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On weighted modulo orientation of graphs

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ABSTRACT

Esperet, de Joannis de Verclos, Le and Thomassé in [SIAM J. Discrete Math., 32(1) (2018), 534–542] introduced the problem that for an odd prime p, whether there exists an orientation D of a graph G for any mapping $f : E(G) \to \mathbb{Z}_p^*$ and any $\mathbb{Z}_p$-boundary b of G, such that under D, at every vertex, the net out f-flow is the same as $b(v)$ in $\mathbb{Z}_p$. Such an orientation D is called an $(f, b; p)$-orientation of G. Esperet et al. indicated that this problem is closely related to mod p-orientations of graphs, including Tutte's nowhere zero 3-flow conjecture. Utilizing properties of additive bases and contractible configurations, we show that every graph G with Euler genus g and edge-connectivity $\kappa'(G)$ admits an $(f, b; p)$-orientation for any mapping $f : E(G) \to \mathbb{Z}_p^*$ and any $\mathbb{Z}_p$-boundary b of G, provided

$$\kappa'(G) \geq \begin{cases} 4p - 6 + \lfloor g/2 \rfloor & \text{if } g \leq 2, \\ (p-2)\lfloor \sqrt{6g + 0.25} + 2.5 \rfloor \\ \quad +1 & \text{if } g \geq 3, \\ p\sqrt{4.98g} & \text{if } g \text{ is sufficiently large.} \end{cases}$$

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1. The problem

We consider finite loopless graphs with possible multiple edges, and follow [3] for generic undefined notation and terms, and [10] for those involving graphs embedded on surfaces. In particular, for a graph G, $\kappa(G)$, $\kappa'(G)$ and $\delta(G)$ denote the connectivity, edge-connectivity and the minimum degree of G, respectively. We write $H \subseteq G$ to mean that H is a subgraph of G. As in [3], (u, v) in a digraph D denotes an arc oriented from u to v, and for a vertex $v \in V$, let

$$E_D^-(v) = \{(u, v) \in D(G) : u \in V(D)\}, \text{ and } E_D^+(v) = \{(v, u) \in D(G) : u \in V(D)\}.$$

The subscript D may be omitted when D is understood from the context. For an integer $k > 0$, let $\mathbb{Z}_k$ denote the (additive) cyclic group of order k. A $\mathbb{Z}_k$**-boundary** of a graph G is a mapping $b : V(G) \to \mathbb{Z}_k$ satisfying $\sum_{v\in V(G)} b(v) \equiv 0 \pmod{k}$. Let $A \subseteq \mathbb{Z}_k$, and define $F(G, A) = \{f : E(G) \to A\}$, and let $\mathbb{Z}_k^* = \mathbb{Z}_k - \{0\}$. Fix an orientation $D = D(G)$ for a graph G. For any $f \in F(G, \mathbb{Z}_k^*)$, define $\partial_D(f) : V(G) \to \mathbb{Z}_k$ as

$$\partial_D(f)(v) = \sum_{e\in E_D^+(v)} f(e) - \sum_{e\in E_D^-(v)} f(e).$$

When the orientation D is understood from the context, we often omit the subscript D in the notation above and write ∂f for $\partial_D(f)$. It is known that for any $f \in F(G, \mathbb{Z}_k^*)$, ∂f is always a $\mathbb{Z}_k$-boundary. Jaeger et al. [13] defined group connectivity of a graph. A graph G is $\mathbb{Z}_k$-connected if for any $\mathbb{Z}_k$-boundary b of G, there exists an orientation D of G and a mapping $f \in F(G, \mathbb{Z}_k^*)$ such that $\partial f \equiv b \pmod{k}$. The following conjecture is proposed in [13] and remains unsolved as of today.

Conjecture 1.1. *Let G be a graph.*
(i) If $\kappa'(G) \geq 3$, then G is $\mathbb{Z}_5$-connected.
(ii) If $\kappa'(G) \geq 5$, then G is $\mathbb{Z}_3$-connected.

Let b be a $\mathbb{Z}_k$-boundary of a graph G. An orientation D of G is a b**-orientation** of G if for the constant mapping $f = 1$, we have $\partial f \equiv b \pmod{k}$. In particular, when $b = 0$, any b-orientation of G is a **mod** k**-orientation** of G. The studies of group connectivity and modulo orientation of graphs are motivated by the most fascinating nowhere zero flow conjectures of Tutte, as shown in the surveys [12,17] as well as in the popular monograph [23], among others. Some of the recent breakthroughs are the following.

Theorem 1.2 (*Lovász, Thomassen, Wu and Zhang [21]*)**.** *Let $k > 0$ be an integer. Every $6k$-edge-connected graph G has a b-orientation for every $\mathbb{Z}_{2k+1}$-boundary b of G.*

Theorem 1.3 (*Han, Li, Wu and Zhang [11], Li [19]*)**.** *Let $k > 0$ be an integer.*
(i) If $k \geq 3$, then there exists a $4k$-edge-connected graph admitting no mod $(2k + 1)$-orientation.
(i) If $k \geq 5$, then there exists a $(4k + 1)$-edge-connected graph admitting no mod $(2k + 1)$-orientation.

In particular, Theorem 1.3 disproved Jaeger's Circular Flow Conjecture, in which Jaeger [12] conjectured that every $4k$-edge-connected graph admits a mod $(2k+1)$-orientation. Further expository of the problem can be found in the informative monograph by Zhang [23]. Aiming at extending Theorem 1.2, Esperet et al. in [9] defined a mod $k\,f$**-weighted** b**-orientation** of a graph G, for a given mapping $f \in F(G, \mathbb{Z}_k^*)$ and a $\mathbb{Z}_k$-boundary b, to be an orientation $D = D(G)$ satisfying $\partial_D(f) \equiv b \pmod{k}$ under D. Throughout the rest of this paper, we shall abbreviate a mod $k\,f$-weighted b-orientation as an $(f, b; k)$**-orientation**. Esperet et al. indicated in [9] that to investigate $(f, b; k)$-orientations of graphs, it is necessary to assume that k is an odd prime number. The following is proved in [9].

Theorem 1.4 (*Esperet, de Joannis de Verclos, Le and Thomassé, [9]*)**.** *Let $p \geq 3$ be a prime number and G be a $(6p^2 - 14p + 8)$-edge-connected graph. Then for any mapping $f \in F(G, \mathbb{Z}_p^*)$ and any $\mathbb{Z}_p$-boundary b of G, G has an $(f, b; p)$-orientation.*

The current study is motivated by Theorems 1.2–1.4. We are going to investigate the relationship between the edge-connectivity of a graph embedded on a 2-manifold and its $(f, b; p)$-orientability over the finite field $\mathbb{Z}_p$. We follow [10] to define a 2**-cell (or cellular) embedding** of a graph G into a closed surface S to be a continuous one-to-one function $i : G \to S$ if every component of $S - i(G)$ is homeomorphic to an open disk. In this paper, all embeddings of graphs are assumed to be 2-cell. We use g to denote the Euler genus of G, which is the minimum integer k such that the graph can be embedded into an orientable surface of genus $k/2$ or into a nonorientable surface of genus k. Our main result is the following.

Theorem 1.5. *Let $p > 0$ be an odd prime, and let G be a graph with Euler genus g and edge connectivity*

$$\kappa'(G) \geq \begin{cases} 4p - 6 + \lfloor g/2 \rfloor & \text{if } g \leq 2, \\ (p-2)\lfloor \sqrt{6g + 0.25} + 2.5 \rfloor + 1 & \text{if } g \geq 3, \\ p\sqrt{4.98g} & \text{if } g \text{ is sufficiently large.} \end{cases} \quad (1)$$

Then for any mapping $f \in F(G, \mathbb{Z}_p^*)$ and any $\mathbb{Z}_p$-boundary b of G, the graph G has an $(f, b; p)$-orientation.

The next section will be focused on developing the needed mechanisms to derive our main result, utilizing additive bases in the linear space of the boundaries of a given graph, and contractible configurations of the related properties. The proof of the main result will be in the last section.

2. Preliminaries

Throughout this section, $\mathbb{F}$, n and p denote a field, a positive integer and an odd prime, respectively. We use $\mathbb{F}^n$ to denote the n-dimensional vector space over $\mathbb{F}$. For a graph G on $n > 0$ vertices, let $Z(G, \mathbb{Z}_k)$ denote the collection of all $\mathbb{Z}_k$-boundaries of G. By definition, $Z(G, \mathbb{Z}_p)$ is isomorphic to $\mathbb{Z}_p^{n-1}$.

2.1. Additive bases of $Z(G, \mathbb{Z}_p)$

Given a subset $S \subseteq \mathbb{Z}_p$, an S**-additive basis** of $\mathbb{Z}_p^n$ is a multiset $\{x_1, x_2, \ldots, x_m\} \subseteq \mathbb{Z}_p^n$ such that for any $x \in \mathbb{Z}_p^n$, there exist scalars $c_i \in S$ such that $x = \sum_{i=1}^m c_i x_i$, which is called an S-linear-combination of x. An **additive basis** is a $\{0, 1\}$-additive basis. As indicated in [13], the mod p-orientation problem of graphs is closely related to the existence of additive bases of vector spaces over $\mathbb{Z}_p$, the field on p elements.

Let $B_1, \ldots, B_t$ be a collection of bases of $\mathbb{F}^n$. Define $\uplus_{i=1}^t B_i$ to be the (multiset) union with repetitions of $B_1, \ldots, B_t$. Let $c(n, \mathbb{F})$ be the smallest positive integer t such that for any t bases $B_1, \ldots, B_t$ of $\mathbb{F}^n$, the multiset $\uplus_{i=1}^t B_i$ is an additive basis of $\mathbb{F}^n$. Define $c(n, p) = c(n, \mathbb{Z}_p)$. An upper bound of $c(c, p)$ was obtained by Alon, Linial and Meshulam [1]. In the following, Theorem 2.1(i) can be derived from Cauchy–Davenport Theorem in [7] (see Theorem 2.4), and Theorem 2.1(ii) verified a former conjecture by H. B. Mann and J. E. Olson.

Theorem 2.1. *Each of the following holds.*
(i) (Davenport [7], see also [2]) If $p \geq 3$ is a prime, then $c(1, p) = p - 1$.
(ii) (Mann and Wou [22]) If $p \geq 3$ is a prime, then $c(2, p) = p - 1$.

We develop some more lemmas for our arguments deployed in this research.

Lemma 2.2. *Let $x, y \in \mathbb{F}$ distinct elements. Then each of the following holds.*
(i) If $A = \{a_1, \ldots, a_m\}$ is an $\{x, y\}$-additive basis of $\mathbb{F}^n$, then $(y - x)A = \{(y - x)a_1, \ldots, (y - x)a_m\}$ is an additive basis of $\mathbb{F}^n$.
(ii) If $A = \{a_1, \ldots, a_m\}$ is an additive basis of $\mathbb{F}^n$, then $(y - x)^{-1}A = \{(y - x)^{-1}a_1, \ldots, (y - x)^{-1}a_m\}$ is an $\{x, y\}$-additive basis of $\mathbb{F}^n$.

Proof. Let β be an arbitrary vector in $\mathbb{F}^n$.
(i) Then $\beta + \sum_{i=1}^{m} xa_i \in \mathbb{F}^n$. As $\{a_1, \dots, a_m\}$ is an $\{x, y\}$-additive basis of $\mathbb{F}^n$, there exist scalars $c_1, \dots, c_m \in \{x, y\}$ such that $\beta + \sum_{i=1}^{m} xa_i = \sum_{i=1}^{m} c_i a_i$. For each $i \in \{1, 2, \dots, m\}$, let $d_i = (y-x)^{-1}(c_i - x)$. Thus if $c_i = x$ then $d_i = 0$, and if $c_i = y$ then $d_i = 1$. It follows that $\beta = (y-x)(y-x)^{-1}\sum_{i=1}^{m}(c_i - x)a_i = \sum_{i=1}^{m} d_i(y-x)a_i$ with $d_i \in \{0, 1\}$, and so $(y-x)A$ is an additive basis of $\mathbb{F}^n$.
(ii) Then $\beta - (y-x)^{-1}\sum_{i=1}^{m} xa_i \in \mathbb{F}^n$. Since $\{a_1, \dots, a_m\}$ is an additive basis of $\mathbb{F}^n$, there exist $c_1, \dots, c_m \in \{0, 1\}$ such that $\beta - (y-x)^{-1}\sum_{i=1}^{m} xa_i = \sum_{i=1}^{m} c_i a_i$. For each $i \in \{1, 2, \dots, m\}$, let $d_i = (y-x)c_i + x$. As $c_i \in \{0, 1\}$, we have $d_i \in \{x, y\}$. It follows that $\beta = \sum_{i=1}^{m}((y-x)c_i + x)(y-x)^{-1}a_i = \sum_{i=1}^{m} d_i(y-x)^{-1}a_i$, and so $(y-x)^{-1}A$ is a $\{x, y\}$-additive basis of $\mathbb{F}^n$. ■

Let G be a connected graph with $n = |V(G)| \geq 1$. For each $e \in E(G)$, define $x_e \in F(G, \mathbb{Z}_p)$ to be the characteristic function of $\{e\}$. Let D be an arbitrary orientation of G. Recall that $Z(G, \mathbb{Z}_p)$ is isomorphic to $\mathbb{Z}_p^{n-1}$. Corollary 2.3 reveals a relationship between additive bases in $Z(G, \mathbb{Z}_p)$ and the existence of an $(f, b; p)$-orientation of G.

Corollary 2.3. *Let $p \geq 3$ be a prime number, and let G be a connected graph with $n = |V(G)|$. The following statements are equivalent.*
(i) For any mapping $f \in F(G, \mathbb{Z}_p^*)$ and any $\mathbb{Z}_p$-boundary b of G, G has an $(f, b; p)$-orientation.
(ii) For any given orientation D_1 of G and for any mapping $f \in F(G, \mathbb{Z}_p^*)$, the multiset $\{f(e)\partial_{D_1}(x_e) : e \in E(G)\}$ is a $\{-1, 1\}$-additive basis of $Z(G, \mathbb{Z}_p)$.
(iii) For any given orientation D_2 of G and for any mapping $f \in F(G, \mathbb{Z}_p^*)$, the multiset $\{2f(e)\partial_{D_2}(x_e) : e \in E(G)\}$ is an additive basis of $Z(G, \mathbb{Z}_p)$.

Proof. The equivalence between (ii) and (iii) is an immediate consequence of Lemma 2.2 by letting $D_1 = D_2$.

It remains to show that equivalence between (i) and (ii). Assume that (i) holds. For any mapping $f \in F(G, \mathbb{Z}_p^*)$ and any $b \in Z(G, \mathbb{Z}_p)$, by (i), G admits an $(f, b; p)$-orientation D. For each $e \in E(G)$, define $c_e = 1$ if e has the same orientation in both D and D_1 and $c_e = -1$ if e is oriented differently in D and in D_1. By definition, we have $\partial_D(f) = b$, and so for each $v \in V(G)$,

$$b(v) = \partial_D(f)(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e) = \sum_{e \in E} c_e f(e)\partial_{D_1}(x_e)(v).$$

Thus b is a $\{1, -1\}$-linear-combination of vectors in $\{f(e)\partial_{D_1}(x_e) : e \in E(G)\}$. By definition, the multiset $\{f(e)\partial_{D_1}(x_e) : e \in E(G)\}$ is a $\{-1, 1\}$-additive basis of $Z(G, \mathbb{Z}_p)$.

Conversely, we assume that the multiset $\{f(e)\partial_{D_1}(x_e) : e \in E(G)\}$ is a $\{-1, 1\}$-additive basis of $Z(G, \mathbb{Z}_p)$. For any $b \in Z(G, \mathbb{Z}_p)$, there exists scalars $c_e \in \{1, -1\}$ such that $b = \sum_{e \in E(G)} c_e f(e)\partial_{D_1}(x_e)$. Let D be an orientation obtained from D_1 such that for any edge $e \in E(G)$, e has the same orientation in D as in D_1 if $c_e = 1$ and e has an orientation in D opposite to its orientation in D_1 if $c_e = -1$. It follows from $b = \sum_{e \in E(G)} c_e f(e)\partial_{D_1}(x_e)$ that $b = \partial_D(f)$, and so D is an $(f, b; p)$-orientation of G. ■

For a multisubset $\{x_1, \dots, x_k\}$ of $\mathbb{Z}_p^*$, define $\Omega(x_1, \dots, x_k) = \{\sum_{i=1}^{k} \ell_i x_i : \ell_i \in \{1, -1\}\}$ to be the set of $\{1, -1\}$-linear combinations of $\{x_1, \dots, x_k\}$. By definition and since $p \geq 3$ is an odd prime,

$$\Omega(x_1, \dots, x_k) = -\Omega(x_1, \dots, x_k), \text{ and so } |\Omega(x_1, \dots, x_k)| \text{ is odd if and only if } 0 \in \Omega(x_1, \dots, x_k). \tag{2}$$

For two nonempty subsets $A, B \in \mathbb{Z}_p$, let $A + B = \{a + b : a \in A, b \in B\}$. The following result was proved by Cauchy [6] in 1813 and was later rediscovered by Davenport [7] in 1935.

Theorem 2.4 (*Cauchy [6] and Davenport [7]*). *Let p be a prime number, and A and B two nonempty subsets of $\mathbb{Z}_p$. Then $|A + B| \geq \min\{p, |A| + |B| - 1\}$.*

Lemma 2.5. *Let p be an odd prime and let k be a positive integer with $1 \leq k < p$. If $\{x_1, \dots, x_k\}$ is a multisubset of $\mathbb{Z}_p^*$, then $|\Omega(x_1, \dots, x_k)| \geq k + 1$.*

Proof. We proceed by induction on k. If $k = 1$, then $\Omega(x_1) = \{x_1, -x_1\}$, and the lemma holds. Let $A = \Omega(x_1, \dots, x_{k-1})$. Then by induction, $|\Omega(x_1, \dots, x_{k-1})| \geq k$. Let $B = \{x_k, -x_k\}$. Note that $\Omega(x_1, \dots, x_k) = A + B$. By Theorem 2.4, $|A+B| \geq \min\{p, |A|+|B|-1\} = \min\{p, k+1\} = k+1$, and so $|\Omega(x_1, \dots, x_k) \geq k+1$. ■

2.2. A family of graphs admitting $(f, b; p)$-orientations

For a graph G and for each edge $uv \in E(G)$, let $[uv]$ denote the set of all (parallel) edges joining the two vertices u and v. If $X \subseteq E(G)$ is an edge subset of a graph G, then the contraction G/X is obtained from G by identifying the two ends of each edge in X and then deleting all the resulting loops. If $X = \{e\}$, we use G/e for $G/\{e\}$. If H is a connected subgraph of G, then we write G/H for $G/E(H)$.

For a prime $p \geq 3$, let $\mathcal{O}_p$ denote the family of connected graphs such that a graph $G \in \mathcal{O}_p$ if and only if G admits an $(f, b; p)$-orientation for any $f \in F(G, \mathbb{Z}_p^*)$ and any $\mathbb{Z}_p$-boundary b. By definition, $K_1 \in \mathcal{O}_p$. For a subgraph H of a graph G, let $A_G(H)$ denote the vertices in $V(H)$ that are adjacent to some vertices in $V(G) - V(H)$ in G. (Vertices in $A_G(H)$ are called the **vertices of attachment of H in** G.) We have the following proposition.

Proposition 2.6. *Let G be a connected graph. Then each of the following holds.*
(i) If $G \in \mathcal{O}_p$ and $e \in E(G)$, then $G/e \in \mathcal{O}_p$.
(ii) If $H \subseteq G$ satisfying $H \in \mathcal{O}_p$ and $G/H \in \mathcal{O}_p$, then $G \in \mathcal{O}_p$.

Proof. (i) Let $e = \{u, v\}$, $G' = G/e$ and w be the vertex in G' onto which e is contracted. Let $f' : E(G') \to \mathbb{Z}_p^*$ and b' be an arbitrary $\mathbb{Z}_p$-boundary of G'. Define mappings f and b as follows:

$$f(h) = \begin{cases} f'(h) & \text{if } h \in E(G') = E(G) - \{e\} \\ 1 & \text{if } h = e. \end{cases} \quad \text{and } b(z) = \begin{cases} b'(z) & \text{if } z \in V(G) - \{u, v\} \\ b'(w) & \text{if } z = u \\ 0 & \text{if } z = v. \end{cases} \tag{3}$$

Thus $f : E(G') \to \mathbb{Z}_p^*$. As $\sum_{z \in V(G)} b(z) = \sum_{z \in V(G')} b'(z) \equiv 0 \pmod p$, b is a $\mathbb{Z}_p$-boundary of G. Since $G \in \mathcal{O}_p$, G admits an $(f, b; p)$-orientation D. Let D' be the restriction of D to $E(G) - \{e\}$. Then D' can be viewed as an orientation of G'. Since

$$\begin{aligned} \partial_{D'} f'(w) &= \sum_{e' \in E_D^+(v) \cup E_D^+(u) - \{e\}} f(e') - \sum_{e' \in E_D^-(v) \cup E_D^-(u) - \{e\}} f(e') \\ &= \partial_D f(u) + \partial_D f(v) = b(u) + b(v) = b'(w), \end{aligned} \tag{4}$$

it follows that $\partial_{D'} f' = b'$, and so D' is an $(f', b'; p)$-orientation of G'. By definition, $G/e \in \mathcal{O}_p$.
(ii) Suppose $H \in \mathcal{O}_p$ and $G/H \in \mathcal{O}_p$. By the definition of contraction, we may assume that H is an induced subgraph of G, and so $E(G)$ is the disjoint union of $E(H)$ and $E(G/H)$. Let v_H be the vertex in G/H onto which H is contracted. We verify the definition to show that $G \in \mathcal{O}_p$.

Arbitrarily take a $\mathbb{Z}_p$-boundary b of G and $f : E(G) \to \mathbb{Z}_p^*$. Let $a_0 = \sum_{v \in V(H)} b(v)$. Define $b_1 : V(G/H) \to \mathbb{Z}_p$ by

$$b_1(z) = \begin{cases} b(z) & \text{if } z \in V(G/H) - \{v_H\} \\ a_0 & \text{if } z = v_H. \end{cases} \tag{5}$$

As b is a $\mathbb{Z}_p$-boundary, we have $\sum_{z \in V(G/H)} b_1(z) = \sum_{z \in V(G)} b(z) = 0$, and so b_1 is a $\mathbb{Z}_p$-boundary of G/H. Let $f_1 : E(G/H) \to \mathbb{Z}_p^*$ be the restriction of f to $E(G/H)$. Since $G/H \in \mathcal{O}_p$, G/H has an $(f_1, b_1; p)$-orientation D_1. Define $b_2 : V(H) \to \mathbb{Z}_p$ by

$$b_2(z) = \begin{cases} b(z) + \sum_{e \in E_{D_1}^-(v_H) \cap E_D^-(z)} f_1(e) - \sum_{e \in E_{D_1}^+(v_H) \cap E_D^+(z)} f_1(e) & \text{if } z \in A_G(H) \\ b(z) & \text{otherwise.} \end{cases} \tag{6}$$

As $a_0 = \sum_{v\in V(H)} b(v)$, we have

$$\sum_{z\in V(H)} b_2(z) = \sum_{z\in V(H)} b(z) + \sum_{e\in E^-_{D_1}(v_H)} f_1(e) - \sum_{e\in E^+_{D_1}(v_H)} f_1(e) = a_0 - \partial_{D_1} f_1(v_H) = 0,$$

and so b_2 is a $\mathbb{Z}_p$-boundary of H. Let $f_2 : E(H) \to \mathbb{Z}_p^*$ be the restriction of f to $E(H)$. Since $H \in \mathcal{O}_p$, H has an $(f_2, b_2; p)$-orientation D_2. Obtain an orientation D of G by taking the union of D_1 and D_2. It remains to show that D is an $(f, b; p)$-orientation of G. For any vertex $z \in V(G) - A_G(H)$, by the definition of D_1 and D_2, we have $\partial_D f(z) = b(z)$. For any vertex $z \in A_G(H)$, by (5) and (6), it follows that

$$\partial_D f(z) = \partial_{D_1} f_1(z) + \partial_{D_2} f_2(z) = \partial_{D_1} f_1(z) + b(z) - \partial_{D_1} f_1(z) = b(z).$$

Therefore $G \in \mathcal{O}_p$. ■

Nonempty families of connected graphs satisfying Proposition 2.6(i) and (ii) are called **complete families** and investigated in [4,5,15]. Complete families have quite a few interesting properties and are associated with certain reduction methods.

Corollary 2.7. *Let G be a connected graph and p be an odd prime. Then $G \in \mathcal{O}_p$ if and only if every block of G is in $\mathcal{O}_p$.*

Proof. Let $B_1, B_2, \ldots, B_c$ be blocks of G. The corollary holds trivially if $c = 1$, and so we assume $c \geq 2$. If $G \in \mathcal{O}_p$, then by Proposition 2.6(i), $B_i = G/(\cup_{j\neq i} B_j) \in \mathcal{O}_p$. Conversely, assume that every $B_i \in \mathcal{O}_p$, we proceed by induction on c to show that $G \in \mathcal{O}_p$. As G/B_c has blocks $B_1, B_2, \ldots, B_{c-1}$ and $B_i \in \mathcal{O}_p$ for each $i \in \{1, \ldots, c-1\}$. By induction on c, we have that $G/B_c \in \mathcal{O}_p$. As $B_c \in \mathcal{O}_p$, by Proposition 2.6(ii) we have that $G \in \mathcal{O}_p$. ■

For a given odd prime p, a graph G is **strongly $\mathbb{Z}_p$-connected** if for any $f : E(G) \to \{1, -1\} \subseteq \mathbb{Z}_p$, and any $\mathbb{Z}_p$-boundary b, G admits an $(f, b; p)$-orientation. The study of strongly $\mathbb{Z}_p$-connected graphs were initiated and investigated in [14,16,18–20], among others. By definition, a graph is strongly $\mathbb{Z}_3$-connected if and only if it is $\mathbb{Z}_3$-connected. Lemma 2.8(i) follows from the definition, and Lemma 2.8(iv) follows from Lemma 2.8(i) and (iii).

Lemma 2.8. *Let p be an odd prime. Each of the following holds.*
(i) Every graph $G \in \mathcal{O}_p$ is strongly $\mathbb{Z}_p$-connected.
(ii) (Jaeger et al. Proposition 2.2 of [13]) A graph G is $\mathbb{Z}_3$-connected if and only if $G \in \mathcal{O}_3$.
(iii) (Proposition 3.9 of [20]) Every strongly $\mathbb{Z}_p$-connected graph contains $p-1$ edge-disjoint spanning trees.
(iv) Every graph in $\mathcal{O}_p$ contains $p-1$ edge-disjoint spanning trees and is thus $(p-1)$-edge-connected.

For an integer $m > 0$ and a graph H, define $H^{(m)}$ to be the graph obtained from H by replacing each edge of H by a set of m parallel edges joining the same pair of vertices. In particular, $K_2^{(m)}$ is a loopless graph on two vertices and m edges. Lemma 2.9 is a consequence of Theorem 2.1(i), Corollary 2.3 and Lemma 2.8(iv).

Lemma 2.9. *Let G be a graph and p be an odd prime. Then $K_2^{(m)} \in \mathcal{O}_p$ if and only if $m \geq p-1$.*

Lemma 2.10 (*Jaeger et al. [13]*). *A graph $G = (V, E)$ is connected if and only if for any $b \in Z(G, \mathbb{Z}_p)$ and for any orientation D, there exists and $f \in F(G, \mathbb{Z}_p)$ such that $\partial f = b$.*

Let $|V(G)| = n$, and let the underlying simple graph of the graph G be C_n, where $V(G) = \{v_j : j \in \mathbb{Z}_n\}$. We denote C_n the cycle with the same vertex set and such that $v_j v_{j+1}$ is an edge for each $j \in \mathbb{Z}_n$. Similarly, we denote $C_n(i_1, \ldots, i_n)$ the graph with the same vertex set and such that $i_j = |[v_j v_{j+1}]|$ for each $j \in \mathbb{Z}_n$. By definition, $C_2(i_1, i_2) = K_2^{(i_1+i_2)}$.

Lemma 2.11. *Let $G = C_n(i_1, i_2, \ldots, i_n)$. If for each $j \in \mathbb{Z}_n$, $i_j \le p-1$, and if $\sum_{j=1}^{n} i_j \ge (n-1)(p-1)$, then $G \in \mathcal{O}_p$.*

Proof. Let $f \in F(G, \mathbb{Z}_p^*)$ and $b \in Z(G, \mathbb{Z}_p)$ be given. We are going to find an orientation D of G such that $\partial_D(f) = b$. Orient the edges of $E(C_n)$ so that for each $j \in \mathbb{Z}_n$, the edge e_j is oriented from v_j to v_{j+1}, and let D_1 denote the resulting orientation of C_n.

By Lemma 2.10, there is a mapping $f_0' \in F(C_n, \mathbb{Z}_p)$ such that $\partial_{D_1} f_0' = b$. For each constant $c \in \{1, \ldots, p-1\}$, let f_c' be the mapping given by $f_c'(e) = f_0'(e) + c$ for any $e \in E(C_n)$. It follows that $\partial_{D_1} f_c' = \partial_{D_1} f_0' = b$.

Fix an arbitrary $j \in \mathbb{Z}_n$, and let $[e_j]$ denote the edges parallel to e_j in G. By assumption, we may denote $[e_j] = \{e_j^1, \ldots, e_j^{i_j}\}$ (with $e_j = e_j^1$). Define a bipartite graph K with vertex bipartition (V_1, V_2), where $V_1 = \{f_0', f_1', \ldots, f_{p-1}'\}$ and $V_2 = \{e_1, e_2, \ldots, e_n\}$ such that f_c' is adjacent to e_j in K if and only if $f_c'(e_j) \notin \Omega(f(e_j^1), \ldots, f(e_j^{i_j}))$. Thus $d_K(e_j) = |\mathbb{Z}_p - \Omega(f(e_j^1), \ldots, f(e_j^{i_j}))|$. By Lemma 2.5 and since $i_j \le p-1$ for each $j \in \mathbb{Z}_n$, we have $\sum_{j=1}^{n} |\Omega(f(e_j^1), \ldots, f(e_j^{i_j}))| \ge \sum_{j=1}^{n} (i_j + 1)$. It follows by the assumption $\sum_{j=1}^{n} i_j \ge (n-1)(p-1)$ that

$$
\begin{aligned}
|E(K)| &= \sum_{j=1}^{n} d_K(e_j) = \sum_{j=1}^{n} |\mathbb{Z}_p - \Omega(f(e_j^1), \ldots, f(e_j^{i_j}))| \\
&= \sum_{j=1}^{n} |\mathbb{Z}_p| - \sum_{j=1}^{n} |\Omega(f(e_j^1), \ldots, f(e_j^{i_j}))| \\
&\le np - \sum_{j=1}^{n} (i_j + 1) \le n(p-1) - \sum_{j=1}^{n} i_j \le p - 1.
\end{aligned}
$$

Hence there exists at least one $c \in \mathbb{Z}_p$ such that f_c' is of degree zero in K. This implies that for any $j \in \mathbb{Z}_n$, we always have $f_c'(e_j) \in \Omega(f(e_j^1), \ldots, f(e_j^{i_j}))$.

Consider a $c \in \mathbb{Z}_p$ such that f_c' is of degree zero in K. We now construct an orientation D of G so that $\partial_D f = b$ to complete the proof. For each $j \in \mathbb{Z}_n$, we orient the edges $\{e_j^1, \ldots, e_j^{i_j}\}$. Since $f_c'(e_j) \in \Omega(f(e_j^1), \ldots, f(e_j^{i_j}))$, by the definition of $\Omega(f(e_j^1), \ldots, f(e_j^{i_j}))$, there exist scalars $\ell_t \in \{1, -1\} \subset \mathbb{Z}_p$ such that $f_c'(e_j) = \sum_{t=1} i_j \ell_t f(e_j^t)$. For each t with $1 \le t \le i_j$, orient e_j^t from v_j to v_{j+1} if $\ell_t = 1$ and from v_{j+1} to v_j if $\ell_t = -1$. Denote the resulting orientation of G by D. By the definition of D, we have

$$
\sum_{e \in E_D^+(v_j) \cap [e_j]} f(e) - \sum_{e \in E_D^-(v_j) \cap [e_j]} f(e) = f_c'(e).
$$

This implies that $\partial_D f = \partial_{D_1} f_0' = b$, and so D is an $(f, b; p)$-orientation of G. This proves the lemma. ■

Corollary 2.12. *Let $G = C_n(i_1, i_2, \ldots, i_n)$. The following are equivalent.*
(i) $G \in \mathcal{O}_p$.
(ii) G has $p-1$ edge-disjoint spanning trees.

Proof. By Lemma 2.8(iv), we have (i) implies (ii). We proceed by induction to prove that (ii) implies (i), and assume that G has $p-1$ edge-disjoint spanning trees. If $n = 2$, then (i) follows from Lemma 2.9. Assume that $n \ge 3$ and that (ii) implies (i) for smaller values of n. If C_n has an edge, say $e_n = v_n v_1$ with $|[e_n]| \ge p-1$, then we induce on $G' = G/[e_n]$. As $G' = C_{n-1}(i_1, i_2, \ldots, i_{n-1})$ and as G' also has $p-1$ edge-disjoint spanning trees, $G' \in \mathcal{O}_p$. By Lemma 2.9 and Proposition 2.6, we have $G \in \mathcal{O}_p$. Therefore, we may assume that $|[e]| \le p-2$ for any $e \in E(C_n)$. Since G has $p-1$ edge-disjoint spanning trees, we have $\sum_{j=1}^{n} i_j = |E(G)| \ge (n-1)(p-1)$, and so by Lemma 2.11, $G \in \mathcal{O}_p$. ■

3. Proof of Theorem 1.5

We first make some remarks before proving Theorem 1.5. In the original version of this paper, for a graph with large Euler genus g, we proved edge connectivity bound $2gp$, roughly, through a different method. A referee of this paper kindly shared his/her ideas to improve the bound from the fact that every simple graph with Euler genus g is $O(\sqrt{g})$-degenerate, which eventually helps us to achieve the current bound $(p-2)\lfloor\sqrt{6g+0.25}+2.5\rfloor+1$ for $g \geq 3$. Digging deeper on those arguments and ideas, with the help of Theorem 3.1, we are also able to get a better bound $p\sqrt{4.98g}$ for a sufficiently large g. We would like to thank the referees for very helpful suggestions.

Theorem 3.1 (*Delcourt and Postle [8]*)**.** *For a sufficiently large integer n, every simple graph on n vertices with minimum degree at least* $0.8274n$ *can be edge-decomposed into triangles if each vertex has degree even and its number of edges is divisible by* 3.

The following is a consequence of Theorem 3.1.

Lemma 3.2. *For a sufficiently large integer n, every simple graph on n vertices with minimum degree at least* $0.8275n$ *can be edge-decomposed into triangles, plus at most* $0.5n+7$ *single edges.*

Proof. Let G be a graph on n vertices with minimum degree at least $0.8275n$. Then G has a Hamiltonian cycle C by Dirac's Theorem. Let T be the set of odd degree vertices in G. Clearly, $|T|$ is even, and so let $|T|=2t$, where $t \geq 0$. We label the vertices of T as $v_1, v_2, \ldots, v_{2t}$ in the cyclic order along the Hamiltonian cycle C. Then for each $1 \leq i \leq t$, there is a path P_i in the cyclic order of C from v_{2i-1} to v_{2i}. Define $X=\cup_{i=1}^{t}E(P_i)$ if $|\cup_{i=1}^{t}E(P_i)| \leq 0.5n$, and $X=E(C)\setminus(\cup_{i=1}^{t}E(P_i))$ otherwise. Then we have $|X| \leq 0.5n$ and each vertex of T has degree odd in X. Let $G_1=G-X$. Then each vertex of G_1 has degree even. If $|E(G_1)|$ is divisible by 3, then let $G_2=G_1$. If $|E(G_1)|$ is not divisible by 3, noting that G_1 contains both 5-cycles and 7-cycles by Turán's Theorem, then we delete the edges of a 5-cycle or a 7-cycle in G_1 to obtain a new graph G_2 whose number of edges is divisible by 3. Now G_2 has minimum degree at least $0.8275n-4 > 0.8274n$, and each vertex of G_2 has degree even. So Theorem 3.1 is applicable for G_2 in any case. Hence $E(G_2)$ can be edge-decomposed into triangles by Theorem 3.1. As $E(G)\setminus E(G_2)$ has at most $0.5n+7$ edges, the lemma follows. ■

Now we are going to prove Theorem 1.5. As Theorem 1.5 holds trivially if $G=K_1$, we assume that $|V(G)| \geq 2$. In the following, we always let $\tilde{G}$ denote the **underlying simple graph** of G. For fixed integer $p \geq 3$, define a function on the interval $[3,\infty)$ as follows.

$$\phi(x)=\frac{2(x-1)}{x-2}p-\frac{2x}{x-2}.$$

As on $[3,\infty)$, the derivative of the function is

$$\phi'(x)=\frac{4-2p}{(x-2)^2}<0,$$

it follows that

$$\phi(x) \text{ is a decreasing function on } [3,\infty). \tag{7}$$

We prove the following equivalent statement of Theorem 1.5.

Theorem 3.3. *Let* $p>0$ *be an odd prime, and let G be a graph with* $\kappa'(G) \geq p-1$. *Then each of the following holds.*
(i) If G has Euler genus $g \leq 2$ *and* $\kappa'(G) \geq 4p-6+\lfloor g/2\rfloor$, *then* $G \in \mathcal{O}_p$.
(ii) If G has Euler genus $g \geq 3$ *and* $\kappa'(G) \geq (p-2)\lfloor 2.5+\sqrt{6g+0.25}\rfloor+1$, *then* $G \in \mathcal{O}_p$.
(iii) If G has sufficiently large Euler genus (independent of p) and $\kappa'(G) \geq p\sqrt{4.98g}$, *then* $G \in \mathcal{O}_p$.

Proof. To prove Theorem 3.3, we argue by contradiction and assume that

$$G \text{ is a counterexample to Theorem 3.3 with } |V(G)| \text{ minimized.} \tag{8}$$

Thus one of (i), (ii) and (iii) holds but $G \notin \mathcal{O}_p$, and so by (8), we have the following claim.

Claim 3.4. *Each of the following holds.*
(i) $\kappa(G) \geq 2$.
(ii) G does not have a nontrivial subgraph H such that $H \in \mathcal{O}_p$.
(iii) G does not have a subgraph isomorphic to a $K_2^{(m)}$ *with* $m \geq p-1$.
(iv) G does not have a subgraph isomorphic to a $C_\ell(i_1, i_2, \ldots, i_\ell)$ *with* $\sum_{j=1}^{\ell} i_j \geq (\ell-1)(p-1)$.

Since $\kappa'(G) \geq p-1 \geq 2$, G is connected. Let $B_1, B_2, \ldots, B_c$ be blocks of G. If $c \geq 2$, then the definition of edge-connectivity implies $\kappa'(G) = \min\{\kappa'(B_i) : 1 \leq i \leq c\}$, and so by (8), each $B_i \in \mathcal{O}_p$. It follows by Corollary 2.7 that $G \in \mathcal{O}_p$, a contradiction to (8). Thus, $c = 1$ and Claim 3.4(i) holds.

Let H be a subgraph of G such that $|V(H)| > 1$ and $H \in \mathcal{O}_p$. Let $G' = G/H$ with Euler genus g'. Then by definition, $\kappa'(G') \geq \kappa'(G)$ and $g \geq g'$. As $|V(H)| > 1$, $|V(G')| < |V(G)|$, and so by (8), $G' \in \mathcal{O}_p$. By Proposition 2.6(ii), we have $G \in \mathcal{O}_p$, a contradiction to (8). Thus Claim 3.4(ii) holds.

By Lemma 2.9, $K_2^{(m)} \in \mathcal{O}_p$ when $m \geq p-1$, and by Lemma 2.11, $C_\ell(i_1, i_2, \ldots, i_\ell) \in \mathcal{O}_p$ when $\sum_{j=1}^{\ell} i_j \geq (\ell-1)(p-1)$. Hence Claim 3.4(iii) and (iv) are consequences of Claim 3.4(ii), and so the claim holds.

Notice that if $n = |V(G)| \leq 3$, then by Claim 3.4(i), we have that the underling simple graph $\tilde{G}$ is isomorphic to K_n. When $n = 2, 3$, the edge connectivity implies that G contains a subgraph in $\mathcal{O}_p$ (as in Claim 3.4(iii) or (iv)), contrary to Claim 3.4(ii). Hence we have

Observation 3.5. $|V(G)| \geq 4$.

By Claim 3.4(iii), for any edge $e \in E(G)$, there are at most $p-2$ edges parallel to e in G; and if G has a subgraph J isomorphic to a $C_\ell(i_1, i_2, \ldots, i_\ell)$, then $|E(J)| \leq (\ell-1)(p-1)-1$. This is a key fact in later proofs.

Let S be a surface of Euler genus g and suppose G is embedded into S in such a way that for each edge $e \in E(G)$, if $[e] = \{e^1, e^2, \ldots, e^s\}$ with $s = |[e]| \geq 2$, then, re-embedding the edges in $[e]$ if needed, the 2-cycles $\{e^1, e^2\}, \{e^2, e^3\}, \ldots, \{e^{s-1}, e^s\}$ are the boundaries of some 2-faces of the embedding.

Define $F(G)$ to be the set of faces of G. For each $f \in F(G)$, we define $d_G(f)$ to be the number of edges incident with f, and for each integer $i \geq 1$, let F_i be the number of faces of degree i in G. A face of degree ℓ is often called an ℓ-face. If the two edges of a 2-face are parallel to or contain an edge of an ℓ-face for some $\ell \geq 3$, then we say this 2-face is related to the ℓ-face, or is a related 2-face of the ℓ-face.

Recall Euler's formula that

$$|V(G)| + |F(G)| - |E(G)| = 2 - g.$$

To find a contradiction, we use a discharging argument. Define k as follows,

$$k = \begin{cases} 4p - 6 + \lfloor g/2 \rfloor & \text{if } g \leq 2, \\ (p-2)\lfloor \sqrt{6g + 0.25} + 2.5 \rfloor + 1 & \text{if } g \geq 3, \\ p\sqrt{4.98g} & \text{if } g \text{ is sufficiently large.} \end{cases} \tag{9}$$

As in a 2-cell embedding of a graph G on a surface, every edge is incident with one or two faces. It follows that every 2-face of G in this 2-cell embedding is related to either one or two faces of degree at least 3. Define, for $i \in \{1, 2\}$,

$$X_i(G) = \{f \in F(G) : f \text{ is a 2-face and is related to } i \text{ faces of degree at least 3 in the embedding.}\}$$

For each face $f \in F(G)$, we assign an initial charge $w(f)$ equaling the degree of f in the embedding. Now we define the discharging rule as follows.

For $\ell \geq 3$ *and* $i \in 1, 2$, *every* ℓ*-face* f *gives* $\dfrac{2(3-i)}{k-2}$ *to each of the 2-faces in* $X_i(G)$ *related to* f.

For any $f \in F(G)$, let $w^*(f)$ be the resulting charge of f after recharging. As every 2-face in $F(G)$ is either in $X_1(G)$ or in $X_2(G)$, by the discharging rule, we conclude that

$$\text{For any 2-face } f \text{ of } G,\ w^*(f) = 2 + \frac{4}{k-2}. \tag{10}$$

For an integer $\ell \geq 3$ and for any $f \in F(G)$ with $d_G(f) = \ell$, let $\bar{E}(f)$ be the set of edges that are in 2-faces related to f or contained in f, and let $E_1(f)$ be the set of edges in 2-faces related to f and in $X_1(G)$. Let Y be the edge-induced graph by $\bar{E}(f) - E_1(f)$ and assume that Y has c components. Note that each component of $\bar{E}(f) - E_1(f)$ is a $C_{\ell_j}(i_1^j, \ldots, i_{\ell_j}^j)$ for $j \in \{1, 2, \ldots, c\}$. Here $C_{\ell_j}(i_1^j, \ldots, i_{\ell_j}^j)$ is a single vertex when $\ell_j = 0$. We may, without loss of generality, assume all those single vertices are $C_{\ell_j}(i_1^j, \ldots, i_{\ell_j}^j)$'s for $j \geq c' + 1$, where $c' \leq c$. Hence $\ell = \sum_{j=1}^{c} \ell_j + 2(c-1) = \sum_{j=1}^{c'} \ell_j + 2c - 2$, and so $\sum_{j=1}^{c'} \ell_j = \ell + 2 - 2c$.

By Claim 3.4(iii) and (iv), $|\bar{E}(f)| \leq (c-1)(p-2) + \sum_{j=1}^{c'}((\ell_j - 1)(p-1) - 1)$. By the discharging rule, for any ℓ-face f of G with $\ell \geq 3$,

$$\begin{aligned}
w^*(f) &\geq \ell - \frac{2}{k-2}\Big[2(c-1)(p-3) + \sum_{j=1}^{c'}((\ell_j - 1)(p-1) - 1 - \ell_j)\Big] \\
&= \ell - \frac{2}{k-2}\Big[2(c-1)(p-3) + (p-2)\sum_{j=1}^{c'} \ell_j - pc'\Big] \\
&= \ell - \frac{2}{k-2}\Big[2(c-1)(p-3) + (p-2)(\ell + 2 - 2c) - pc'\Big] \\
&= \ell - \frac{2}{k-2}\Big[-pc' - 2c + \ell(p-2) + 2\Big].
\end{aligned}$$

By the definition of 2-cell embedding and 2-connectivity of G, one has $c \geq c' \geq 1$. Hence, for any ℓ-face f of G with $\ell \geq 3$,

$$w^*(f) \geq \ell - \frac{2}{k-2}[-p - 2 + \ell(p-2) + 2] = \ell - (\ell p - 2\ell - p)\frac{2}{k-2}. \tag{11}$$

By (1), we have that $\kappa'(G) \geq k$. Then $2|E(G)| \geq \kappa'(G)|V(G)| \geq k|V(G)|$. It follows from Euler's formula $|V(G)| + |F(G)| - |E(G)| = 2 - g$ that $\frac{k}{k-2}(|F(G)| - 2 + g) \geq |E(G)|$, and so

$$\sum_{i\geq 2}\left(2 + \frac{4}{k-2}\right)f_i - \frac{2k(2-g)}{k-2} = \frac{2k}{k-2}(|F(G)| - 2 + g) \geq 2|E(G)| = \sum_{f\in F(G)} w(f) = \sum_{f\in F(G)} w^*(f). \tag{12}$$

Case A $g \in \{0, 1, 2\}$.

Then $\kappa'(G) \geq k = 4p - 6 + \lfloor g/2 \rfloor \geq 4p - 6$. Let $k' = 4p - 6$. By (7), for any $f \in F(G)$ with $d_G(f) = \ell \geq 3$, we have

$$k \geq k' = 4p - 6 = \phi(3) \geq \phi(\ell) = \frac{2(\ell - 1)}{\ell - 2}p - \frac{2\ell}{\ell - 2} = \frac{2\ell p - 2p - 2\ell}{\ell - 2}, \tag{13}$$

which is equivalent to $(k' - 2)\ell - 2(\ell p - 2\ell - p) \geq 2k'$. Hence

$$\ell - (\ell p - 2\ell - p)\frac{2}{k' - 2} \geq \frac{2k'}{k' - 2}. \tag{14}$$

If $g = 0, 1$, then $k' = k$, and so by (11) and (14) we have for any $f \in F(G)$ with $d_G(f) = \ell \geq 3$, $w^*(f) \geq \frac{2k}{k-2} = 2 + \frac{4}{k-2}$. This, together with (10), implies $\sum_{f\in F(G)} w(f) = \sum_{f\in F(G)} w^*(f) \geq \sum_{i\geq 2}(2 + \frac{4}{k-2})f_i$, contrary to (12). Thus the theorem must hold in Case A with $g = 0, 1$.

Now assume that $g = 2$. Then $k > k' = 4p - 6 = \phi(3)$. It follows by (13) and by $k > k'$ that (14) holds with strict inequality if we replace k' by k in (14). This leads to $\ell - (\ell p - 2\ell - p)\frac{2}{k-2} > \frac{2k}{k-2}$. This, together with (11), implies that for any $f \in F(G)$ with $d_G(f) = \ell \geq 3$, $w^*(f) > \frac{2k}{k-2} = 2 + \frac{4}{k-2}$. Thus, in conjunction with (10), we have

$$\sum_{f \in F(G)} w(f) = \sum_{f \in F(G)} w^*(f) > \sum_{i \geq 2}(2 + \frac{4}{k-2})f_i,$$

contrary to (12). This settles Case A.

In the rest of the arguments, we let $\delta = \delta(\tilde{G})$ to be the minimum degree of $\tilde{G}$, the underling simple graph of G. By Claim 3.4(iii), for any edge $e \in E(G)$ there are at most $p-2$ edges parallel to edge e. Hence the minimum degree of G is at most $(p-2)\delta$. This provides the following observation.

Observation 3.6. $(p-2)\delta \geq \kappa'(G) \geq k$.

Case B $g \geq 3$.

In this case, by (9) and Observation 3.6, we have

$$\delta(\tilde{G}) = \delta \geq \lfloor\sqrt{6g + 0.25} + 2.5\rfloor + \frac{1}{p-2} > \lfloor\sqrt{6g + 0.25} + 2.5\rfloor.$$

Note that δ is a positive integer. Thus we have

$$\delta > \sqrt{6g + 0.25} + 2.5. \tag{15}$$

Since $\tilde{G}$ is a simple graph, every face of the embedding of $\tilde{G}$ has degree at least 3, and so $2|E(\tilde{G})| = \sum_{f \in F(\tilde{G})} d(f) \geq 3|F(\tilde{G})|$. Note that the Euler genus of $\tilde{G}$ is the same as the Euler genus of G. Applying Euler's formula $|V(\tilde{G})| + |F(\tilde{G})| - |E(\tilde{G})| = 2 - g$ for $\tilde{G}$, we have $\frac{2}{3}|E(\tilde{G})| \geq |F(\tilde{G})| = 2 - g + |E(\tilde{G})| - |V(\tilde{G})|$, which gives

$$g - 2 \geq \frac{1}{3}|E(\tilde{G})| - |V(\tilde{G})| = \frac{1}{3}|V(\tilde{G})|(\frac{|E(\tilde{G})|}{|V(\tilde{G})|} - 3) \geq \frac{1}{3}(\delta(\tilde{G}) + 1)(\frac{\delta(\tilde{G})}{2} - 3).$$

Combining with (15), it follows that $g - 2 \geq \frac{1}{6}(\delta^2 - 5\delta - 6) > \frac{1}{6}[(\sqrt{6g + 0.25} + 2.5)^2 - 5(\sqrt{6g + 0.25} + 2.5) - 6] = g - 2$, a contradiction. This settles Case B.

Case C g is sufficiently large.

For any ℓ-face f of G with $\ell \geq 3$, by (11), we have

$$w^*(f) \geq \ell(1 - \frac{2(p-2)}{k-2}) + \frac{2p}{k-2} \geq 3(1 - \frac{2(p-2)}{k-2}) + \frac{2p}{k-2} = \frac{3k - 4p + 6}{k-2}.$$

Thus, by (10) and (12), we have

$$\sum_{i \geq 2}\left(2 + \frac{4}{k-2}\right)f_i - \frac{2k(2-g)}{k-2} \geq \sum_{f \in F(G)} w^*(f) \geq (2 + \frac{4}{k-2})f_2 + \sum_{i \geq 3}\frac{3k - 4p + 6}{k-2}f_i,$$

which gives $\frac{2k(g-2)}{k-2} \geq \frac{k-4p+6}{k-2}\sum_{i \geq 3} f_i$ and

$$2k(g-2) \geq (k - 4p + 6)\sum_{i \geq 3} f_i. \tag{16}$$

Notice that, since G is embedded into S, the embedding of $\tilde{G}$ on S may be obtained from embedding G by deleting parallel edges. So for any $\ell \geq 3$, each ℓ-face of G is exactly an ℓ-face of $\tilde{G}$. Hence we have $\sum_{i \geq 3} f_i = |F(\tilde{G})| = 2 - g + |E(\tilde{G})| - |V(\tilde{G})|$. By (16), we have

$$2k(g-2) \geq (k - 4p + 6)(2 - g + |E(\tilde{G})| - |V(\tilde{G})|). \tag{17}$$

If $|V(\tilde{G})| > \frac{\delta}{0.828} + 6$, then it follows from (17) and Observation 3.6 that

$$\begin{aligned}
2k(g-2) &\geq (k-4p+6)(2-g+|E(\tilde{G})|-|V(\tilde{G})|)\\
&\geq (k-4p+6)(2-g+\frac{\delta}{2}|V(\tilde{G})|-|V(\tilde{G})|)\\
&\geq (k-4p+6)\left(2-g+(\frac{\delta}{2}-1)(\frac{\delta}{0.828}+6)\right)\\
&\geq (k-4p+6)\left(2-g+(\frac{k}{2(p-2)}-1)(\frac{k}{0.828(p-2)}+6)\right).
\end{aligned}$$

Since $k = p\sqrt{4.98g}$, $\frac{k}{p-2} > \sqrt{4.98g}$, and g is sufficiently large, we further obtain from the above inequality that

$$\begin{aligned}
2p\sqrt{4.98g}(g-2) &\geq (k-4p+6)\left(2-g+(\frac{k}{2(p-2)}-1)(\frac{k}{0.828(p-2)}+6)\right)\\
&> (p\sqrt{4.98g}-4p+6)\left(2-g+(0.5\sqrt{4.98g}-1)(\frac{\sqrt{4.98g}}{0.828}+6)\right)\\
&> (p\sqrt{4.98g}-4p+6)(2-g+3.007g)\\
&> 2.006gp\sqrt{4.98g},
\end{aligned}$$

a contradiction.

Assume instead that $|V(\tilde{G})| \leq \frac{\delta}{0.828} + 6 < \frac{\delta}{0.8275}$. Then $\delta(\tilde{G}) = \delta \geq 0.8275|V(\tilde{G})|$ and $|V(\tilde{G})| \geq \delta + 1 \geq \sqrt{4.98g}$ is sufficiently large. Hence Lemma 3.2 is applicable to $\tilde{G}$. It follows by Lemma 3.2 that $\tilde{G}$ can be decomposed into edge-disjoint triangles, plus at most $0.5|V(G)| + 7$ single edges. By Claim 3.4(iv), each such triangle of $\tilde{G}$ corresponds to at most $2p-3$ edge of G, and each single edge corresponds to at most $p-2$ edge of G. As there are at most $\frac{1}{3} \cdot \frac{|V(G)|(|V(G)|-1)}{2}$ such triangles in $\tilde{G}$, this gives an estimation on the number of edges in G as follows:

$$|E(G)| \leq (2p-3) \cdot \frac{|V(G)|(|V(G)|-1)}{6} + (p-2) \cdot (0.5|V(G)|+7) < \frac{2p|V(G)|^2}{6}.$$

Hence we have

$$|V(\tilde{G})| = |V(G)| > \frac{6|E(G)|}{2p|V(G)|} \geq \frac{3k}{2p} = \frac{3p\sqrt{4.98g}}{2p} = 1.5\sqrt{4.98g}.$$

Thus, by (17) and since $\frac{\delta}{2} \geq \frac{k}{2(p-2)} > \frac{1}{2}\sqrt{4.98g} + 1$, we obtain a contradiction as follows:

$$\begin{aligned}
2p\sqrt{4.98g}(g-2) = 2k(g-2) &\geq (k-4p+6)(2-g+\frac{\delta}{2}|V(\tilde{G})|-|V(\tilde{G})|)\\
&> (k-4p+6)(2-g+(\frac{\delta}{2}-1)\cdot 1.5\sqrt{4.98g})\\
&> (p\sqrt{4.98g}-4p+6)(2-g+0.75\cdot 4.98g)\\
&> 2.5gp\sqrt{4.98g},
\end{aligned}$$

a contradiction. This completes the proof for this case and justifies Theorem 3.3. ■

Theorem 1.4 indicates that if the edge connectivity of a graph G is at least some quadratic function of p, then G is in $\mathcal{O}_p$. In view of our main result, we believe that it is possible that a linear function would suffice. We conclude this paper with the following conjecture.

Conjecture 3.7. *There exists a constant c independent of p such that every cp-edge-connected graph is in $\mathcal{O}_p$.*

CRediT authorship contribution statement

Jian-Bing Liu: Methodology, Writing - original draft. **Ping Li:** Conceptualization, Funding acquisition. **Jiaao Li:** Methodology, Validation, Writing - review & editing. **Hong-Jian Lai:** Conceptualization, Validation, Investigation, Supervision.

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