)}{\mu_{n-1}(G)}$		Theorem 1.13
2.3333	>	2	>	1.3725	4.3508	<	4.6417

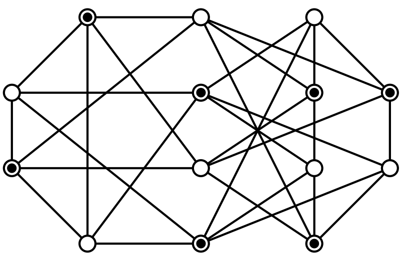

Fig. 2. The graph G in Example 5.2.

Example 5.3. Let G be the bipartite graph in Fig. 3 obtained from the union of two $K_{a,2a}$ and two $K_{a,a}$, where $n = |V(G)| = 6a$ $(a \geq 2)$, $g(G) = 4$, $\Delta(G) = \delta(G) = \kappa(G) = 2a$ and $N(\delta, g) = 2\delta = 4a$. The following table illustrates the lower bounds on $\mu_{n-1}(G)$ of Theorem 1.6 and Theorem 1.10 for $k = 2a$. Since the characteristic polynomial of the Laplacian matrix of G is

$$\lambda(\lambda - a)^2(\lambda - 2a)^{6(a-1)}(\lambda - 3a)^2(\lambda - 4a),$$

we have $\mu_{n-1}(G) = a$.

Theorem 1.6		$\mu_{n-1}(G)$		Theorem 1.10	$\kappa(G)$
$\frac{6a^2(2a-1)}{(2a+1)(4a-1)}$	$>$	a	$>$	$\frac{12a^2(2a-1)}{6a(4a+1)-1}$	$2a$

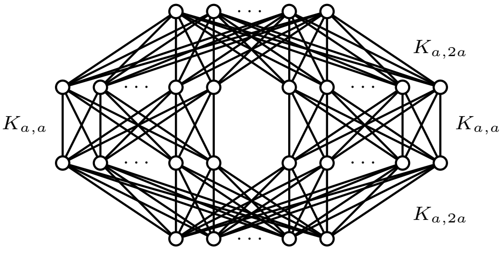

Fig. 3. The graph G in Example 5.3.

Example 5.4. Let G be the 3-regular graph in Fig. 4, where $n = |V(G)| = 20$, $\Delta(G) = \delta(G) = \kappa(G) = 3$, $g(G) = 5$, $N(\delta, g) = 10$ and $\mu_{n-1}(G) = 0.2215$. The following table shows the lower bounds on $\mu_{n-1}(G)$ of Theorem 1.6 and Theorem 1.10 for $k = 2$.

Theorem 1.6		$\mu_{n-1}(G)$		Theorem 1.10	$\kappa(G)$
0.3297	$>$	0.2215	$>$	0.1583	2

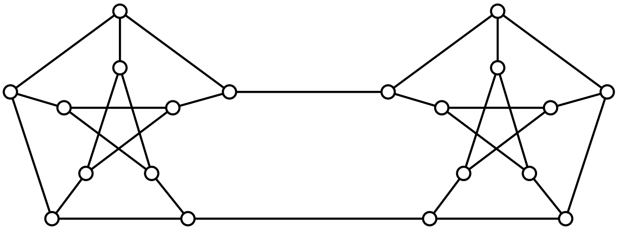

Fig. 4. The graph G in Example 5.4.

Example 5.5. Let G be the graph in Fig. 5, where $n = |V(G)| = 20$, $\Delta(G) = 4$, $\delta(G) = \kappa(G) = 3$, $g(G) = 5$, $N(\delta, g) = 10$, $f(\delta, g) = 8$ and $\mu_{n-1}(G) = 0.4158$. The following table shows the lower bounds on $\mu_{n-1}(G)$ of Theorem 1.5 and Theorem 1.8 for $k = 3$.

Theorem 1.5		$\mu_{n-1}(G)$		Theorem 1.8	$\kappa'(G)$
0.4167	$>$	0.4158	$>$	0.4	3

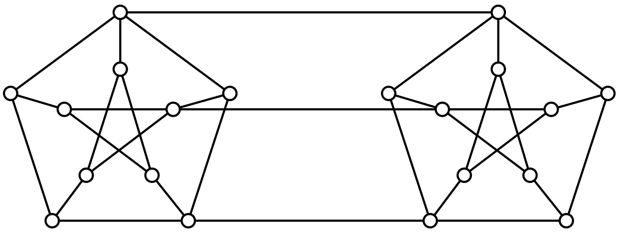

Fig. 5. The graph G in Example 5.5.

Example 5.6. Let G be the graph in Fig. 6 obtained from the disjoint union of two copies of $K_{5,5,5}$ by adding a matching, where $n = |V(G)| = 30$, $\Delta(G) = 11$, $\delta(G) = \kappa'(G) = 10$, $g(G) = \omega(G) = 3$, $N(\delta, g) = 11$, $\varphi(\delta, 3) = 15$ and $\mu_{n-1}(G) = 1.27158$. The following table shows the lower bounds on $\mu_{n-1}(G)$ of Theorem 1.8 and Theorem 1.9 for $k = 10$.

Theorem 1.8		$\mu_{n-1}(G)$		Theorem 1.9	$\kappa'(G)$
1.2919	>	1.27158	>	1.2	10

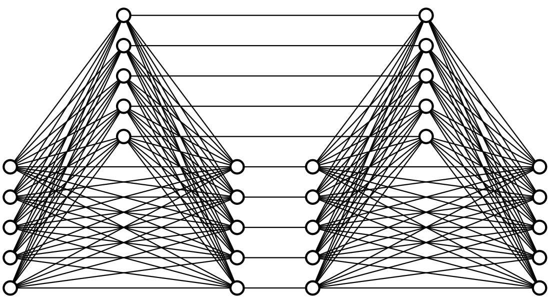

Fig. 6. The graph G in Example 5.6.

Example 5.7. Let $K_{6,6,6}$ be the complete 3-partite graph with partition $U = U_1 \cup U_2 \cup U_3$, where $U_i = \{u_{6i-5}, u_{6i-4}, u_{6i-3}, u_{6i-2}, u_{6i-1}, u_{6i}\}$ for $i = 1, 2, 3$, and $K_{5,5,5}$ be the complete 3-partite graph with partition $V = V_1 \cup V_2 \cup V_3$, where $V_i = \{v_{5i-4}, v_{5i-3}, v_{5i-2}, v_{5i-1}, v_{5i}\}$ for $i = 1, 2, 3$. Let G be the graph in Fig. 7 obtained from the disjoint union of $K_{6,6,6}$ and $K_{5,5,5}$ by adding the edges u_6v_{j+5}, $u_{12}v_j$ and $u_{18}v_{j+5}$ for each $j = 1, 2, 3, 4, 5$, where $n = |V(G)| = 33$, $\Delta(G) = 17$, $\delta(G) = 10$, $\kappa(G) = 3$, $g(G) = \omega(G) = 3$, $N(\delta, g) = 11$, $\frac{2r\delta}{r-1} = 30$ and $\mu_{n-1}(G) = 1.27637$. The following table gives the lower bounds on $\mu_{n-1}(G)$ of Theorem 1.10 and Theorem 1.12 for $k = 3$.

Theorem 1.10		$\mu_{n-1}(G)$		Theorem 1.12	$\kappa(G)$
1.3138	>	1.27637	>	1.1242	3

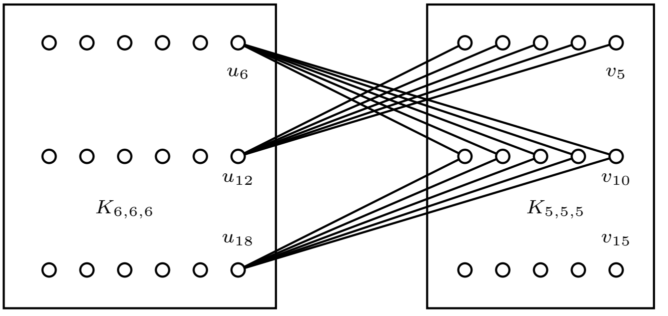

Fig. 7. The graph G in Example 5.7.

Declaration of competing interest

There is no competing interests.

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