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On the extremal sizes of maximal graphs without $(k+1)$-connected subgraphs

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ABSTRACT

Let G be a graph and let $\kappa(G)$ be the vertex-connectivity of G. The maximum subgraph connectivity of G is $\overline{\kappa}(G) = \max\{\kappa(H) : H \subseteq G\}$. A simple graph G is vertex-k-maximal if $\overline{\kappa}(G) \leq k$, but for any $e \in E(G^c)$, $\overline{\kappa}(G+e) \geq k+1$. Mader conjectured that every vertex-k-maximal simple graph of order n satisfies $|E(G)| \leq \frac{3}{2}(k-\frac{1}{3})(n-k)$. We prove the following.

(i) Every vertex-k-maximal simple graph of order n satisfies $|E(G)| \geq (n-k)k + \frac{k(k-1)}{2}$. This lower bound is best possible.

(ii) For every integer m in the range $2n-3 \leq m \leq 5n/2 - \frac{21+(-1)^{n+1}}{4}$ there exists a vertex 2-maximal graph of order n with m edges.

1. Introduction

Throughout this paper, we consider finite simple graphs. Undefined terms and notation will follow [5]. In particular, $\kappa(G)$ denotes the connectivity of a graph G, and G^c is the complement of G. We use $H \subseteq G$ to mean that H is a subgraph of G. If $X \subseteq E(G^c)$, then $G+X$ is the simple graph with vertex set $V(G)$ and edge set $E(G) \cup X$. We will use $G+e$ for $G+\{e\}$. If $W \subseteq V(G)$ or if $W \subseteq E(G)$, then $G[W]$ denotes the subdigraph of G induced by W. For $v \in V(G)$, define $G-v = G[V(G)-v]$, and

$$N_G(v) = \{u \in V(G) : uv \in E(G)\} \text{ and } E_G(v) = \{e \in E(G) : \exists u \in N_G(u), e = uv\}.$$

Matula [16] first explicitly studied the quantity $\overline{\kappa}(G) = max\{\kappa(H) : H \subseteq G\}$. For an integer $k > 0$, a simple graph G with $|V(G)| \geq k+1$ is **vertex** k**-maximal** if $\overline{\kappa}(G) \leq k$ but for any edge $e \in E(G^c)$, $\overline{\kappa}(G+e) > k$. By definition, a vertex k-maximal graph on $n = k+1$ vertices must be K_{k+1}. Mader [11] constructed an infinite family of graph without $(k+1)$-connected subgraphs and with a large number of edges.

Example 1.1 (*Mader [11]*). Let k, n, q and r be nonnegative integers with $n = kq + r$ and $0 \leq r \leq k$. Let $G_{n,k}$ be a graph with vertex set $\cup_{i=0}^{q} V_i$, where $V_0, \ldots, V_q$ are pairwise disjoint vertex subsets satisfying each of the following:
(a) $|V_0| = \cdots = |V_{q-1}| = k$, while $|V_q| = r$.
(b) V_0 is an independent set in $G_{n,k}$; and for $1 \leq i \leq q$, V_i is a clique in $G_{n,k}$.
(c) Every vertex in V_0 is adjacent to every vertex in $\cup_{i=1}^{q} V_i$, and $G_{n,k}$ has no other edges.

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Then each of the following holds.
(i) V_0 is a vertex cut with $|V_0| = k$, and every component of $G_{n,k} - V_0$ has at most k vertices.
(ii) $G_{n,k}$ is a vertex k-maximal graph.
(iii) $|E(G_{n,k})| \le \frac{3}{2}(k - \frac{1}{3})(n - k)$, where the equality holds if $r = 0$.

Mader in [11] investigated the extremal size a vertex-k-maximal graph on n vertices may have, and he conjectured that, for large order of graphs, the graphs in Example 1.1 would in fact present the best possible upper bound for the size of a vertex k-maximal graph.

Conjecture 1.2 (*Mader [11]*)**.** *Let $k \ge 2$ be an integer. Then for sufficiently large n, every vertex k-maximal graph on n vertices satisfies $|E(G)| \le \frac{3}{2}(k - \frac{1}{3})(n - k)$.*

There have been some progress towards Conjecture 1.2. To the best of our knowledge, there has been little studies on the characterizations of the extremal graphs for the cases that the conjecture is proved. Apparently the structures of these extremal graphs are quite elusive and not easy to be determined.

Theorem 1.3. *Let $k > 0$ be an integer, and G be a vertex k-maximal simple graph of order n.*
(i) (Mader [10], see also [11]) Conjecture 1.2 holds for $k \le 6$.
(ii) (Mader [10], see also [11]) For sufficiently large n, every vertex k-maximal graph on n vertices satisfies $|E(G)| \le (1 + \frac{1}{\sqrt{2}})k(n - k)$.
(iii) (Yuster [18]) If $n \ge \frac{9k}{4}$, then every vertex k-maximal graph on n vertices satisfies $|E(G)| \le \frac{193}{120}k(n - k)$.
(iv) (Bernshteyn and Kostochka [4]) If $n \ge \frac{5k}{2}$, then every vertex k-maximal graph on n vertices satisfies $|E(G)| \le \frac{19}{12}k(n - k)$.

The related studies on the maximum subgraph edge-connectivity and its extremal problems related to edge k-maximal graphs have been conducted by quite a few researchers, as seen in [7,9–15,17], among others. The corresponding digraph problems have been investigated recently, which can be found in [1–3,8], among others.

These motivate the current research. The objective of this study is to determine the best possible lower bound of the extremal size a vertex-k-maximal graph on n vertices. A main result of this paper is the following.

Theorem 1.4. *Let k, n be positive integers with $n \ge k + 1$, and let G be a simple graph on n vertices. Each of the following holds.*
(i) If G is vertex-k-maximal, then $|E(G)| \ge (n - k)k + \frac{k(k-1)}{2}$.
(ii) This lower bound is best possible in the sense that there exists an infinite family of vertex-k-maximal simple graphs on n vertices with $|E(G)| = (n - k)k + \frac{k(k-1)}{2}$.

Characterizing the extremal vertex k-maximal graphs reaching the upper or lower bound seems to be a difficult problem for a generic value of k. While the upper bound of vertex 2-maximal graphs has been obtained by Mader in [10], (see also [11]), a question to be answered is, what are the vertex 2-maximal graphs whose sizes attain the upper bound? Another objective of this research is to determine all possible integer values which can be the size of a vertex 2-maximal graphs, and to characterize the extremal vertex 2-maximal graphs. In particular, the following is proved in this paper.

Theorem 1.5. *Let m and n be integers with $n \ge 3$. Then there exists a vertex 2-maximal graph G of order n such that $m = |E(G)|$ if and only if $2n - 3 \le m \le \frac{5n}{2} - \frac{21+(-1)^{n+1}}{4}$. Moreover, all vertex 2-maximal graphs on n vertices with $\frac{5n}{2} - \frac{21+(-1)^{n+1}}{4}$ edges are recursively characterized.*

In Section 2, we will prove Theorem 1.4. Section 3 will be devoted to the discussion of the extremal structures of vertex 2-maximal graphs and the proof of Theorem 1.5.

2. The lower bound of sizes of vertex-k-maximal graphs

Throughout this section, we assume that k is an integer with $k \ge 2$. As in [5], a **clique** in a graph G is a set of mutually adjacent vertices. A clique of size k is often referred to as a k-clique. A vertex cut X of a graph G is a **peripheral cut** if there exists a vertex $v \in V(G)$ such that $X = N_G(v)$. We adopt the following definition of a k-tree in [6].

Definition 2.1. For any integer $k > 0$, we define the family of k-tree, denoted by $\mathcal{T}^k$, to be the graph family that contains the complete graph K_k, such that for $n \ge k$, a graph G on $n+1$ vertices is in $\mathcal{T}^k$ if and only if there exists a vertex $v \in V(G)$ satisfying both of the following:
(T1) $G - v \in \mathcal{T}^k$, and
(T2) $N_G(v)$ is a clique in $G - v$.

We often use $T_{k,n}$ to denote a generic k-tree on n vertices. In particular, $T_{k,k} = K_k$ and $T_{k,k+1} = K_{k+1}$.

Lemma 2.2. *Let k, n be integers with $n \ge k + 1 > 2$. Every k-tree $T_{k,n}$ is a vertex k-maximal graph.*

Proof. We adopt the notation in Definition 2.1. It is routine to argue by induction by using Definition 2.1 to show that for any $T_{k,n} \in \mathcal{T}^k$,

$$\kappa(T_{k,n}) = k, \text{ and every minimum vertex-cut of } T_{k,n} \text{ is peripheral and is a } k\text{-clique.} \tag{1}$$

Claim 1. $\overline{\kappa}(T_{k,n}) = k$.

Claim 1 holds trivially if $n = k+1$ as $T_{k,k+1} = K_{k+1}$. Assume that $n \geq k+2$ and Claim 1 holds for smaller values of n. Let H be a subgraph of $T_{k,n}$ such that $\kappa(H) = \overline{\kappa}(T_{k,n})$. If $\kappa(H) = k$, then done. Assume that $\kappa(H) \geq k+1$. By (1), $T_{k,n}$ contains a peripheral cut $N_{T_{k,n}}(v)$ for some $v \in V(T_{k,n})$. Since $\kappa(H) \geq k+1$, $v \notin V(H)$. Hence H is a subgraph of $T_{k,n-1} = T_{k,n} - v$. By induction, $\kappa(H) \leq \overline{\kappa}(T_{k,n-1}) \leq k$, contrary to the assumption that $\kappa(H) \geq k+1$. This justifies the claim.

By Claim 1, it remains to show that for any $e = xy \in E((T_{k,n})^c)$, $\overline{\kappa}(T_{k,n}+e) \geq k+1$. As $T_{k,k+1} = K_{k+1}$, we may assume that $n \geq k+2$ and Lemma 2.2 holds for smaller values of n. By (1), $T_{k,n}$ contains a peripheral cut $N_{T_{k,n}}(u)$ for some $u \in V(T_{k,n})$. We again denote $T_{k,n-1} = T_{k,n} - u$. Let $e = xy \in E((T_{k,n})^c)$.

If $xy \in E((T_{k,n-1})^c)$, then by induction, $T_{k,n-1} + xy$ contains a $(k+1)$-connected subgraph, and so Lemma 2.2 holds. Hence we assume that $x = u$ and $y = v \in V(T_{k,n-1})$. If $\kappa(T_{k,n} + e) \geq k+1$, then $\overline{\kappa}(T_{k,n} + e) \geq \kappa(T_{k,n} + e) \geq k+1$, and so we are done. Assume, by (1), that $\kappa(T_{k,n} + e) = k$, and $T_{k,n} + e$ has a vertex k-cut X. Then X is also a vertex cut of $T_{k,n}$. By (1), there exists a vertex $w \in V(T_{k,n})$ such that $X = N_{T_{k,n}}(w)$. If $w = v$ or $w = u$, since X is a vertex cut of $T_{k,n} + e$, then there exists another vertex $w' \notin \{u, v\}$, such that $X = N_{T_{k,n}}(w')$. Let $T'_{k,n-1} = T_{k,n} - w'$. Then by induction, $T'_{k,n-1}$ is vertex k-maximal. Since $w' \notin \{u, v\}$, $xy \in E((T'_{k,n-1})^c)$, and so $\overline{\kappa}(T_{k,n} + e) \geq \overline{\kappa}(T'_{k,n-1} + e) \geq k+1$. Hence we must have $w \notin \{u, v\}$. Let $T'_{k,n-1} = T_{k,n} - w$. Then by induction, $T'_{k,n-1}$ is vertex k-maximal. Since $w \notin \{u, v\}$, $xy \in E((T'_{k,n-1})^c)$, and so $\overline{\kappa}(T_{k,n} + e) \geq \overline{\kappa}(T'_{k,n-1} + e) \geq k+1$. This completes the proof of Lemma 2.2. ■

Lemma 2.3. *Let k, n be integers with $n \geq k+1 > 2$. If G is a vertex k-maximal graph with $n = |V(G)|$, then $\overline{\kappa}(G) = \kappa(G) = k$.*

Proof. Let G be a vertex k-maximal graph. By definition, $\overline{\kappa}(G) \leq k$. Hence it suffices to show that $\kappa(G) \geq k$. As K_{k+1} is the only vertex k-maximal graph on $k+1$ vertices, we assume that $n \geq k+2$, and so G is not a complete graph. Arguing by contradiction, we assume that $\kappa(G) = r < k$, and so as G is not a complete graph, G has a vertex r-cut S. Let C_1 be a component of $G - S$, and $C_2 = G - (S \cup V(C_1))$. Hence there must be a vertex $v_1 \in V(C_1)$ and a vertex $v_2 \in V(C_2)$ such that $e = v_1v_2 \notin E(G)$,

Since G is vertex k-maximal, $\overline{\kappa}(G + e) \geq k+1$. Hence $G + v_1v_2$ contains a subgraph H with $\kappa(H) = \overline{\kappa}(G+e) \geq k+1$. Since $\overline{\kappa}(G) \leq k$, H cannot be a subgraph of G, and so $e \in E(H)$. As $V(H) \cap V(C_1) \neq \emptyset$ and $V(H) \cap V(C_2) \neq \emptyset$, it follows that $V(H) \cap S$ is a vertex cut of $H - e$.

If $|V(C_1)| = |V(C_2)| = 1$, then $n = |X| + 2 \leq k+1$, contrary to the assumption of $n \geq k+2$. Hence for some $i \in \{1, 2\}$, $|V(C_i)| \geq 2$. Let $S' = (V(H) \cap S) \cup \{v_i\}$. Since $|V(C_i)| \geq 2$ and since $V(H) \cap S$ is a vertex cut of $H - e$, it follows that S' is a vertex cut of H, and so we obtain a contradiction:

$$k+1 > r+1 \geq |S| + 1 \geq |V(H) \cap S| + 1 = |S'| \geq \kappa(H) \geq k+1.$$

This contradiction shows that the lemma must hold. ■

Following [5], if H, K are subgraphs of a graph G, then $H \cup K$ is the subgraph of G with $V(H \cup K) = V(H) \cup V(K)$ and $E(H \cup K) = E(H) \cup E(K)$. Let G be a vertex k-maximal graph with $|V(G)| \geq k+2$. By Lemma 2.3, $\overline{\kappa}(G) = \kappa(G) = k$. For any vertex cut S of G with $|S| = k$, $G - S$ has two vertex disjoint subgraphs L_1 and L_2 such that

$$\begin{aligned} &G_1 := G[S \cup V(L_1)] \text{ and } G_2 := G[S \cup V(L_2)] \text{ are connected,} \\ &|S| = \kappa(G), G = G_1 \cup G_2 \text{ and } V(G_1) \cap V(G_2) = S. \end{aligned} \tag{2}$$

A triple (S, G_1, G_2) of a graph G is a **separation triple** of G if it satisfies (2).

Lemma 2.4. *Let k, n be integers with $k \geq 2$ and $n \geq k+1$, and G be a vertex k-maximal graph on n vertices. Let (S, G_1, G_2) be a separation triple of G.*
(i) If $e \in E(G_1^c) \cup E(G_2^c)$, then any subgraph H of $G+e$ with $\kappa(H) \geq k+1$ is either a subgraph of $G_1 + e$ or a subgraph of $G_2 + e$.
(ii) If $G[S]$ is a clique, then each of G_1 and G_2 is a vertex k-maximal graph.

Proof. Let $e \in E(G_1^c)$. Since G is vertex k-maximal, $\overline{\kappa}(G+e) \geq k+1$. Let H be a subgraph of $G+e$ with $\kappa(H) \geq k+1$. We are to show that H is either a subgraph of G_1 or a subgraph of G_2. By contradiction, assume that we have $V(H) \cap (V(G_1) - S) \neq \emptyset$ and $V(H) \cap (V(G_2) - S) \neq \emptyset$. This, together with the fact $e \in E(G_1^c)$, implies that $S \cap V(H)$ is a vertex cut of H. Hence we reach a contradiction:

$$k = |S| \geq |S \cap V(H)| \geq \kappa(H) = k+1.$$

This contradiction indicates that we cannot have both $V(H) \cap (V(G_1) - S) \neq \emptyset$ and $V(H) \cap (V(G_2) - S) \neq \emptyset$. Hence either $V(H) \cap (V(G_1) - S) = \emptyset$, whence H is a subgraph of $G_2 + e$; or $V(H) \cap (V(G_2) - S) = \emptyset$, whence H is a subgraph of $G_1 + e$. This proves (i).

To prove (ii), we assume that $G[S]$ is a clique. If G_1 is a clique, then by Lemma 2.3, G_1 must be a K_{k+1}, and so by definition, G_1 is vertex k-maximal. Assume that G_1 is not a complete graph. Since $G[S]$ is a clique, every edge in $E(G_1^c)$ is incident with a vertex in $V(G_1) - S$. Since G_1 is not a complete graph, for any edge $e \in E(G_1^c)$, as G is vertex k-maximal, $G + e$ contains a $(k+1)$-connected subgraph H, and $e \in E(H)$. Hence $V(H) \cap (V(G_1) - S) \neq \emptyset$. By Lemma 2.4(i), H is a subgraph of $G_1 + e$, and so by definition, G_1 is vertex k-maximal. By symmetry, we can similarly show that G_2 is also vertex k-maximal. This proves (ii). ■

For an integer $n \geq k+1$, define

$$f(n,k) = \min\{|E(G)| : G \text{ is simple, and vertex-}k\text{-maximal with } n = |V(G)|\}.$$

The goal of this section is to determine the value of $f(n,k)$, thereby proving Theorem 1.4. Theorem 2.5 is the main result of this section.

Theorem 2.5. *For integers k, n with $k \geq 2$ and $n \geq k+1$,*

$$f(n,k) = (n-k)k + \frac{k(k-1)}{2}.$$

Proof. By Lemma 2.2, we have, for any $n \geq k+1$, $f(n,k) \leq (n-k)k + \frac{k(k-1)}{2}$. Thus to prove the theorem, it remains to show that for $n \geq k+1$,

$$f(n,k) \geq (n-k)k + \frac{k(k-1)}{2}. \tag{3}$$

As K_{k+1} is the only vertex k-maximal graph on $n = k+1$ vertices, we assume that $n \geq k+2$, and that (3) holds for smaller values of n.

Let G be a vertex k-maximal of order n. Since $n \geq k+2$, G is not a complete graph. By Lemma 2.3, $\overline{\kappa}(G) = \kappa(G) = k$, and so G has a separation triple (S, G_1, G_2) with $|S| = k$. Let $n_1 = |G_1|$ and $n_2 = |G_2|$.

If $G[S] = K_k$, then by Lemma 2.4, both G_1 and G_2 are vertex k-maximal. It follows by induction that both $|E(G_1)| \geq (n_1 - k)k + \frac{k(k-1)}{2}$ and $|E(G_2)| \geq (n_2 - k)k + \frac{k(k-1)}{2}$. Thus (3) holds by induction.

$$\begin{aligned}
|E(G)| &= |E(G_1)| + |E(G_2)| - \frac{k(k-1)}{2} \\
&\geq (n_1 - k)k + \frac{k(k-1)}{2} + (n_2 - k)k + \frac{k(k-1)}{2} - \frac{k(k-1)}{2} \\
&= (n_1 + n_2 - 2k)k + k(k-1) - \frac{k(k-1)}{2} = (n-k)k + \frac{k(k-1)}{2}.
\end{aligned}$$

Therefore we assume that

$$G[S] \text{ is not a } k\text{-clique}. \tag{4}$$

Since G is vertex k-maximal, for any $e \in E((G[S])^c)$, there exists a $(k+1)$-connected subgraph H of $G+e$. By Lemma 2.4, H is either a subgraph of $G_1 + e$ or a subgraph of $G_2 + e$. Define

$$\begin{aligned}
E_1 &= \{e : e \in E((G[S])^c) \text{ and } \overline{\kappa}(G_2 + e) = k\}, \\
E_2 &= \{e : e \in E((G[S])^c) \text{ and } \overline{\kappa}(G_1 + e) = k\}.
\end{aligned} \tag{5}$$

Claim 2. *Each of the following holds.*
(i) $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 \subset E((G[S])^c)$.
(ii) If $G_1 + E_1$ is a complete graph, then $E(G[S]^c) = E_1$; if $G_1 + E_1$ is not a complete graph, then there exists a subset $E_1' \subseteq E_1$ such that $G_1 + E_1'$ is vertex k-maximal.
(iii) If $G_2 + E_2$ is a complete graph, then $E(G[S]^c) = E_2$; if $G_2 + E_2$ is not a complete graph, then there exists a subset $E_2' \subseteq E_2$ such that $G_2 + E_2'$ is vertex k-maximal.

By (5), we have $E_1 \cup E_2 \subset E((G[S])^c)$. Since G is vertex k-maximal, we have $E_1 \cap E_2 = \emptyset$, and so Claim 2(i) must hold.

By symmetry, it suffices to prove one of Claim 2(ii) and (iii). Assume first that $G_1 + E_1$ is a complete graph. Since $E(G_1) \subseteq E(G)$ and since $G_1 + E_1$ is a complete graph, $(G_1 + E_1)[S]$ is a complete graph with $E((G[S])^c) = (G_1 + E_1)[S] - E(G_1) = E_1$.

Next, we assume that $G_1 + E_1$ is not a complete graph. Take an arbitrary edge $e = xy \in E((G_1 + E_1)^c)$. Then $e \in E(G^c)$, and so as G is vertex k-maximal, $G + e$ has a $(k+1)$-connected subgraph H with $e \in E(H)$. If $\{x, y\} \cap (V(G_1) - S) \neq \emptyset$, then by Lemma 2.4, H is a subgraph of $G_1 + e$. If $\{x, y\} \subseteq S$, then as $e \notin E_1$, by (5), once again H must be a subgraph of $G_1 + e$. Since $G_1 + e$ is a subgraph of $(G_1 + E_1) + e$, we conclude that for any edge $e \in E((G_1 + E_1)^c)$, $(G_1 + E_1) + e$ contains a $(k+1)$-connected subgraph H with $E(H) \cap E_1 = \emptyset$. If $\overline{\kappa}(G_1 + E_1) \leq k$, then by definition, $G_1 + E_1$ is vertex k-maximal. Now assume that $\overline{\kappa}(G_1 + E_1) \geq k+1$. Since $\overline{\kappa}(G_1) \leq k$, there existsa maximum subset $E_1' \subseteq E_1$ such that $\overline{\kappa}(G_1 + E_1') \leq k$. It follows by the maximality of E_1' and by the definition of vertex k-maximal graphs that $G_1 + E_1'$ is vertex k-maximal. This verifies Claim 2.

By Claim 2(i) and by the definition of $(G[S])^c$, we have

$$|E_1| + |E_2| + |E(G[S])| \le |E((G[S])^c)| + |E(G[S])| = |E(K_k)| = \frac{k(k-1)}{2}. \tag{6}$$

If both $G_1 + E_1$ and $G_2 + E_2$ are vertex k-maximal, then by (6), it follows by induction that (3) holds:

$$\begin{aligned} |E(G)| &= |E(G_1)| + |E(G_2)| - |E(G[S])| \\ &= |E(G_1+E_1)| - |E_1| + |E(G_2+E_2)| - |E_2| - |E(G[S])| \\ &\ge k(n_1-k) + \frac{k(k-1)}{2} + k(n_2-k) + \frac{k(k-1)}{2} - |E_1| - |E_2| - |E(G[S])| \\ &\ge k(n_1+n_2-2k) + k(k-1) - \frac{k(k-1)}{2} \\ &= k(n-k) + \frac{k(k-1)}{2}. \end{aligned}$$

Hence we may assume that

$$\text{at least one of } G_1 + E_1 \text{ and } G_2 + E_2 \text{ is not vertex } k\text{-maximal.} \tag{7}$$

Case 1. Exactly one of $G_1 + E_1$ and $G_2 + E_2$ is vertex k-maximal.

By symmetry, we assume that G_1+E_1 is vertex k-maximal and G_2+E_2 is not vertex k-maximal. If G_2+E_2 is a complete graph, then by Claim 2, we have $E((G[S])^c) = E_2$, and so $E_1 = \emptyset$. By assumption, G_1 is vertex k-maximal. It follows by induction, by (6) and by $n_2 \ge k+1$ that

$$\begin{aligned} |E(G)| &= |E(G_1)| + |E(G_2)| - |E(G[S])| \\ &= |E(G_1)| + |E(G_2+E_2)| - |E_2| - |E(G[S])| \\ &\ge k(n_1-k) + \frac{k(k-1)}{2} + \frac{1}{2}(n_2)(n_2-1) - |E_2| - |E(G[S])| \\ &= k(n_1-k) + \frac{k(k-1)}{2} + \frac{1}{2}(n_2-k)(n_2-k-1) \\ &\quad +(n_2-k)k + \frac{k(k-1)}{2} - |E_2| - |E(G[S])| \\ &\ge k(n_1-k) + \frac{k(k-1)}{2} + (n_2-k)k + \frac{k(k-1)}{2} - |E_2| - |E(G[S])| \\ &\ge k(n_1+n_2-2k) + k(k-1) - \frac{k(k-1)}{2} = k(n-k) + \frac{k(k-1)}{2}. \end{aligned}$$

Therefore we assume that G_2+E_2 is not a complete graph. Take an arbitrary edge $e = xy \in E((G_2+E_2)^c)$. Then $e \in E(G^c)$, and so as G is vertex k-maximal, $G+e$ has a $(k+1)$-connected subgraph H with $e \in E(H)$. If $\{x, y\} \cap (V(G_2)-S) \ne \emptyset$, then by Lemma 2.4, H is a subgraph of $G_2 + e$. Assume that $x, y \in S$. Then as $e \notin E_2$, by (5), once again H must be a subgraph of $G_2 + e$. Thus Claim 2 is applicable.

By Claim 2, for some edge subset $E_2' \subset E_2$, $G_2 + E_2'$ is vertex k-maximal. By (6) and by induction on $G_1 + E_1$ and $G_2 + E_2'$, we have

$$\begin{aligned} |E(G)| &= |E(G_1)| + |E(G_2)| - |E(G[S])| \\ &= |E(G_1+E_1)| - |E_1| + |E(G_2+E_2')| - |E_2'| - |E(G[S])| \\ &\ge k(n_1-k) + \frac{k(k-1)}{2} + (n_2-k)k + \frac{k(k-1)}{2} - |E_1| - |E_2'| - |E(G[S])| \\ &\ge k(n_1+n_2-2k) + k(k-1) - \frac{k(k-1)}{2} = k(n-k) + \frac{k(k-1)}{2}. \end{aligned}$$

Hence (3) holds in Case 1.

Case 2. Neither $G_1 + E_1$ nor $G_2 + E_2$ is vertex k-maximal.

By Claim 2, if both $G_1 + E_1$ and $G_2 + E_2$ are complete graphs, then $E_1 = E((G[S])^c) = E_2$. By Claim 2(i), we must have $E_1 = E_2 = \emptyset$, and so either both G_1 and G_2 are vertex k-maximal, contrary to (7), or $G[S]$ is a complete graph, contrary to (4). Hence by symmetry, we may assume that $G_1 + E_1$ is not a complete graph.

If $G_2 + E_2$ is a complete graph, then by Claim 2 $E_1 = \emptyset$ and $E_2 = E((G[S])^c)$. For any $e \in E((G[S])^c)$, since G is a vertex k-maximal graph, $G + e$ contains a $(k+1)$-connected subgraph H. By (5) and since $E_1 = \emptyset$, this H must be a subgraph of $G_1 + e$. This, together with Lemma 2.3, implies that G_1 is a vertex k-maximal graph, contrary to the assumption of Case 2.

Therefore, we conclude that each of $G_1 + E_1$ and $G_2 + E_2$ is not a complete graph. By Claim 2, there exist $E_1' \subseteq E_1$ and $E_2' \subseteq E_2$ such that $G_1 + E_1'$ and $G_2 + E_2'$ are vertex k-maximal. By (6) and by induction,

$$\begin{aligned}
|E(G)| &= |E(G_1)| + |E(G_2)| - |E(G[S])| \\
&= |E(G_1 + E_1')| - |E_1'| + |E(G_2 + E_2')| - |E_2'| - |E(G[S])| \\
&\geq k(n_1 - k) + \frac{k(k-1)}{2} + (n_2 - k)k + \frac{k(k-1)}{2} - |E_1'| - |E_2'| - |E(G[S])| \\
&\geq k(n_1 + n_2 - 2k) + k(k-1) - \frac{k(k-1)}{2} = k(n-k) + \frac{k(k-1)}{2}.
\end{aligned}$$

Hence (3) holds in Case 2 as well, and so (3) is justified by induction.

This proves Theorem 2.5. ■

3. Characterization of extremal vertex 2-maximal graphs

The main goal of this section is to determine the structures of all vertex 2-maximal graphs with maximum possible edges, as well as the possible values of sizes of vertex 2-maximal graphs. We start defining our notation in the discussion.

Definition 3.1. Let k and n be integers with $n \geq k + 1$.
(i) Let $\mathcal{G}(n, k)$ be the family of all vertex k-maximal graphs on n vertices, and $\mathcal{G}(k) = \cup_{n \geq k+1} \mathcal{G}(n, k)$ be the family of all vertex k-maximal graphs with at least $k + 1$ vertices.
(ii) Define,

$$\mathcal{MG}(n, k) = \{G : G \in \mathcal{G}(n, k) \text{ and } |E(G)| = \max\{|E(L)| : L \in \mathcal{G}(n, k)\}\},$$

and

$$\mathcal{SG}(n, k) = \{G : G \in \mathcal{G}(n, k) \text{ and } |E(G)| = \min\{|E(L)| : L \in \mathcal{G}(n, k)\}\}.$$

Let $F(n, 2) = \max\{|E(L)| : L \in \mathcal{G}(n, 2)\}$. The purpose of this section is to characterize all vertex 2-maximal graphs whose on n vertices with size $F(n, 2)$. The next lemma is a consequence of Lemma 2.4, whose proof is omitted.

Lemma 3.2. *Let G be a vertex 2-maximal graph with $n = |V(G)| \geq 4$. Then for any separation triple (S, G_1, G_2) with $S = \{z_1, z_2\}$, one of the following must hold.*
(i) The edge $z_1z_2 \in E(G)$ and each of G_1 and G_2 is vertex 2-maximal.
(ii) The edge $z_1z_2 \notin E(G)$ and for each $i \in \{1, 2\}$, either G_i or $G_i + z_1z_2$ is vertex 2-maximal.

We shall present a graph construction such that all graphs in $\mathcal{G}(2)$ can be recursively built with this construction.

Definition 3.3. Let $G(u, v)$ denote a graph G with two distinguished vertices $u, v \in V(G)$, and $G_1 = G_1(u_1, v_1)$ and $G_2 = G_2(u_2, v_2)$ be two vertex disjoint graphs.
(i) Suppose that $u_1v_1 \notin E(G_1)$ and $u_2v_2 \notin E(G_2)$. Define $[G_1(u_1, v_1), G_2(u_2, v_2)]_2$ to be the graph obtained from the disjoint union of $G_1(u_1, v_1)$ and $G_2(u_2, v_2)$ by identifying u_1 and u_2, v_1 and v_2, respectively. When we do not emphasize the distinguished vertices, or when the distinguished vertices are understood in the context, we often use $[G_1, G_2]_2$ for $[G_1(u_1, v_1), G_2(u_2, v_2)]_2$. Thus one can also view $[G_1, G_2]_2$ as a family of graphs.
(ii) Suppose that $u_1v_1 \notin E(G_1)$ and $u_2v_2 \in E(G_2)$. Define

$$[G_1(u_1, v_1), G_2(u_2, v_2)]^2 = [G_1(u_1, v_1), G_2(u_2, v_2) - \{u_2v_2\}]_2 + u_2v_2.$$

(iii) Let w_1, w_2 denote the two vertices of degree 1 in $K_{1,2}$. Define $K_4^- = [K_3(u_1, v_1), K_{1,2}(w_1, w_2)]_2$.

3.1. Characterization of vertex 2-maximal graphs with maximum sizes

We start with a family of examples, which attain maximum sizes among all vertex 2-maximal graphs with given order.

Example 3.4. Let n be an integer with $n \geq 4$, and let $t = \lfloor \frac{n-2}{2} \rfloor$. Let $G(n)$ be the graph satisfying the following.
(i) $V(G(n)) = \{z_1, z_2\} \cup \{v_1, v_2, \ldots, v_{n-2}\}$,
(ii) $E(G(n)) = \{z_1v_i, z_2v_i : 1 \leq i \leq n-2\} \cup \{v_jv_{t+j} : 1 \leq j \leq t\}$.

Thus $G(4) = K_4^-$ and $G(5) = [G(4)(z_1, z_2), K_{1,2}(u_2, v_2)]_2$. By definition, it is routine to verify that $\overline{\kappa}(G(n)) = \kappa(G(n)) = 2$. When n is even, $\delta(G(n)) = 3$; and when n is odd, v_{n-2} is the only vertex of degree 2, with all other vertices having degree at least 3. The only minimum vertex cut of $G(n)$ is $S = \{z_1, z_2\}$. By as $\kappa(K_4) = 3$, by definition, $G(4) \in \mathcal{G}(2)$. Similarly, $G(5) \in \mathcal{G}(2)$. Assuming that $n \geq 6$ and $G(n') \in \mathcal{G}(2)$, for any $n' < n$. Take any edge $e \in E((G(n))^c)$. If $e = z_1z_2$, then $G(n) + e$ contains a K_4, and so $\overline{\kappa}(G(n) + e) = 3$. Assume that $e \neq z_1z_2$, and so by symmetry, we further assume that $z_1 = v_1$, and $z_2 = v_i$ for some $i \geq 2$ and $i \neq t + 1$. Since $n \geq 6$, there must be an j with $2 \leq j$ such that $i \notin \{j, t + j\}$. Let $G' = G(n) - \{v_j, v_{t+j}\}$. Then $G' \cong G(n-2)$ and $e \in E((G')^c)$. By induction, $\overline{\kappa}(G(n) + e) \geq \overline{\kappa}(G' + e) \geq 3$. By definition, $G(n) \in \mathcal{G}(2)$. A direct computation yields

$$F(n, 2) \geq |E(G(n))| = \frac{5n}{2} - \frac{21 + (-1)^{n+1}}{4}.$$

Example 3.4 motivates the following theorem, which is a slightly enhanced version of Theorem 1.3(i) with $k = 2$.

Theorem 3.5. *For $n \geq 4$,*

$$F(n, 2) = \frac{5n}{2} - \frac{21 + (-1)^{n+1}}{4}. \tag{8}$$

Moreover, $\mathcal{MG}(4, 2) = \{K_4^-\}$ and $\mathcal{MG}(5, 2) = \{[K_{1,2}(w_1, w_2), K_4^-(u_1, v_1)]_2 : u_1, v_1 \in V(K_4^-)\} \cup \{[K_{1,2}(w_1, w_2), K_4^-(u_1, v_1)]^2 : u_1, v_1 \in V(K_4^-)\}$. For $n \geq 6$, $G \in \mathcal{MG}(n, 2)$ if and only if for any separation triple (S, G_1, G_2) of G with $|S| = 2$, one of the following holds.
(i) $n \equiv 1 \pmod 2$, $G_1 = K_3$, $G_2 \in \mathcal{MG}(n-1, 2)$, $|E(G[S])| = 1$ and $G \in [K_{1,2}(w_1, w_2), G_2]^2$.
(ii) $n \equiv 1 \pmod 2$, $G_1 = K_{1,2}$, $G_2 \in \mathcal{MG}(n-1, 2)$, $|E(G[S])| = 0$ and $G \in [K_{1,2}(w_1, w_2), G_2]_2$.
(iii) $n \equiv 1 \pmod 2$, $G_1, G_2 \in \mathcal{MG}(2)$ such that for some $i \in \{1, 2\}$, $|V(G_i)| \equiv 0 \pmod 2$ and $|V(G_{3-i})| \equiv 1 \pmod 2$, $E(G[S]) = \emptyset$ and $G \in [G_1, G_2]_2$.
(iv) $n \equiv 0 \pmod 2$, $G_1, G_2 \in \mathcal{MG}(2)$ with $|V(G_1)| \equiv |V(G_2)| \equiv 0 \pmod 2$, and $G \in [G_1, G_2]_2$ such that $E(G[S]) = \emptyset$.

Proof. By Example 3.4, to justify (8), it suffices to show that for any $G \in \mathcal{G}(2)$ with $n = |V(G)|$, we always have

$$|E(G)| \leq \frac{5n}{2} - \frac{21 + (-1)^{n+1}}{4}. \tag{9}$$

When $n \in \{4, 5\}$, it is routine to show that

$$\mathcal{MG}(4, 2) = \{K_4^-\} \text{ and } \mathcal{MG}(4, 5) = \{[K_4^-(u_1, v_1), K_{1,2}(w_1, w_2)]_2 : u_1, v_1 \in V(K_4^-)\},$$

and so (9) holds for $n \in \{4, 5\}$. Therefore, we assume that $n \geq 6$ and Theorem 3.5 holds for smaller values of n.

Let $G \in \mathcal{G}(n, 2)$ be a graph. As $n \geq 6$, G is not a complete graph. Hence G has a vertex cut of size 2. Let (S, G_1, G_2) be a separation triple of G with $S = \{u_0, v_0\}$. Let $n_1 = |V(G_1)|$ and $n_2 = |V(G_2)|$. Thus $n = n_1 + n_2 - 2$. Without loss of generality, we assume that $n_1 \leq n_2$.

Claim 3. *If $u_0v_0 \in E(G)$, then (9) holds. Moreover, equality in (9) holds if and only if Theorem 3.5(i) holds.*

By Lemma 3.2, both G_1 and G_2 are vertex 2-maximal. Assume first that $n_1 \geq 4$. Then as G_1 and G_2 are vertex 2-maximal, neither G_1 nor G_2 is a complete graph. By induction and by the fact that n and $n_1 + n_2$ have the same parity,

$$\begin{aligned}
|E(G)| &= |E(G_1)| + |E(G_2)| - |\{u_0v_0\}| \\
&= \frac{5n_1}{2} - \frac{21 + (-1)^{n_1+1}}{4} + \frac{5n_2}{2} - \frac{21 + (-1)^{n_2+1}}{4} - 1 \\
&= \frac{5(n_1 + n_2 - 2)}{2} - \frac{21 + (-1)^{n+1}}{4} + \left(4 + \frac{21 + (-1)^{n+1}}{4} - \sum_{i=1}^{2} \frac{21 + (-1)^{n_i+1}}{4}\right) \\
&= \frac{5(n_1 + n_2 - 2)}{2} - \frac{21 + (-1)^{n+1}}{4} - \frac{5 + (-1)^{n_1+1} + (-1)^{n_2+1} - (-1)^{n+1}}{4} \\
&< \frac{5n}{2} - \frac{21 + (-1)^{n+1}}{4}.
\end{aligned} \tag{10}$$

Now assume that $n_1 = 3$. Then $G_1 = K_3$ and as $n \geq 5$, we have $n = n_2 + 1 > n_2 \geq 4$. In this case, by induction on G_2,

$$\begin{aligned} |E(G)| &= |E(G_1)| + |E(G_2)| - |\{u_0v_0\}| = |E(G_2)| + 2 \\ &= \frac{5n_2}{2} - \frac{21+(-1)^{n_2+1}}{4} + 2 \\ &= \frac{5n}{2} - \frac{21+(-1)^{n+1}}{4} + 2 - \frac{5}{2} + \frac{21+(-1)^{n+1}}{4} - \frac{21+(-1)^{n_2+1}}{4} \\ &= \frac{5n}{2} - \frac{21+(-1)^{n+1}}{4} - \frac{2+(-1)^{n_2+1}-(-1)^{n+1}}{4} \\ &\leq \frac{5n}{2} - \frac{21+(-1)^{n+1}}{4}. \end{aligned} \tag{11}$$

As $u_0v_0 \in E(G)$, the last inequality in (11) is an equality if and only if Theorem 3.5(i) holds. This proves the claim.

By Claim 3, in the rest of the arguments, we assume that $u_0v_0 \notin E(G)$. By Lemma 3.2, for each $i \in \{1, 2\}$, either G_i or $G_i + u_0v_0$ is vertex 2-maximal.

Claim 4. *If $u_0v_0 \notin E(G)$, $n_1 \geq 4$, and for some $i \in \{1, 2\}$, both G_i and $G_{3-i} + u_0v_0$ are vertex 2-maximal, then (9) holds with strict inequality.*

Without lost of generality, we assume that G_1 and $G_2 + u_0v_0$ are vertex 2-maximal. By induction, we have, as in (10), that $|E(G)| = |E(G_1)| + |E(G_2)| - 1 < \frac{5n}{2} - \frac{21+(-1)^{n+1}}{4}$. Thus the claim must hold.

Claim 5. *If $u_0v_0 \notin E(G)$, $n_1 = 3$, and both $G_1 + u_0v_0$ and G_2 are vertex 2-maximal, then (9) holds. Moreover, equality in (9) holds if and only if Theorem 3.5(ii) holds.*

As $n_1 = 3$ and $G_1 + u_0v_0 \in \mathcal{G}(2)$, we have $G_1 \cong K_{1,3}$. As $n \geq 6$, we have $n_2 = n - 1 \geq 5$. By induction, we have, as in (11),

$$\begin{aligned} |E(G)| &= |E(G_1 + u_0v_0)| + |E(G_2)| - |\{u_0v_0\}| = |E(G_2)| + 2 \\ &= \frac{5n}{2} - \frac{21+(-1)^{n+1}}{4} - \frac{2+(-1)^{n_2+1}-(-1)^{n+1}}{4} \leq \frac{5n}{2} - \frac{21+(-1)^{n+1}}{4}. \end{aligned} \tag{12}$$

As $u_0v_0 \notin E(G)$ and $G_1 \cong K_{1,2}$, the last inequality in (12) is an equality if and only if Theorem 3.5 (ii) holds. This proves the claim.

Claim 6. *If $u_0v_0 \notin E(G)$ and G_1 and G_2 are both vertex 2-maximal, then (9) holds. Moreover equality in (9) holds if and only if Theorem 3.5(iii) or (iv) holds.*

If G_1 and G_2 are both vertex 2-maximal, then by induction,

$$\begin{aligned} |E(G)| &= |E(G_1)| + |E(G_2)| = \frac{5n_1}{2} - \frac{21+(-1)^{n_1+1}}{4} + \frac{5n_2}{2} - \frac{21+(-1)^{n_2+1}}{4} \\ &= \frac{5(n_1+n_2-2)}{2} - \frac{21+(-1)^{n+1}}{4} + \left(5 + \frac{21+(-1)^{n+1}}{4} - \sum_{i=1}^{2} \frac{21+(-1)^{n_i+1}}{4}\right) \\ &= \frac{5(n_1+n_2-2)}{2} - \frac{21+(-1)^{n+1}}{4} - \frac{1+(-1)^{n_1+1}+(-1)^{n_2+1}-(-1)^{n+1}}{4} \\ &\leq \frac{5n}{2} - \frac{21+(-1)^{n+1}}{4}. \end{aligned} \tag{13}$$

As G_1 is vertex 2-maximal and $u_0v_0 \notin E(G)$, we have $n_1 \geq 4$. As $u_0v_0 \notin E(G)$, the equality holds in (13) if and only if either $n \equiv n_1 + n_2 \equiv 1 \pmod 2$ and Theorem 3.5(iii) holds or $n \equiv n_1 \equiv n_2 \equiv 0 \pmod 2$ and Theorem 3.5(iv) holds. This justifies Claim 6.

Claim 7. *If $u_0v_0 \notin E(G)$, $n_1 \geq 4$, and both $G_1 + u_0v_0$ and $G_2 + u_0v_0$ are both vertex 2-maximal, then (9) holds with strict inequality.*

Since $G_1 + u_0v_0$ and $G_2 + u_0v_0$ are both vertex 2-maximal and $n_1 \geq 4$, by induction, we have

$$\begin{aligned} |E(G)| &= |E(G_1 + u_0v_0)| + |E(G_2 + u_0v_0)| - 2 \\ &= \frac{5n_1}{2} - \frac{21 + (-1)^{n_1+1}}{4} + \frac{5n_2}{2} - \frac{21 + (-1)^{n_2+1}}{4} - 2 \\ &= \frac{5(n_1 + n_2 - 2)}{2} - \frac{21 + (-1)^{n+1}}{4} + \left(3 + \frac{21 + (-1)^{n+1}}{4} - \sum_{i=1}^{2} \frac{21 + (-1)^{n_i+1}}{4}\right) \\ &< \frac{5n}{2} - \frac{21 + (-1)^{n+1}}{4}. \end{aligned} \tag{14}$$

The theorem now follows from the claims above. ■

The next corollary follows immediately from Theorem 3.5.

Corollary 3.6. *Let G be a graph on $n \geq 3$ vertices. Then $G \in \mathcal{MG}(2)$ if and only if one of the following holds.*
(i) $G \in \{K_3, K_4^-\} \cup \{[K_{1,2}(w_1, w_2), K_4^-(u_1, v_1)]_2 : u_1, v_1 \in V(K_4^-)\} \cup \{[K_{1,2}(w_1, w_2), K_4^-(u_1, v_1)]^2 : u_1, v_1 \in V(K_4^-)\}$.
(ii) For some integer $t \geq 3$, $n = 2t$ and there exist $G_1, G_2 \in \mathcal{MG}(2)$ satisfying $|V(G_1)| \equiv |V(G_2)| \equiv 0$ (mod 2) and $G \in [G_1, G_2]_2$.
(iii) For some integer $t \geq 3$, $n = 2t + 1$, and either there exist $G_1, G_2 \in \mathcal{MG}(2)$ satisfying $|V(G_1)| + |V(G_2)| \equiv 1$ (mod 2) and $G \in [G_1, G_2]_2$; or there exists a $G_2 \in \mathcal{MG}(n-1, 2)$ such that $G \in [K_{1,2}(w_1, w_2), G_2]_2 \cup [K_{1,2}(w_1, w_2), G_2]^2$.

3.2. Size range of vertex 2-maximal graphs

The goal of this subsection is to determine the size range of all vertex 2-maximal graphs with given order. We first present an example which indicates possible values which are attained by the sizes of vertex 2-maximal graphs of order n.

Example 3.7. Let n and ℓ be integers with $n \geq 4$ and $\ell = \lceil \frac{1}{2}n - 2 \rceil$. For any integer r with $0 \leq r \leq \ell$. We define a graph $G = G(n, r)$ with vertex set

$$V(G) = \{z_1, z_2\} \cup \{u_i, v_i : 1 \leq i \leq r + 1\} \cup \{w_1, w_2, \ldots, w_{n-2r-4}\},$$

and edge set

$$E(G) = \{z_1u_i, z_2u_i, z_1v_i, z_2v_i, u_iv_i : 1 \leq i \leq r + 1\} \cup \{w_ju_1, w_jv_1 : 1 \leq j \leq n - 2r - 4\}.$$

Then each of the following holds.
(i) Let $L = G - \{w_1, w_2, \ldots, w_{n-2r-4}\}$. Then $L \in \mathcal{G}(2)$.
(ii) $G \in \mathcal{G}(2)$.
(iii) $|E(G)| = 2n - 3 + r$.

Proof. By Example 3.4, $L \in \mathcal{G}(2)$, and so (i) holds.

Let $e = x_1x_2 \in E(G^c)$ be an edge not in $E(G)$. If $x_1, x_2 \in V(L)$, then by (i), $G + e$ has a 3-connected subgraph. If $x_1, x_2 \in \{w_1, w_2, \ldots, w_{n-2r-4}\}$, then as $G[\{u_1, v_1\} \cup \{w_1, w_2, \ldots, w_{n-2r-4}\}] \cong K_{2,n-2r-4} + u_1v_1$, $G[\{u_1, v_1\} \cup \{w_1, w_2, \ldots, w_{n-2r-4}\}] + e$ contains a K_4. Now assume that $x_1 \in \{w_1, w_2, \ldots, w_{n-2r-4}\}$ and $x_2 \in V(G) - \{w_1, w_2, \ldots, w_{n-2r-4}\}$. Without lost of generality, assume that $x_1 = w_1$. If $x_2 \in \{z_1, z_2\}$, say $x_2 = z_1$, then $G[\{z_1, u_1, v_1, w_1\}] \cong K_4$ is 3-connected. Hence we assume that $x_2 \in \{u_iv_i : 2 \leq i \leq r + 1\}$. By symmetry, assume that $x_2 = u_2$. Then it is routine to verify that $G[\{z_1, z_2, w_1, u_1, v_1, u_2, v_2\}]$ is 3-connected. By definition, $G \in \mathcal{G}(2)$ and so (ii) follows.

As a direct computation yields (iii), we have justified the conclusions of Example 3.7. ■

Example 3.7 motivates the following main result of this subsection.

Theorem 3.8. *Let m and n be integers with $n \geq 3$. The following are equivalent.*
(i) There exists a graph $G \in \mathcal{G}(n, 2)$ with $m = |E(G)|$.
(ii) $2n - 3 \leq m \leq \frac{5n}{2} - \frac{21+(-1)^{n+1}}{4}$.

Proof. By Example 3.7, it is known that (ii) implies (i). Conversely, for any graph $G \in \mathcal{G}(n, 2)$, by Theorem 2.5 with $k = 2$ and Theorem 3.5, we conclude that $2n - 3 \leq m \leq \frac{5n}{2} - \frac{21+(-1)^{n+1}}{4}$, and so (i) implies (ii). ■

CRediT authorship contribution statement

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