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Contractible graphs for flow index less than three

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ABSTRACT

For a graph G, the *flow index* $\phi(G)$ is the smallest rational number $t > 0$ such that the graph has a circular t-flow. Li et al. (2018) recently proved that $\phi(G) < 3$ for any 8-edge-connected graph G, and conjectured that 6-edge-connectivity would suffice. Here we present a contraction method to investigate this problem and apply it to verify this conjecture for certain 6-edge-connected graph families, including chordal graphs, graphs with few odd vertices, and graphs with small independence number.

1. Introduction

This paper studies finite loopless graphs, which may contain parallel edges. Given integers $k \geq 2d > 0$, a *circular k/d-flow* of a graph G consists of an orientation D of G together with a mapping $f : E(G) \mapsto \{\pm d, \pm(d+1), \ldots, \pm(k-d)\}$ such that for any vertex the total incoming flow equals the total outgoing flow. The $d = 1$ case is known as *nowhere-zero k-flow*, introduced by Tutte [18]. Goddyn, Tarsi, Zhang [3] showed that every graph admitting a circular t-flow must have a circular s-flow for any rational numbers $s \geq t \geq 2$. Thus the circular flow property is monotonic. Following [3,13], define the *flow index* $\phi(G)$ of a graph G to be the infimum of all rational numbers r such that G has a circular r-flow. It is proved in [3] that this flow index exists as a rational number for any finite bridgeless graph. Tutte's 5-flow conjecture asserts that every bridgeless graph G satisfies $\phi(G) \leq 5$, and Seymour [16] proved that $\phi(G) \leq 6$ for any bridgeless graph G. Tutte's 3-flow conjecture states that $\phi(G) \leq 3$ for any 4-edge-connected graph G, while Jaeger [7] proved that every 4-edge-connected graph G satisfies $\phi(G) \leq 4$ and Lovász, Thomassen, Wu and Zhang [14] showed $\phi(G) \leq 3$ for any 6-edge-connected graph G.

Extending Tutte's flow conjectures, Jaeger [8] proposed a circular flow conjecture that $\phi(G) \leq 2 + 1/p$ for any $4p$-edge-connected graph G. The weak version of this conjecture was established by Thomassen [17]. Later, Lovász et al. [14]

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improved this result to show that $\phi(G) \le 2 + 1/p$ for any $6p$-edge-connected graph G. However, Jaeger's circular flow conjecture was disproved in [6] recently by constructing infinitely many counterexamples for every $p \ge 3$. The cases when $p = 1, 2$, which are more important due to their connection to Tutte's flow conjectures, remain open.

Tutte initiated the investigation on the relationship between the edge connectivity and the existence of flows. Tutte's 5-flow conjecture, 3-flow conjecture, and Jaeger's circular flow conjecture for $p = 2$ are proposed for 2-edge-connected, 4-edge-connected, and 8-edge-connected graphs, respectively. Motivated by those conjectures as well as the main result in [14], a new flow conjecture for 6-edge-connected graphs was proposed in [13].

Conjecture 1.1 (*[13]*). *For every* 6*-edge-connected graph G, the flow index $\phi(G) < 3$.*

It is known that $\phi(K_6) = 3$ and an infinite family of 5-edge-connected planar graphs with flow index exactly 3 is also constructed in [13]. Hence in Conjecture 1.1 the edge-connectivity condition cannot be relaxed. The authors in [13] verified Conjecture 1.1 within 8-edge-connected graphs.

Theorem 1.2 (*[13]*). *For every* 8*-edge-connected graph G, the flow index $\phi(G) < 3$.*

In this paper, we develop a contraction method to study the $\phi < 3$ problem and Conjecture 1.1. This method is also related to some Hamiltonian property of graphs, as to be seen in Section 2. We also apply this contraction method to verify Conjecture 1.1 for certain graph families.

Theorem 1.3. *Let G be a* 6*-edge-connected graph.*
(i) If G has at most 12 *vertices of odd degree, then $\phi(G) < 3$.*
(ii) If G has the independence number $\alpha(G) \le 2$, then $\phi(G) < 3$.

Note that the family of graphs with independence number at most two is as rich as triangle-free graphs, since simple graphs with independence number at most two are exactly complement of triangle-free simple graphs.

A simple graph G is *chordal* if each cycle of length at least 4 contains a chord. Formed by perfect elimination ordering, the family of chordal graphs is an important graph class in graph algorithm theory, since various NP-complete graph problems can be efficiently solved for chordal graphs. Applying our reduction methods, Conjecture 1.1 is verified for chordal graphs in a strong sense.

Theorem 1.4. *If G is a simple* 5*-connected chordal graph other than K_6, then $\phi(G) < 3$.*

The rest of the paper is organized as follows: We first develop contraction and reduction methods in Section 2, and then apply those tools to prove Theorem 1.4 in Section 3. The proof of Theorem 1.3, which involves additional orientation techniques from Hakimi's orientation theorem [4], will be presented in Sections 4 and 5.

2. Preliminaries on orientations and reduction methods

For a graph G, let $\delta(G)$ and $\alpha(G)$ denote the minimum degree and the independence number, respectively. Given two nonempty vertex subsets $A, B \subseteq V(G)$, let $[A, B]_G = \{ab \in E(G) | a \in A, b \in B\}$. When $A = \{a\}$ or $B = \{b\}$, we use $[a, B]_G$ or $[A, b]_G$ for $[A, B]_G$, respectively. Define $\partial_G(A) = [A, V(G) - A]_G$ and $d(A) = |\partial_G(A)|$. The neighborhood of a vertex v is the set $N_G(v) = \{x | vx \in \partial_G(v)\}$. When the graph G is understood from the contest, the subscript G is often omitted.

In a given graph G, a function $\beta : V(G) \to \mathbb{Z}_3$ is called a *boundary function* if $\sum_{x\in V(G)} \beta(x) = 0 \pmod 3$. The set of all boundary functions in G is denoted by $Z(G, \mathbb{Z}_3)$. We call an orientation D of G a β*-orientation* if for any vertex $x \in V(G)$, $d^+_D(x) - d^-_D(x) \equiv \beta(x) \pmod 3$. The special case of β-orientation with $\beta(x) = 0$ for any $x \in V(G)$ is known as a *modulo* 3*-orientation* of G. The following proposition is well-known (cf. [8,18]).

Proposition 2.1 (*[18]*). *A graph G has $\phi(G) \le 3$ if and only if G admits a modulo* 3*-orientation.*

In [13], a relation between flow index strictly less than 3 and strongly connected orientations is established.

Theorem 2.2 (*[13]*). *A graph G satisfies $\phi(G) < 3$ if and only if it has a strongly connected modulo* 3*-orientation.*

Group connectivity, introduced by Jaeger et al. [9], is a useful tool in studying nowhere-zero 3-flows and modulo 3-orientations. A graph G is called $\mathbb{Z}_3$*-connected* if for every $\beta \in Z(G, \mathbb{Z}_3)$, there exists a β-orientation in G. A similar concept to study the problems for flow index strictly less than 3 was introduced in [11].

Definition 2.3. A graph G is called **strongly-connected $\mathbb{Z}_3$-contractible** if there exists a strongly-connected β-orientation in G for any $\beta \in Z(G, \mathbb{Z}_3)$. Let $\mathcal{S}_3$ denote the family of all strongly-connected $\mathbb{Z}_3$-contractible graphs.

For notational convenience, a strongly-connected β-orientation is called a **β-SCO**, and a strongly-connected $\mathbb{Z}_3$-contractible graph is also called an **$\mathcal{S}_3$-graph** in the rest of this paper. That is, an $\mathcal{S}_3$-graph is a graph such that for any boundary function $\beta \in Z(G, \mathbb{Z}_3)$, one can always find a β-SCO, i.e., a strongly-connected orientation D with $d_D^+(x) - d_D^-(x) \equiv \beta(x) \pmod 3$, $\forall x \in V(G)$.

Given $B \subset E(G)$, the contraction G/B is the graph obtained from G by identifying the two ends of each edge in B, and then removing the resulting loops. We use G/H for $G/E(H)$ if H is a connected subgraph of G.

Let G be a graph with boundary $\beta \in Z(G, \mathbb{Z}_3)$. If H is a subgraph of G and $G' = G/H$, then the corresponding boundary β' of G' is defined by

$$\beta'(x) = \begin{cases} \beta(x), & \text{if } x \in V(G) \setminus V(H), \\ \sum_{v \in V(H)} \beta(v), & \text{if } x = v_H, \end{cases} \tag{1}$$

where v_H is the vertex in G' onto which H is contracted. This notation will be used through out the rest of the paper.

Some useful properties of $\mathcal{S}_3$-graphs can be seen in the following proposition, which are associated with some contraction and reduction methods. Families of connected graphs satisfying those properties are called *complete families* and have been investigated in [1,2].

Proposition 2.4 (*[11]*). *Each of the following holds.*
(C1) $K_1 \in \mathcal{S}_3$.
(C2) *If $G \in \mathcal{S}_3$ and $e \in E(G)$, then $G/e \in \mathcal{S}_3$.*
(C3) *For a subgraph H of G, if both $H \in \mathcal{S}_3$ and $G/H \in \mathcal{S}_3$, then $G \in \mathcal{S}_3$.*

To apply this reduction idea in a wider class of graphs, we will also need the following weak version of contractible graphs, which plays a significant role in our later proofs. In [11], a similar reduction idea with closure operation was introduced to study the flow index problem of complementary graphs. The new weakly contractible concept in this paper is much more general. For a graph H, we call G a **proper supergraph** of H, if $H \subseteq G$ and there exist two distinct vertices $u, v \in V(H)$ with a (u, v)-path in $G - E(H)$.

Definition 2.5. A graph H is called **weakly contractible**, if for any proper supergraph G of H and any boundary β of G, any corresponding β'-SCO of G/H can be extended to a β-SCO of G. Let $\mathcal{W}_3$ denote the family of all weakly contractible graphs. A weakly contractible graph is also called a $\mathcal{W}_3$**-graph** for short.

If $H \in \mathcal{W}_3$ and G is a proper supergraph of H, then we also say that H is a **proper** $\mathcal{W}_3$**-subgraph** of G. By Definitions 2.3 and 2.5, together with Theorem 2.2, it is routine to verify the following fact.

Proposition 2.6. *Let H be a proper $\mathcal{W}_3$-subgraph of a graph G. Then each of the following holds.*
(i) $\phi(G) < 3$ if and only if $\phi(G/H) < 3$.
(ii) $G \in \mathcal{S}_3$ if and only if $G/H \in \mathcal{S}_3$.

Clearly, we have $\mathcal{S}_3 \subseteq \mathcal{W}_3$. The following proposition characterizes the relation between $\mathcal{S}_3$-graphs and $\mathcal{W}_3$-graphs.

Proposition 2.7. *A graph $H \in \mathcal{W}_3$ if and only if $H + xy \in \mathcal{S}_3$ for any two distinct vertices $x, y \in V(H)$.*

Proof. If $H \in \mathcal{W}_3$, we let $G = H + xy$ for some distinct vertices $x, y \in V(H)$. Note that G is a proper supergraph of H and $G/H = K_1$. By Definition 2.5, for any boundary β of G, the graph $G/H = K_1$ can be extended to a β-SCO of G. Thus $G = H + xy \in \mathcal{S}_3$.

Conversely, assume that for any two distinct vertices $x, y \in V(H)$, it holds that $H+xy \in \mathcal{S}_3$. Let G be a proper supergraph of H. Given $\beta \in Z(G, \mathbb{Z}_3)$, we are to show that any corresponding β'-SCO D' of G/H can be extended to a β-SCO in G. The orientation D' of G/H results in an orientation D_1 of $G - E(H)$ (after uncontracting the subgraph H and arbitrarily orienting the edges in $E(G[V(H)]) - E(H)$ if any). We claim that

$$\text{there exist distinct vertices } x, y \in V(H) \text{ such that there is a directed path from } x \text{ to } y \text{ in } D_1. \tag{2}$$

Since G is a proper supergraph of H, there is a path P in $G - E(H)$ connecting two distinct vertices $u, v \in V(H)$. If each edge in P is contained in a directed cycle under orientation D_1, then there exists a directed path between u and v, and hence (2) holds. Otherwise, there is an edge e_0 in P that is not contained in any directed cycle under orientation D_1. Notice that $D' = D'(G/H)$ is a strongly connected orientation, and so it must have a directed cycle C containing the edge e_0 in D'. Since $E(C)$, the edge set of C, is not a directed cycle in D_1, it must be a directed path from x to y, where $x, y \in V(H)$. This proves (2).

By (2) there exist distinct vertices $x, y \in V(H)$ with a directed path P_{xy} from x to y in D_1. By assumption, we have $H + xy \in \mathcal{S}_3$. The orientation D_1 restricted to $G - E(H)$ is called a β_1-orientation. Define a boundary function β_2 of H by

$$\beta_2(v) = \begin{cases} \beta(x) - \beta_1(x) + 1, & \text{if } v = x, \\ \beta(y) - \beta_1(y) - 1, & \text{if } v = y, \\ \beta(v) - \beta_1(v), & \text{otherwise.} \end{cases}$$

Then, similar to the arguments as in Proposition 2.4, it is routine to verify that $\beta_2 \in Z(H, \mathbb{Z}_3)$. Since $H + xy \in \mathcal{S}_3$, there exists a β_2-SCO D_2 of $H + xy$. We may further assume this new added edge $D_2(xy)$ is oriented from x to y. (Otherwise, revise the orientation of a directed cycle containing xy, and it still gives a β_2-SCO.) Denote D_2' to be the restriction of D_2 in H. Then $D_1 \cup D_2'$ is a β-orientation of G. Notice that $D_2' \cup P_{xy}$ is strongly-connected by (2), and $(D_1 \cup D_2')/(D_2' \cup P_{xy}) = D'/P_{xy}$ is also strongly connected. Hence we deduce that $D_1 \cup D_2'$ is strongly connected. Therefore, $D_1 \cup D_2'$ is a β-SCO of G, extended from $D'(G/H)$. By Definition 2.5, we have $H \in \mathcal{W}_3$. ■

Let $u_1v, u_2v \in E(G)$. Denote $G_{[v,u_1u_2]}$ to be the graph obtained from $G - \{u_1v, u_2v\}$ by connecting a new edge u_1u_2. The operation of obtaining $G_{[v,u_1u_2]}$ from G is referred as *lifting* the edges u_1v, u_2v in G.

Lemma 2.8. *Let G be a connected graph and let $u_1v, u_2v \in E(G)$.*
(i) If $G_{[v,u_1u_2]} \in \mathcal{S}_3$ ($G_{[v,u_1u_2]} \in \mathcal{W}_3$, resp.), then $G \in \mathcal{S}_3$ ($G \in \mathcal{W}_3$, resp.).
(ii) If $d(v) \geq 4$ and $G_{[v,u_1u_2]} - v \in \mathcal{S}_3$, then $G \in \mathcal{S}_3$.
(iii) Assume that $d(v) \geq 5$ and $E(v) \setminus \{u_1v, u_2v\}$ contains at least two non-parallel edges. If $G_{[v,u_1u_2]} - v \in \mathcal{W}_3$, then $G \in \mathcal{W}_3$.

Proof. It is straightforward to verify (i) by definitions.

(ii) Given $\beta \in Z(G, \mathbb{Z}_3)$, we can orient the edge set $E(v) \setminus \{u_1v, u_2v\}$ to archive the boundary $\beta(v)$ since $d(v) \geq 4$. Delete vertex v and change the boundaries of vertices adjacent to v according to the oriented edges in $E(v) \setminus \{u_1v, u_2v\}$. Let β' be the resulting boundary of $G' = G_{[v,u_1u_2]} - v$. Since $G' \in \mathcal{S}_3$, there exists a β'-SCO of G' with a directed edge u_1u_2. Then in G, orient u_1v, vu_2 to perform as the directed edge u_1u_2 and add the oriented edges $E(v) \setminus \{u_1v, u_2v\}$. This gives a β-SCO of G.

(iii) We shall show that $G + xy \in \mathcal{S}_3$ for any two distinct vertices $x, y \in V(G)$. Hence $G \in \mathcal{W}_3$ by Proposition 2.7. Denote $G' = G_{[v,u_1u_2]} - v$. If $x, y \in V(G')$, then $G' + xy \in \mathcal{S}_3$ by Proposition 2.7. Thus $G + xy \in \mathcal{S}_3$ by Lemma 2.8(ii). Otherwise, we have $y \in V(G')$ and $x = v$. By assumption there exists $zv \in E(v) \setminus \{u_1v, u_2v\}$ with $z \neq y$. Since $G' = G_{[v,u_1u_2]} - v \in \mathcal{W}_3$, we have $G' + yz \in \mathcal{S}_3$. Hence $G_{[x,u_1u_2]} + xy \in \mathcal{S}_3$ by Lemma 2.8(ii), and so $G + xy \in \mathcal{S}_3$ by Lemma 2.8(i). This shows $G \in \mathcal{W}_3$ by Proposition 2.7. ■

3. Contractible graphs and reduced graphs

There are also some $\mathcal{S}_3$-graphs that can be obtained from Hamiltonian properties.

Proposition 3.1. *(i) If G contains a Hamiltonian cycle C such that $G - E(C)$ is $\mathbb{Z}_3$-connected, then $G \in \mathcal{S}_3$.*
(ii) If for any two distinct vertices x, y of a graph H, there is a Hamiltonian path P_{xy} such that $H - E(P_{xy})$ is $\mathbb{Z}_3$-connected, then $H \in \mathcal{W}_3$.

Proof. (i) Let $\beta \in Z(G, \mathbb{Z}_3)$. As $G - E(C)$ is $\mathbb{Z}_3$-connected, there exists a β-orientation of $G - E(C)$. We take a directed Hamiltonian cycle by orienting C. This, together with the β-orientation of $G - E(C)$, provides a β-SCO of G. Hence $G \in \mathcal{S}_3$.

(ii) For any two distinct vertices $x, y \in V(H)$, it follows by (i) that $H + xy \in \mathcal{S}_3$. Therefore, $H \in \mathcal{W}_3$ by Proposition 2.7. ■

Note that in Proposition 3.1(i)–(ii) the conditions of Hamiltonian cycle and Hamiltonian path can be replaced by similar weak conditions of spanning closed trail and spanning trail, respectively.

Denote tK_2 to be the graph obtained from K_2 by adding t parallel edges.

Corollary 3.2. *(i) The graph $tK_2 \in \mathcal{S}_3$ if and only if $t \geq 4$. The complete graph $K_n \in \mathcal{S}_3$ if and only if $n \geq 7$.*
(ii) The graph $tK_2 \in W_3$ if and only if $t \geq 3$. The complete graph $K_n \in \mathcal{W}_3$ if and only if $n \geq 6$.

Proof. This follows by Proposition 3.1. Note that the graph obtained from K_6 by deleting a Hamiltonian path is a Prism graph plus an edge, which is $\mathbb{Z}_3$-connected as shown by Luo et al. in Lemma 2.2 of [15]. ■

Since K_1 is an $\mathcal{S}_3$-graph by definition, each vertex in G lies in a maximal $\mathcal{S}_3$-subgraph. Let $H_1, \ldots, H_k$ be all the maximal $\mathcal{S}_3$-subgraphs of G. Then those $H_1, \ldots, H_k$ are pairwise vertex-disjoint by Proposition 2.4. We denote $G' = G/(\cup_{i=1}^{c} E(H_i))$, called the $\mathcal{S}_3$**-reduction** of G. If $G = G'$, then G is also called an $\mathcal{S}_3$**-reduced graph**. By Proposition 2.4, $G \in \mathcal{S}_3$ if and only if its $\mathcal{S}_3$-reduction is K_1, and $\phi(G) < 3$ if and only if its $\mathcal{S}_3$-reduction G' satisfies $\phi(G') < 3$. This is a standard reduction method for complete families.

However, the graph family $\mathcal{W}_3$ is not a complete family since it does not satisfy (C3) property similarly as in Proposition 2.4. To see this, take a graph G consisting of three vertices u, v, w with 3 parallel edges between u and v, 3 parallel edges between v and w, and no edge between u and w. Then $G/uv \in \mathcal{W}_3$ and $G[\{u, v\}] = 3K_2 \in \mathcal{W}_3$ by Corollary 3.2(ii). But $G \notin \mathcal{W}_3$ as $G + uv \notin \mathcal{S}_3$ and by Proposition 2.7.

In view of Proposition 2.6, we can also use a similar reduction method in certain proper supergraphs. That is, if G contains a proper $\mathcal{W}_3$-subgraph H, then we only need to work on G/H to seek flow index $\phi < 3$ property or $\mathcal{S}_3$-property. By Propositions 2.4 and 2.6, we directly obtain the following lemma.

Lemma 3.3. *If for any edge $e \in E(G)$, there is a subgraph H_e of G containing the edge e such that either $H_e \in \mathcal{S}_3$ or H_e is a proper $\mathcal{W}_3$-subgraph of G, then $G \in \mathcal{S}_3$.*

We also need the following well-known characterization of chordal graphs (cf. [10]).

Lemma 3.4 ([10]). *A simple graph G is chordal if and only if every minimal vertex-cut induces a complete subgraph of G.*

We shall use these lemmas to verify Theorem 3.5, which implies Theorem 1.4 immediately.

Theorem 3.5. *If G is a simple 5-connected chordal graph other than K_6, then $G \in \mathcal{S}_3$.*

Proof. If G is a complete graph, say $G = K_m$, then we have $m \geq 7$, and so $G \in \mathcal{S}_3$ and $\phi(G) < 3$ by Corollary 3.2. Thus we assume G is not a complete graph. In the following, we call a subgraph H *good* for convenience if either $H \in \mathcal{S}_3$ or H is a proper $\mathcal{W}_3$-subgraph of G. Note that any K_6 in G is a proper $\mathcal{W}_3$-subgraph of G since G is 5-connected, and $K_m \in \mathcal{S}_3$ when $m \geq 7$ by Corollary 3.2. Thus any complete subgraph on at least 6 vertices in G is good. Now we are to prove that for every edge $e = uv \in E(G)$ there exists a good subgraph H_e of G containing the edge e. This would imply that $G \in \mathcal{S}_3$ and $\phi(G) < 3$ by Lemma 3.3. Considering two possibilities of $N(u)$ and $N(v)$, we shall show that such good graph H_e can always be found.

We first assume that both $N(u) = V(G) \setminus \{u\}$ and $N(v) = V(G) \setminus \{v\}$. Then there must exist two distinct vertices x, y with $xy \notin E(G)$, since G is not a complete graph. Let $Q \subset N(x)$ be a minimal vertex-cut separating x and y. By Lemma 3.4, $G[Q]$ is a complete graph on at least 5 vertices. Since $N(u) = V(G) \setminus \{u\}$ and $N(v) = V(G) \setminus \{v\}$, we know that $u, v \in Q$, and so let $H_e = G[Q \cup \{x\}]$. Then H_e is a complete graph on at least 6 vertices, which is a good subgraph containing uv.

Otherwise, assume that either $N(u) \neq V(G) \setminus \{u\}$ or $N(v) \neq V(G) \setminus \{v\}$. By symmetry, we assume $N(u) \neq V(G) \setminus \{u\}$ and there exists a vertex w with $uw \notin E(G)$. Thus a minimal vertex-cut R separating u and w is contained in $N(u)$. By Lemma 3.4, $G[R \cup \{u\}]$ is a complete graph on at least 6 vertices. If $v \in R$, we choose $H_e = G[R \cup \{u\}]$ as a good subgraph by Corollary 3.2(i)–(ii). Hence assume that $v \notin R$ for any minimal vertex R contained in $N(u)$ which separates u and w. Now we further claim $N(v) \subseteq N(u) \cup \{u\}$. Otherwise, there exists a vertex z with $vz \in E(G)$ and $uz \notin E(G)$. Thus $N(u)$ contains a minimal vertex-cut separating u, w and containing the vertex v, which is a contradiction. This shows that $N(v) \subseteq N(u) \cup \{u\}$. Hence a minimal vertex-cut S is contained in $N(v)$, which separates v and w. By Lemma 3.4, $G[S \cup \{v\}]$ is a complete graph on at least 6 vertices. If $u \in S$, we choose $H_e = G[S \cup \{v\}] \in \mathcal{W}_3$ to be a good subgraph. If $u \notin Y$, then $H_e = G[S \cup \{u, v\}]$ is a complete graph on at least 7 vertices, which is a good subgraph.

Therefore, for any edge $e = uv$, a good subgraph H_e containing e can be found in any case. Hence $G \in \mathcal{S}_3$ by Lemma 3.3. ∎

4. Graphs with few odd vertices

A simple observation in [11] shows that an $\mathcal{S}_3$-graph cannot be too sparse, and similar property follows for a $\mathcal{W}_3$-graph by Proposition 2.7.

Lemma 4.1. *(i) [11] Let G be an $\mathcal{S}_3$-graph on n vertices. Then $|E(G)| \geq 3n - 2$.*
(ii) Let G be a $\mathcal{W}_3$-graph on n vertices. Then $|E(G)| \geq 3n - 3$.

Proof. By Proposition 2.7 and by (i), we immediately have that $|E(G)| \geq 3n - 3$ for any $\mathcal{W}_3$-graph G on n vertices. ∎

A graph G is called $\mathcal{W}_3$**-reduced** if for any $H \subseteq G$ with $|V(H)| > 1$, we have $H \notin \mathcal{W}_3$. Let $m(n, \mathcal{W}_3)$ denote the maximum number of edges in a $\mathcal{W}_3$-reduced graph on n vertices. Some bounds of $m(n, \mathcal{W}_3)$, when n is small, would be very helpful in later proofs.

Lemma 4.2. *We have $m(2, \mathcal{W}_3) = 2$, $m(3, \mathcal{W}_3) = 5$, $m(4, \mathcal{W}_3) = 8$, $m(5, \mathcal{W}_3) \leq 12$, and $m(6, \mathcal{W}_3) \leq 17$.*

Proof. Let G_3 be a $\mathcal{W}_3$-reduced graph with $|V(G)| = 3$ and with maximum number of edges. By Corollary 3.2(ii), the edge multiplicity is at most 2 in G_3. If $G_3 = 2C_3$, then by Proposition 3.1(ii), we have $G_3 \in \mathcal{W}_3$, a contradiction. Thus $|E(G_3)| = 5$ and $m(3, \mathcal{W}_3) = 5$.

Now we are to prove $m(4, \mathcal{W}_3) = 8$. By contradiction, suppose that G_4 is a $\mathcal{W}_3$-reduced graph on 4 vertices with $|E(G_4)| = 9$. Since $m(3, \mathcal{W}_3) = 5$ and deleting any vertex of G_4 still results in a $\mathcal{W}_3$-reduced graph, we have $\delta(G_4) = 4$. Thus we have $G_4 \in \{G_4^1, G_4^2, G_4^3\}$ as in Fig. 1. By applying Proposition 2.6(ii), adding any edge to each of those graphs results in an $\mathcal{S}_3$-graph. Hence $G_4 \in \mathcal{W}_3$ by Proposition 2.7, a contradiction. This proves $m(4, \mathcal{W}_3) = 8$.

Since $3K_2 \in \mathcal{W}_3$ and $2C_3 \in \mathcal{W}_3$, the following graphs in Fig. 2 are all the $\mathcal{W}_3$-reduced graphs with 4 vertices and 8 edges.

Now we are to prove $m(5, \mathcal{W}_3) \leq 12$. By contradiction, suppose that G_5 is a $\mathcal{W}_3$-reduced graph on 5 vertices with $|E(G_5)| = 13$. Thus $\delta(G_5) \leq 5$ and denote $d(v) = \delta(G_5)$. Since $m(4, \mathcal{W}_3) = 8$ and $G_5 - v$ is a $\mathcal{W}_3$-reduced graph, we have $d(v) = 5$. Moreover, $G_5 - v$ is isomorphic to one of the graphs in Fig. 2, where x, y are two specified vertices. Since $\delta(G_5) = 5$, we have $vx, vy \in E(G_5)$. Obtain a graph G' from G_5 by lifting two edges xv, vy and deleting the vertex v. Thus $|E(G')| = 9$ and $G' \in \{G_4^1, G_4^2, G_4^3\}$, which implies $G' \in \mathcal{W}_3$. By Lemma 2.8(iii), we have $G_5 \in \mathcal{W}_3$, a contradiction.

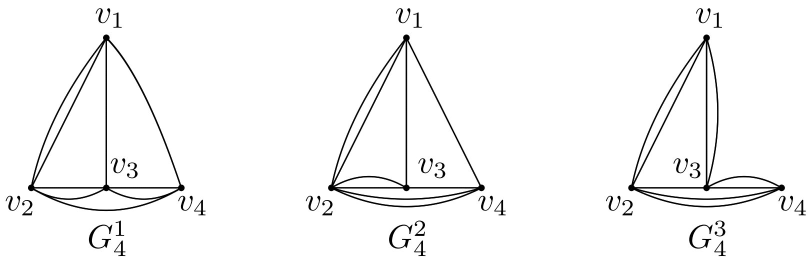

Fig. 1. The graphs G_4^1, G_4^2, G_4^3.

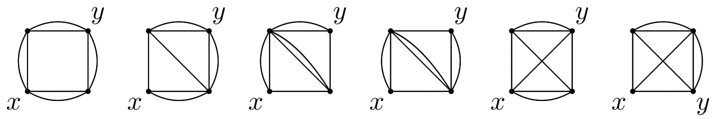

Fig. 2. All the $\mathcal{W}_3$-reduced graphs with 4 vertices and 8 edges.

Now we show that $m(6, \mathcal{W}_3) \le 17$. Suppose, for contradiction, that G_6 is a $\mathcal{W}_3$-reduced graph on 6 vertices with $|E(G_6)| = 18$. Thus $\delta(G_6) \le 6$. Denote $d(v) = \delta(G_6)$. Then $G - v$ is a $\mathcal{W}_3$-reduced graph with $18 - \delta(G_6) \ge 12$ edges. Since $m(5, \mathcal{W}_3) \le 12$, we have $\delta(G_6) = 6$, and so G_6 is 6-regular. Moreover, $G - v$ has 12 edges, and hence it contains a vertex, say u, of degree 4. $G - v - u$ is a $\mathcal{W}_3$-reduced graph on 4 vertices with 8 edges, which is isomorphic to one of the graphs in Fig. 2. Since G_6 is 6-regular, each of u and v is connected to $G - v - u$ with four edges. As G_6 is a $\mathcal{W}_3$-reduced graph which contains no $3K_2$ nor $2C_3$, we can check that either $ux, uy \in E(G_6)$ or $vx, vy \in E(G_6)$. By symmetry, we assume $ux, uy \in E(G_6)$. Obtain a graph G' from G_6 by lifting two edges xu, uy and deleting the vertex u. Then $G' - v \in \{G_4^1, G_4^2, G_4^3\}$, which implies $G' - v \in \mathcal{W}_3$. Clearly, G' is a proper supergraph of $G' - v$. As $G'/(G' - v) = 4K_2 \in \mathcal{S}_3$ and by Proposition 2.6, we have $G' \in \mathcal{S}_3$. Thus $G_6 \in \mathcal{S}_3$ by Lemma 2.8(ii), leading to a contradiction. ■

Theorem 4.3 (*Hakimi [4]*)**.** *For a graph G, let $\ell : V(G) \mapsto \mathbb{Z}$ be a function with $\sum_{x \in V(G)} \ell(x) = 0$ such that $\ell(u) \equiv d_G(u) \pmod 2$ for any $u \in V(G)$. Then the following statements are equivalent.*
(i) There is an orientation D of G such that $d_D^+(v) - d_D^-(v) = \ell(v)$ for any $v \in V(G)$.
(ii) For any $S \subset V(G)$, $|\sum_{v \in S} \ell(v)| \le |\partial_G(S)|$.

By modifying Hakimi Theorem above, we obtain the following useful tool immediately.

Theorem 4.4. *For a graph G, let $\ell : V(G) \mapsto \mathbb{Z}$ be a function with $\sum_{v \in V(G)} \ell(v) = 0$ such that $\ell(v) \equiv d_G(v) \pmod 2$, $\forall v \in V(G)$. Then G has a strongly connected orientation D such that $d_D^+(v) - d_D^-(v) = \ell(v)$, $\forall v \in V(G)$ if and only if*

$$|\sum_{v \in S} \ell(v)| < |\partial_G(S)|, \forall S \subset V(G).$$

Proof. "$\Rightarrow$" Let D be a strongly connected orientation of the graph G such that $d_D^+(v) - d_D^-(v) = \ell(v)$, $\forall v \in V(G)$. Thus any vertex set $S \subset V(G)$ satisfies $\partial_D^+(S) > 0$ and $\partial_D^-(S) > 0$, which implies that $|\sum_{v \in S} \ell(v)| = |\partial_D^+(S) - \partial_D^-(S)| < |\partial_G(S)|$.

"$\Leftarrow$" By Theorem 4.3, there exists an orientation D such that $d_D^+(v) - d_D^-(v) = \ell(v)$, $\forall v \in V(G)$. Furthermore, for any set $S \subset V(G)$, since $|\sum_{v \in S} \ell(v)| = |\partial_D^+(S) - \partial_D^-(S)| < |\partial_G(S)|$, we have both $\partial_D^+(S) > 0$ and $\partial_D^-(S) > 0$. Thus D is a strongly connected orientation. ■

Recall that $G_{[v,u_1u_2]}$ denotes the graph obtained from $G - \{u_1v, u_2v\}$ by adding a new edge u_1u_2. A graph is *odd-*$(2t+1)$*-edge-connected* if the smallest odd-edge-cut has cardinality at least $2t + 1$.

Lemma 4.5 (*Zhang [19]*)**.** *Let G be an odd-$(2k+1)$-edge-connected graph, and let $x \in V(G)$ with $d(x) \notin \{2, 2k+1\}$. Then there exist two edges $u_1x, u_2x \in E_G(x)$ such that $G_{[v,u_1u_2]}$ is odd-$(2k+1)$-edge-connected.*

We shall apply the above lemmas to prove the following stronger version of Theorem 1.3(i).

Theorem 4.6. *(i) For every odd-7-edge-connected graph G with at most 12 vertices of odd degree, we have $\phi(G) < 3$.*
(ii) For every odd-7-edge-connected graph G with $|V(G)| \le 13$, we have $\phi(G) < 3$.

Proof. Clearly, (ii) follows from (i) since a graph G with $|V(G)| \le 13$ has at most 12 vertices of odd degree. It remains to prove (i). Let G be a counterexample with $|E(G)| + |V(G)|$ as small as possible. Assume that there exists a vertex $v \in V(G)$

with $d(v) \neq 7$ or $d(v)$ is even. By Lemma 4.5, there exists a pair of edges $u_1v, u_2v \in E(G)$ such that $G_{[v,u_1u_2]}$ remains odd-edge-connectivity 7. By minimality, $\phi(G_{[v,u_1u_2]}) < 3$ and we could extend the strongly connected modulo 3-orientation of $G_{[v,u_1u_2]}$ to the graph G, and so $\phi(G) < 3$, which is a contradiction. Therefore, we may assume that G is a 7-regular graph and $|V(G)|$ is even.

Let $[U, W]$ be the maximal edge-cut of G with $|U| \leq |W|$. This implies that for any $x \in U$ and any $y \in W$, we have $|[x, W]| \geq 4$ and $|[y, U]| \geq 4$. Thus

$$4|W| \leq |[U, W]| = 7|U| - 2|E(G[U])| \leq 7|U|. \tag{3}$$

By Theorem 4.4, we must have that for any function $\ell : V(G) \mapsto \{3, -3\}$ with $\sum_{v\in V(G)} \ell(v) = 0$, there exists an $S \subset V(G)$ such that

$$|S| \leq \frac{|V(G)|}{2} \text{ and } |\sum_{v\in S} \ell(v)| \geq |\partial_G(S)|. \tag{4}$$

Denote $S \cap U = S_U$ and $S \cap W = S_W$. By (4), we have

$$S_U \neq \emptyset \text{ and } S_W \neq \emptyset. \tag{5}$$

We first show that $G[S]$ is a $\mathcal{W}_3$-reduced graph. Otherwise, let $H \in \mathcal{W}_3$ be a nontrivial subgraph graph of $G[S]$. Clearly, G is a proper supergraph of H since G is 2-connected. By minimality, $\phi(G/H) < 3$, and it follows from Proposition 2.6(i) that $\phi(G) < 3$, a contradiction. This shows that $G[S]$ is a $\mathcal{W}_3$-reduced graph, and so $|E(G[S])| \leq m(|S|, \mathcal{W}_3)$. Hence $|\partial_G(S)| = 7|S| - 2|E(G[S])| \geq 7|S| - 2m(|S|, \mathcal{W}_3)$, and it follows from Lemma 4.2 that

$$|\partial_G(S)| \geq \begin{cases} 7 & \text{if } |S| = 1, \\ 10 & \text{if } |S| = 2, \\ 11 & \text{if } |S| = 3, \end{cases} \quad \text{and} \quad |\partial_G(S)| \geq \begin{cases} 12 & \text{if } |S| = 4, \\ 11 & \text{if } |S| = 5, \\ 8 & \text{if } |S| = 6. \end{cases} \tag{6}$$

By (4) and (6), we must have $4 \leq |S| \leq \frac{1}{2}|V(G)|$.

Assume that $|V(G)| = 8$. By (3), we have $|U| = 3$, $|W| = 5$ or $|U| = |W| = 4$. Define a function ℓ such that four vertices in W having value 3, and the rest vertices of G having value -3. Then there is an $S \subset V(G)$ satisfying (4) and (5). By (6), we have $|S| = 4$ and $|\sum_{v\in S} \ell(v)| = 12$. This implies that $S \subset W$ or $(V(G) \setminus S) \subset W$, a contradiction to (5). Therefore, we must have $|V(G)| = 10$ or 12.

Case 1: $|V(G)| = 10$.

By (3), we have $4(10 - |U|) = 4|W| \leq |[U, W]| \leq 7|U|$, which shows $4 \leq |U| \leq 5$.

Subcase 1.1: $|U| = |W| = 5$.

Set $\ell(v) = 3$ for every $v \in U$ and $\ell(w) = -3$ for every $w \in W$. If $|S| = 4$, by (4) and (6), we have $|\sum_{v\in S} \ell(v)| = 12$, and so $S \subset U$ or $S \subset W$, a contradiction to (5). If $|S| = 5$, by (5), we have $|S_U| \neq 0$ and $|S_W| \neq 0$. Hence $|\sum_{v\in S} \ell(v)| \leq 9 < 11 \leq |\partial_G(S)|$ by (6), which is a contradiction to (4).

Subcase 1.2: $|U| = 4$ and $|W| = 6$.

By (3), there is a vertex $w_1 \in W$ such that $|[w_1, U]| = 4$. Otherwise, we have $30 = 5|W| \leq |[U, W]| \leq 7|U| = 28$, which is a contradiction. Set $\ell(w_1) = -3$, $\ell(w) = 3$ for every $w \in W \setminus \{w_1\}$ and $\ell(v) = -3$ for every $v \in U$.

If $|S| = 4$, then $|S_U| = 3$ and $S_W = \{w_1\}$, by (4) and (5). Thus we have $|E(G[U])| \leq 2$ by (3). Hence $|\partial_G(S)| = 7|S| - 2|E(G[S])| = 28 - 2(|[w_1, U]| + |E(G[U])|) \geq 28 - 2(4 + 2) = 16 > |\sum_{v\in S} \ell(v)| = 12$, contrary to (4).

If $|S| = 5$, then we have $|\sum_{v\in S} \ell(v)| \geq |\partial_G(S)| \geq 11$ by (4) and (6). Thus $|\sum_{v\in S} \ell(v)| = 15$, and so $S = U \cup \{w_1\}$ or $S = W - \{w_1\}$. By (5), we must have $S = U \cup \{w_1\}$. But now $|\partial_G(S)| = 7|S| - 2|E(G[S])| \geq 35 - 2(|[w_1, U]| + |E(G[U])|) \geq 35 - 2(4 + 2) = 23 > 15 = |\sum_{v\in S} \ell(v)|$, which is a contradiction to (4).

Case 2: $|V(G)| = 12$

In this case we have $5 \leq |U| \leq 6$ by (3).

Subcase 2.1: $|U| = 5$ and $|W| = 7$.

Denote w_1 to be the vertex such that $|[w, U]|$ is minimized among all $w \in W$. Since $\frac{|[U,W]|}{|W|} \leq \frac{7|U|}{|W|} = 5$, we have $|[w_1, U]| \leq 5$. Set $\ell(v) = 3$ for every $v \in U$, $\ell(w_1) = 3$ and $\ell(u) = -3$ for every $u \in W \setminus \{w_1\}$. Applying (3) again, we have the following holds: *If* $|[w_1, U]| \leq 4$, *then we have* $|E(G[U])| \leq 3$; *otherwise* $|[w_1, U]| = 5$, *then* $|E(G[U])| = 0$ *and* $|[w, U]| = 5$ *for every* $w \in W$. Thus in any case we have

$$|[w_1, U]| + |E(G[U])| \leq 7. \tag{7}$$

If $|S| = 4$ or 5, then $|S_U| = |S| - 1$ and $S_W = \{w_1\}$, by (4) and (5). By (7), we have

$$|\partial_G(S)| = 7|S| - 2|E(G[S])| \geq 28 - 2(|[w_1, U]| + |E(G[U])|) = 7|S| - 2(7) \geq 3|S| + 2 > |\sum_{v\in S} \ell(v)|,$$

a contradiction to (4).

If $|S| = 6$, then $|\sum_{v\in S} \ell(v)| \geq |\partial_G(S)| \geq 8$ by (6). Thus $|\sum_{v\in S} \ell(v)| = 12$ or 18. By (5), we must have $|\sum_{v\in S} \ell(v)| = 12$, which implies $|S_U| = 4$, $|S_W| = 2$ and $w_1 \in S_W$.

When $|[w_1, U]| = 4$, we have $|\partial_G(S)| = 7|S| - 2|E(G[S])| \geq 42 - 2(|E(G[U])| + |[w_1, U]| + 7) \geq 42 - 2(3 + 4 + 7) = 14 > \sum_{v\in S} \ell(v) = 12$, contrary to (4). When $|[w_1, U]| = 5$, we have $|\partial_G(S)| = 7|S| - 2|E(S)| \geq 42 - 2(|E(G[U])| + 5 + 5) = 42 - 20 = 22 > \sum_{v\in S} \ell(v) = 12$, again a contradiction to (4).

Subcase 2.2: $|X| = |Y| = 6$.

In this subcase, set $\ell(v) = 3$ for every $v \in U$ and $\ell(w) = -3$ for every $w \in W$. By (5), we have $|\sum_{v\in S} \ell(v)| \leq 3|S| - 6$. Thus $|S| > 5$ by (6), and so $|S| = 6$. Then either $|S_U| = 5$, $|S_W| = 1$ or $|S_U| = 4$, $|S_W| = 2$. In the former case, we have $|\partial_G(S)| \geq 4|S_U| - 7 = 13 > |\sum_{v\in S} \ell(v)| = 12$, contradicting (4). In the later case, $|\partial_G(S)| \geq 8 > |\sum_{v\in S} \ell(v)| = 6$ by (6), again a contradiction. The proof is completed. ■

5. Graphs with small independence number

We shall adopt similar ideas as in [5,12] to study properties of reduced graphs and to prove Theorem 1.3(ii). Let G be a graph and $\beta \in Z(G, \mathbb{Z}_3)$. For a vertex set $A \subset V(G)$, denote $\beta(A) \equiv \sum_{v\in A} \beta(v) \pmod 3$ and $d(A) = |\partial_G(A)|$. Define an integer-valued mapping $\tau : 2^{V(G)} \mapsto \{0, \pm 1, \pm 2, \pm 3\}$ as follows: for each vertex set $A \subset V(G)$,

$$\tau(A) \equiv \begin{cases} \beta(A) \pmod 3; \\ d(A) \pmod 2. \end{cases}$$

Theorem 5.1 ([13]). *Let G be a graph with $\beta \in Z(G, \mathbb{Z}_3)$ and $z_0 \in V(G)$. Denote D_{z_0} to be a pre-orientation of $E(z_0)$. Assume that*
(i) $|V(G)| \geq 3$;
(ii) $d(z_0) \leq 4 + |\tau(z_0)|$ and $d^+_{D_{z_0}}(z_0) - d^-_{D_{z_0}}(z_0) \equiv \beta(z_0) \pmod 3$ under the orientation D_{z_0};
(iii) $d(A) \geq 6 + |\tau(A)|$ for each vertex subset A not containing z_0 with $1 \leq |A| \leq |V(G)| - 2$.
Then the pre-orientation D_{z_0} of $E(z_0)$ can be extended to a β-orientation D of G such that $D(G - z_0)$ is strongly connected.

For a 8-edge-connected graph G, the fact of $d(A) \geq 8$ implies $d(A) \geq 6 + |\tau(A)|$ for each $A \subset V(G)$. Thus Theorem 5.1 implies Theorem 1.2 immediately and it also shows the following stronger theorem.

Theorem 5.2 ([13]). *Let G be a 8-edge-connected graph. Then $G \in \mathcal{S}_3$.*

Now we prove the main result of this section.

Theorem 5.3. *Let t be an integer with $t \geq 2$. The following are equivalent.*
(i) For every odd-7-edge-connected graph G with $\alpha(G) \leq t$, the flow index $\phi(G) < 3$.
(ii) For every odd-7-edge-connected graph G with $\alpha(G) \leq t$ and $|V(G)| \leq 8t - 3$, the flow index $\phi(G) < 3$.

Proof. It suffices to show that "(ii)⇒(i)". Let G be a counterexample to (i) with $|V(G)|$ minimized. If $|V(G)| \leq 8t - 3$, then we are done by (ii). If $|V(G)| > 8t - 3$, we shall derive a contradiction below.

Firstly, G **contains no** K_6. If G contains a subgraph $H \cong K_6$, then, clearly, G/H is still odd-7-edge-connected and $\alpha(G/H) \leq t$. By the minimality of G, we have $\phi(G/H) < 3$. Since G is odd-7-edge-connected, G is a proper supergraph of H. Since $\phi(G/H) < 3$ and by Proposition 2.6(ii) and Corollary 3.2(ii), we have $\phi(G) < 3$, a contradiction.

Secondly, G **is an** $\mathcal{S}_3$**-reduced graph.** Assume not, and let $H \in \mathcal{S}_3$ be a subgraph of G on at least two vertices. Then $\phi(G/H) < 3$ by minimality. Hence we have $\phi(G) < 3$ by Proposition 2.4, a contradiction. Note that any subgraph of G is also an $\mathcal{S}_3$-reduced graph.

Thirdly, G **and each subgraph** H **of** G **have minimal degree at most** 7. By contradiction, suppose that H is an $\mathcal{S}_3$-reduced graph with $\delta(H) \geq 8$. By Theorem 5.2, H is not 8-edge-connected. Among all the edge-cuts $|\partial_H(S)| \leq 7$, choose the one with $|S|$ as small as possible. Denote $S^c = V(H) \setminus S$. Let z_0 be the contracted vertex which S^c corresponds to in $H/H[S^c]$. Let H' be the graph obtained from $H/H[S^c]$ by adding $7 - d_{H/H[S^c]}(v)$ edges between z_0 and S. Now we prove that $H' - z_0 = H[S] \in \mathcal{S}_3$ by Theorem 5.1. Define $\beta(z_0) = 3$. By $d_{H'}(z_0) = 7$, we obtain $\tau(z_0) = 3$. Orient the edges in $E_{H'}(z_0)$ with the orientation D_{z_0} such that $d^+_{D_{z_0}}(z_0) = 5$ and $d^-_{D_{z_0}}(z_0) = 2$. For any $\beta' \in Z(H' - z_0, \mathbb{Z}_3)$, define $b(v) = d^+_{D_{z_0}}(v) - d^-_{D_{z_0}}(v)$ for every $v \in N_{H'}(z_0)$ and

$$\beta(x) = \begin{cases} \beta'(x) + b(x), & x \in N_{H'}(z_0), \\ \beta(z_0), & x = z_0, \\ \beta'(x), & \text{otherwise}. \end{cases}$$

Thus we have $\beta \in Z(H', \mathbb{Z}_3)$. For any $A \subset V(H')$ with $|V(H') \setminus A| > 1$, we have $d(A) \geq 6 + |\tau(A)|$ since $d(A) \geq 8$. By Theorem 5.1, D_{z_0} can be extended to an orientation D of H' which agrees the boundary β and satisfies $H' - z_0$ is strongly connected under D. Let D' be the restriction of D on $H' - z_0$. Thus D' is a β'-SCO of $H' - z_0 = H[S]$. Hence $H[S] \in \mathcal{S}_3$, a contradiction to the fact that H is an $\mathcal{S}_3$-reduced graph.

Finally, we are ready to derive a contradiction. Let x_1 be a minimal degree vertex in $G = G_1$. Then we have $d_{G_1}(x_1) \leq 7$ by the third statement. Delete x_1 and all of its neighbors to obtain a graph G_2. Then $\alpha(G_2) \leq t - 1$. Otherwise, we obtain an independent set of size $t + 1$ in G from the vertex x_1 and an independent set of size t in G_2, contradicting to $\alpha(G) \leq t$. Thus there is a minimal degree vertex x_2 in G_2 with $d_{G_2}(x_2) \leq 7$ by the third statement. Delete x_2 and all of its neighbors to obtain a graph G_3. Keep on this process until the resulting graph G_s satisfies $\alpha(G_s) = 1$, where $s \leq t$. Since G has no K_6, G_s contains at most 5 vertices. Thus G has at most $8(s - 1) + 5 \leq 8t - 3$ vertices, a contradiction to (ii). This proves the theorem. ■

Note that Theorem 1.3(i) is a special case of Theorem 4.6(i), and Theorem 1.3(ii) follows from Theorems 5.3 and 4.6(ii). In fact, we obtain the following stronger version of Theorem 1.3(ii) concerning odd edge-connectivity.

Corollary 5.4. *For every odd-7-edge-connected graph G with $\alpha(G) \leq 2$, the flow index $\phi(G) < 3$.*

CRediT authorship contribution statement

Miaomiao Han: Conceptualization, Methodology, Formal analysis, Visualization, Writing - original draft. **Hong-Jian Lai:** Methodology, Supervision, Writing - review & editing, Project administration. **Jiaao Li:** Methodology, Formal analysis, Writing - original draft. **Yezhou Wu:** Conceptualization, Visualization, Writing - review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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