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Chvátal–Erdős Conditions and Almost Spanning Trails

Lan Lei[1] · Xiaomin Li[1] · Xiaoling Ma[2] · Mingquan Zhan[3] · Hong-Jian Lai[4]

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Abstract

Let $\alpha'(G), ess'(G), \kappa(G), \kappa'(G), N_G(v)$ and $D_i(G)$ denote the matching number, essential edge connectivity, connectivity, edge connectivity, the set of neighbors of v in G and the set of degree i vertices of a graph G, respectively. For $u, v \in V(G)$, define $u \sim v$ if and only if $u = v$ or both $u, v \in D_2(G)$ and $N_G(u) = N_G(v)$. Then, $\sim$ is an equivalence relation, and $[v]$ denotes the equivalence class containing v. A subgraph H of G is almost spanning if $H \subseteq G - D_1(G), \bigcup_{j\geq 3} D_j(G) \subseteq V(H)$ and for any $v \in D_2(G)$, $|[v] - V(H)| \leq 1$. The line graph version of Chvátal–Erdős theorem for a connected graph G are extended as follows.

(i) If $ess'(G) \geq \alpha'(G)$, then G has an almost spanning closed trail.
(ii) If $ess'(G) \geq \alpha'(G) - 1$, then G has an almost spanning trail.
(iii) If $ess'(G) \geq \alpha'(G) + 1$, then for $e, e' \in E(G - D_1(G))$, $G - D_1(G)$ has an almost spanning trail starting from e and ending at e'.

Keywords Chvátal–Erdős theorem · Supereulerian · Collapsible · Essential edge connectivity · Matching number

Mathematics Subject Classification 05C76 · 05C07 · 05C45

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1 Introduction

Graphs considered here are finite and loopless. We follow [3] for undefined terms and notation. As in [3], for a graph G, let $\alpha(G)$, $\alpha'(G)$, $\kappa(G)$ and $\kappa'(G)$ denote the stability number (also called the independence number), matching number, connectivity and edge connectivity of G, respectively. A cycle on n vertices is often called an n-cycle. The *girth* of G, denoted by $g(G)$, is the length of a shortest cycle of G. For a subset $X \subseteq V(G)$ or $X \subseteq E(G)$, $G[X]$ is the subgraph of G induced by X. A path from a vertex u to a vertex v is referred as to a (u, v)-path. As in [3], a graph G is *Hamiltonian* if G has a spanning cycle, and is *Hamilton-connected* if for any pair of distinct vertices u and v, G contains a spanning (u, v)-path. The *line graph* of a graph G, written $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. For a graph G, let $O(G)$ denote the set of odd degree vertices of G and G is *Eulerian* if G is connected with $O(G) = \emptyset$. A graph is *supereulerian* if it has a spanning closed trail. An edge cut X of G is *essential* if $G - X$ has at least two nontrivial components. For an integer $k > 0$, a graph G is *essentially k-edge-connected* if G is connected and does not have an essential edge cut X with $|X| < k$. For a connected graph G, let $ess'(G)$ be the largest integer k such that G is essentially k-edge-connected, if at least one such k exists, or $ess'(G) = |E(G)| - 1$ if for any integer k, G does not have an essential edge cut.

This research is motivated by the following well-known theorems of Chvátal and Erdős on Hamiltonian graphs.

Theorem 1.1 (Chvátal and Erdős [14]) *Let G be a graph with at least three vertices.*

(i) If $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian.
(ii) If $\kappa(G) \geq \alpha(G) - 1$, then G has a Hamiltonian path.
(iii) If $\kappa(G) \geq \alpha(G) + 1$, then G is Hamilton-connected.

There have been researches on conditions analogous to this Chvátal–Erdős Theorem to assure the existence of spanning trails in a graph utilizing relationship among independence number, matching number and edge connectivity, as seen in [1,16,18] and [27], among others. Given a trail $T = v_0e_1v_1 \ldots e_{n-1}v_{n-1}e_nv_n$ in a graph G, we often refer this trail as a (v_0, v_n)-trail to emphasize the end vertices, or as an (e_1, e_n)-trail to emphasize the end edges. The vertices $v_1, v_2, \ldots, v_{n-1}$ are the *internal vertices* of T. As a vertex may occur more than once in a trail, when either v_0 or v_n occurs in the trail as a v_i with $0 < i < n$, it is also an internal vertex by definition. A trail T of G is *dominating* if every edge of G is incident with an internal vertex of T, is *spanning* if T is dominating with $V(T) = V(G)$. A Eulerian subgraph (a closed trail) H of G is *dominating* if $E(G - V(H)) = \emptyset$. Harary and Nash-Williams discovered a close relationship between dominating Eulerian subgraphs and hamiltonian line graphs.

Theorem 1.2 (Harary and Nash-Williams [15]) *Let G be a connected graph with at least three edges. The line graph $L(G)$ is hamiltonian if and only if G has a dominating Eulerian subgraph.*

Following the same idea of Theorem 1.2, the following have been observed.

Proposition 1.3 *Let G be a connected graph with at least three edges.*

(i) The line graph $L(G)$ has a Hamilton path if and only if G has a dominating trail.
(ii) (Theorem 1.5 of [19]) The line graph $L(G)$ is Hamilton-connected if and only if for any edges $e, e' \in E(G)$, G has a dominating (e, e')-trail.

By the definitions of line graphs and essential edge connectivity, for a connected graph G,

$$\kappa(L(G)) = ess'(G) \text{ and } \alpha(L(G)) = \alpha'(G). \tag{1}$$

Therefore by Theorem 1.2, Proposition 1.3 and (1), the line graph version of Theorem 1.1 can be stated as follows.

Theorem 1.4 (Chvátal and Erdős [14]) *Let G be a connected graph with $|E(G)| \geq 3$.*

(i) If $ess'(G) \geq \alpha'(G)$, then G has a dominating Eulerian subgraph.
(ii) If $ess'(G) \geq \alpha'(G) - 1$, then G has a dominating trail.
(iii) If $ess'(G) \geq \alpha'(G) + 1$, then for any edges $e, e' \in E(G)$, G has a dominating (e, e')-trail.

Our goal is to extend Theorem 1.4. Let G be a connected graph. For an integer $i \geq 0$, define

$$D_i(G) = \{v \in V(G) : d_G(v) = i\} \text{ and } d_i(G) = |D_i(G)|.$$

For a subset $X \subseteq V(G)$, define $N_G(X) = \{y \in V(G) - X$ for some $x \in X, xy \in E(G)\}$. When $X = \{v\}$, we use $N_G(v)$ for $N_G(\{v\})$. For $u, v \in V(G)$, define a relation $u \sim v$ if and only if either $u = v$ or both $u, v \in D_2(G)$ and $N_G(u) = N_G(v)$. It is routine to verify that this is an equivalent relation. The equivalence class containing v will be denoted by $[v]$, and the equivalence classes are called the *D_2-equivalent classes*. A subgraph H of G is *almost spanning* if

(AS1) $H \subseteq G - D_1(G)$,
(AS2) $\bigcup_{j \geq 3} D_j(G) \subseteq V(H)$,
(AS3) For any $v \in D_2(G)$, $|[v] - V(H)| \leq 1$.

Let $e = u_1v_1$ and $e' = u_2v_2$ be two edges of G. If $e \neq e'$, then the graph $G(e,\ e')$ is the graph obtained from G by replacing $e = u_1v_1$ with a path $u_1v_ev_1$ and by replacing $e' = u_2v_2$ with a path $e' = u_2v_{e'}v_2$, where v_e, $v_{e'}$ are two new vertices not in $V(G)$. If $e = e'$, then $G(e,\ e')$, also denoted by $G(e)$ in this case, is obtained from G by replacing $e = u_1v_1$ with a path $u_1v_ev_1$. As defined in [22], a graph G is *strongly spanning trailable* if for any $e, e' \in E(G)$, $G(e, e')$ has a $(v_e, v_{e'})$-trail T with $V(G) = V(T) - \{v_e, v_{e'}\}$. By definition, every strongly spanning trailable graph is spanning trailable. As observed in [23] (also in Chapter 1 of [29]), the Wagner graph H_8 (see Fig. 1 below) is spanning trailable but not strongly spanning trailable.

By definition, given a graph G, every spanning (open or closed) trail of G is also almost spanning, and every almost spanning (open or closed) trail of G is also dominating. Furthermore, it is routine to verify that if for $e,\ e' \in E(G - D_1(G))$, $G(e, e')$ has an almost spanning $(v_e,\ v_{e'})$-trail, then for any $e, e' \in E(G)$, G has a dominating (e, e')-trail. In these sense, the following main result of this paper extends Theorem 1.4.

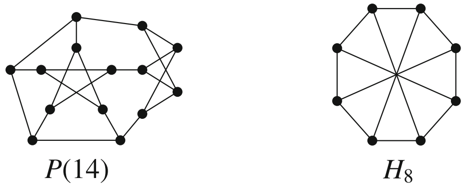

Fig. 1 $P(14)$ and H_8

Theorem 1.5 *Let G be a connected graph. Each of the following holds.*

(i) If $ess'(G) \geq \alpha'(G)$, then G has an almost spanning closed trail.
(ii) If $ess'(G) \geq \alpha'(G) - 1$, then G has an almost spanning trail.
(iii) If $ess'(G) \geq \alpha'(G) + 1$, then for $e,\ e' \in E(G - D_1(G))$, $G(e, e')$ has an almost spanning $(v_e,\ v_{e'})$-trail.

In Sect. 2, we display the mechanism we will use in our arguments. Then, we provide some auxiliary results that will be applied in Sect. 3 to prove our main results. The main results will be proved in the last section.

2 Preliminaries

Before obtaining the proof of main theorem, we introduce some notations. For a subset $Y \subseteq E(G)$, the *contraction* G/Y is the graph obtained from G by identifying the two ends of each edge in Y and then by deleting the resulting loops. If H is a subgraph of G, we often use G/H for $G/E(H)$. For a vertex $v \in V(G/X)$, we define $PI_G(v)$ to be the contraction preimage of v in G. A graph G is called *collapsible* if for any $R \subseteq V(G)$ with $|R|$ is even, G has a spanning subgraph S_R with $O(S_R) = R$. By definition, collapsible graphs are supereulerian. In [5], Catlin showed that every graph G has a unique collection of maximal collapsible subgraphs $H_1, H_2, \cdots, H_c$. The *reduction* in G, denoted by G', is the graph $G/(H_1 \cup H_2 \cup \cdots \cup H_c)$. A graph G is *reduced* if $G' = G$. The following theorem summarizes some properties of collapsible graphs and reduced graphs.

Theorem 2.1 (Catlin [5]) *Let G be a connected graph, H be a collapsible subgraph of G and let G' be the reduction in G. Each of the following holds:*

(i) (Theorem 8 of [5]) G is collapsible if and only if G/H is collapsible. In particular, G is collapsible if and only if $G' = K_1$.
(ii) (Theorem 5 of [5]) G is reduced if and only if G has no nontrivial collapsible subgraphs.
(iii) (Theorem 8 of [5]) G is supereulerian (respectively, has a spanning trail) if and only if G/H is supereulerian (respectively, has a spanning trail) .
(iv) (Corollary of [5]) Any subgraph of a reduced graph is reduced.

Let $F(G)$ be the minimum number of extra edges that must be added to G so that the resulting graph has two-edge-disjoint spanning trees. Hence, a graph G has two-edge-disjoint spanning trees if and only if $F(G) = 0$. Following the notation in [8], define

$$\gamma(G) = \max\left\{\frac{|X|}{|V(G[X])|-1} : \emptyset \neq X \subseteq E(G)\right\}. \quad (2)$$

Catlin initiated the study and applications of collapsible graphs and the related reduction method. Let $\mathcal{N}$ be a collection of graphs. A graph G is $\mathcal{N}$-*clear* if G does not have a (not necessary induced) subgraph isomorphic to a member in $\mathcal{N}$. Let $K_{3,3}^-$ denote the graph obtained from $K_{3,3}$ by deleting an edge. Basically, studies on reduced graphs are using the properties stated in Theorem 2.2 (i) below.

Theorem 2.2 *Let G be a connected graph. Then,*

(i) (Catlin [4] *and Theorem 8 of* [5]) *If G is reduced with $|V(G)| \geq 3$, then G is $\{K_{3,3}^-\}$-clear, $g(G) \geq 4$ and $\gamma(G) < 2$. As a consequence of $\gamma(G) < 2$, $\delta(G) \leq 3$.*
(ii) (Catlin, Theorem 7 of [4], *see also Corollary 2.13 of [21]) If $\gamma(G) \leq 2$, then $F(G) = 2(|V(G)| - 1) - |E(G)|$.*
(iii) (Catlin [5]*) If $F(G) = 0$, or if $F(G) \leq 1$ and $\kappa'(G) \geq 2$, then G is collapsible;*
(iv) (Catlin et al., Theorem 1.3 of [9]*) If G is reduced and $F(G) \leq 2$, then $G \in \{K_1, K_2\} \cup \{K_{2,t} : t \geq 1\}$.*
(v) (Li et al., Lemma 2.2 of [19]*) If G is collapsible, then for any $u, v \in V(G)$, G has a spanning (u, v)-trail.*
(vi) Suppose that $F(G) = 0$. For any $e', e'' \in E(G)$, $G(e', e'')$ has a spanning $(v_{e'}, v_{e''})$-trail if and only if $\{e', e''\}$ is not an edge cut of G. In particular, if $\kappa'(G) \geq 3$, then G is strongly spanning trailable.

Proof It suffices to prove (vi). Let $e', e'' \in E(G)$. By definition, if $\{e', e''\}$ is an edge cut of G, then $G(e', e'')$ cannot have a spanning $(v_{e'}, v_{e''})$-trail. Conversely, we assume that $\{e', e''\}$ is not an edge cut of G. As $F(G) = 0$, we have $F(G(e', e'')) \leq 2$, and so by Theorem 2.2 (iv), either $G(e', e'')$ is collapsible, whence by Theorem 2.2 (v) that $G(e', e'')$ has a spanning $(v_{e'}, v_{e''})$-trail; or the reduction in $G(e', e'')$ is a $K_{2,t}$ for some integer $t \geq 2$. Since G has two-edge-disjoint spanning trees, both $v_{e'}$ and $v_{e''}$ must be vertices of degree 2 in this $K_{2,t}$. Since $\{e', e''\}$ is not an edge cut of G, we must have $t \geq 3$, and so $K_{2,t}$ has a spanning $(v_{e'}, v_{e''})$-trail. By Theorem 2.1(iii), $G(e', e'')$ has a spanning $(v_{e'}, v_{e''})$-trail. □

Theorem 2.2 (vi) improved Theorem 4 of [7]. Let P(10) denote the Petersen graph and $P(14)$ be the 3-regular graph formed by blowing up a vertex of $P(10)$ by a $K_{2,3}$. We follow [25] to denote the Wagner graph by H_8. Both $P(14)$ and H_8 are depicted in Fig. 1. Let P^n be a path of order n.

Theorem 2.3 (Chen and Chen, Theorem 1.1 of [10]) *Let G be a 3-edge-connected graph with at most 15 vertices. Let G' be the reduction in G. Then, each of the following holds:*

(i) If $|V(G)| \leq 13$, then either G is supereulerian or $G' \cong P(10)$.
(ii) If $|V(G)| \leq 14$, then either G is supereulerian or $G' \in \{P(10), P(14)\}$.
(iii) If $|V(G)| = 15$, G is not supereulerian and $G' \notin \{P(10), P(14)\}$, then G is a 2-connected and essentially 4-edge-connected reduced graph with girth at least 5 and $V(G) = D_3(G) \cup D_4(G)$, such that $D_4(G)$ is a stable set with $|D_4(G)| = 3$.

Theorem 2.4 (Chen et al., Corollary 4.10 of [13]) *Let G be a connected graph and G' be the reduction in G. If $|V(G)| \leq 15$ and $\kappa'(G) \geq 3$, then G is supereulerian if and only if $G' \notin \{P(10), P(14)\}$.*

Some prior results on reduced graphs of small orders are given in the following theorem:

Theorem 2.5 *Let G be a simple connected graph of order n.*

(i) (Chen [11]). If $n \leq 7$, $\kappa'(G) \geq 2$, and $|D_2(G)| \leq 2$, then G is collapsible.
(ii) (Catlin [6]). If $n \leq 8$, $\kappa'(G) \geq 2$ and $|D_2(G)| \leq 1$, then G is collapsible.
(iii) (Chen [10]). If $n \leq 9$, $\kappa'(G) \geq 2$ and $|D_2(G)| \leq 2$, then $G' \in \{K_1, K_{2,3}\}$. Furthermore, if $g(G) \geq 4$, then G is collapsible.

In the following, we summarize prior results on the relationship between $ess'(G)$ and $\alpha'(G)$ which may warrant the existence of (possibly open) spanning trails.

Theorem 2.6 *Let G be a connected graph. Each of the following holds.*

(i) (Zhan [32]) If $\kappa'(G) \geq 3$ and $ess'(G) \geq 7$, then G has two-edge-disjoint spanning tree.
(ii) (Chen et al., Theorem 4.4 of [13]) If G is reduced, $n = |V(G)|$ and $\delta(G) \geq 3$, then $\alpha'(G) \geq \min\{\frac{n}{2}, \frac{n+5}{3}\}$.
(iii) (Theorem 2 of [18]) If $\kappa'(G) \geq 2$ and $\alpha'(G) \leq 2$, then G is supereulerian if and only if G is not $K_{2,t}$ for some odd number t.

Recently, Li et al. [30] further improved Theorem 2.6(iii) and proved the following Theorem 2.8(i). Here, we first describe the graph family $\mathcal{F}'$, which is the excluded graph family stated in Theorem 2.8(i).

Definition 2.7 [11] (The families $\mathcal{F}$ and $\mathcal{F}'$). Let $i, s_1, s_2, s_3, m, n, t$ be integers with $t \geq 2$ and $i,\ m,\ n \geq 1$.

(i) Let $M \cong K_{1,3}$ with center a and ends a_1, a_2, a_3. Define $K_{1,3}(s_1, s_2, s_3)$ to be the graph obtained from M by adding s_i vertices with neighbors a_i, a_{i+1}, where $i \equiv 1,\ 2,\ 3 \pmod 3$. Define $C^6(s_1, s_2, s_3) = K_{1,3}(s_1, s_2, s_3) - a$.
(ii) Let m and n be two positive integers, $H_1 \cong K_{2,m}$ and $H_2 \cong K_{2,n}$ be two complete bipartite graphs. Let u_1 and v_1 be two nonadjacent vertices of degree m in H_1 and u_2 and v_2 be two nonadjacent vertices of degree n in H_2. Define $S(m, n)$ to be the graph obtained from H_1 and H_2 by identifying u_1 with u_2 and by adding a new edge v_1v_2 joining v_1 and v_2. As an example, $S(1, 1)$ is the 5-cycle.
(iii) Let $K_{2,3}(1, 2, 2)$ be the union of three internally disjoint (u, w)-paths of lengths 2,3 and 3, respectively.

In Fig. 2, we depict some graphs in Definition 2.7 with small parameters.
Define

$$\mathcal{F} = \left\{K_{2,3}(1,2,2)\right\} \bigcup \left\{K_{2,2t+1} : t \geq 1\right\} \bigcup \left\{K_{1,3}\left(s, s', s''\right), C^6\left(s, s', s''\right) : s > s' > 0, s'' \geq 0\right\}$$

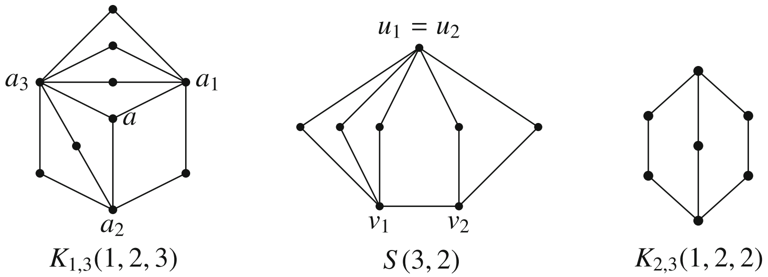

Fig. 2 Some graphs in Definition 2.7 with small parameters

$$\bigcup \{S_{m,n} : m, n \geq 1\},$$
$$\mathcal{F}' = \{G \in \mathcal{F} : G\ is\ non\ supereulerian\}. \tag{3}$$

The following former results are useful.

Theorem 2.8 *Let G be a connected graph. Each of the following holds.*

(i) (Li et al., Theorem 1.3 of [30]) If $\kappa'(G) \geq 2$ and $\alpha'(G) \leq 3$, then G is supereulerian if and only if the reduction in G is not a member in $\mathcal{F}'$.

(ii) (Chen et al., Theorem 4.9 of [13]) Suppose that $n = |V(G)|$, $\kappa'(G) \geq 3$, and G' be the reduction in G. If $\alpha'(G) \leq 7$, then G is supereulerian if and only if $G' \notin \{P(10), P(14)\}$.

Lemma 2.9 *If G is a graph satisfying $\kappa'(G) \geq 2$, $g(G) \geq 4$, $\gamma(G) < 2$, $d_2(G) \leq 2$, and $n = |V(G)| \leq 10$, then $n = 10$ and either G is collapsible or G is reduced with $d_2(G) = 2$ and $d_3(G) = 8$.*

Proof Throughout the proof, we use $d_i = d_i(G)$. As $\kappa'(G) \geq 2$ and $d_2 \leq 2$, $G \notin \{K_2\} \cup \{K_{2,t} : t \geq 1\}$. By Theorem 2.2(ii), we have $2n - |E(G)| - 2 = F(G) \geq 3$, and so

$$\begin{cases} 2d_2 + 3d_3 + \cdots + (n-1)d_{n-1} \leq 4n - 10 \\ d_2 + d_3 + \cdots + d_{n-1} = n. \end{cases} \tag{4}$$

It is routine to show that when $n \leq 6$, system (4) has no integral solutions, and so the lemma holds for $n \leq 6$. Let G' be the reduction in G. If G' is a $K_{2,t}$ for some $t \geq 2$, then since $g(G) \geq 4$ and by the definition of collapsible graphs, every nontrivial vertex of G' must contain at least 6 vertices in G, and so by $d_2 \leq 2$, exactly 2 vertices in G' must be trivial vertices. It follows that $G' = K_{2,3}$ with exactly one vertex $v_0 \in D_2(G')$ being a nontrivial vertex in the contraction. But then, $H = PI_G(v_0)$ satisfies the hypotheses of the lemma with $|V(H)| \leq n - |V(K_{2,3} - v_0)| \leq 10 - 4 = 6$. It is known that no such H exists. Hence, we must have $G = G'$ and so G is reduced and the parameters of G must satisfy system (4). It is now routine, for example, examining each value of $n \in \{7, 8, 9, 10\}$, to see that system (4) has no integral solution except that when $n = 10$, $d_2 = 2$ and $d_3 = 8$. □

Lemma 2.10 *(Li et al. Lemma 2.2(iv) of [22] and Wang [29]) Let G be a connected graph with $n = |V(G)| \geq 3$ and $\kappa'(G) \geq 3$. If $n \leq 11$, then for any $e \in E(G)$, then either $G(e)$ is collapsible or $n = 11$ and $G(e) \cong P(10)(e)$.*

3 Proof of the Main Results

Let G be a graph with $ess'(G) \geq 3$. The *core* of G is obtained from $G - D_1(G)$ by contracting exactly one edge xy or yz for each path xyz in G with $d_G(y) = 2$. Throughout this section, we use G_0 to denote the core of G. As $G - D_1(G)$ is also the graph formed by contracting all edges incident with a vertex in $D_1(G)$, G_0 is a contraction of G. Observation (5) follows from the definitions.

$$ess'(G_0) \geq ess'(G), \kappa'(G_0) \geq \kappa'(G), \text{ and } \alpha'(G_0) \leq \alpha'(G), \tag{5}$$

We start with some lemmas.

Lemma 3.1 *(Shao [26]) Let G be a connected nontrivial graph with $ess'(G) \geq 3$. Each of the following holds.*

(i) The core G_0 is uniquely determined by G and $\kappa'(G_0) \geq 3$.
(ii) If G_0 is supereulerian, then $L(G)$ is Hamiltonian.
(iii) (see also Lemma 2.9 of [17]) If G_0 is strongly spanning trailable, then $L(G)$ is Hamilton-connected.

Lemma 3.2 *If G be a graph with $ess'(G) \geq \max\{3, \alpha'(G)\}$, then G_0 is supereulerian.*

Proof By (5), $ess'(G_0) \geq ess'(G)$, $\alpha'(G) \geq \alpha'(G_0)$. Since $ess'(G) \geq \alpha'(G)$, it follows that $ess'(G_0) \geq \alpha'(G_0)$. By Lemma 3.1(i), $ess'(G_0) \geq \kappa'(G_0) \geq 3$. If $ess'(G_0) \geq 7$, then by Theorem 2.6(i), G_0 has two-edge-disjoint spanning tree, and so by Theorem 2.2(iii), G_0 is supereulerian.

Assume that $3 \leq ess'(G_0) \leq 6$. Let G_0' be the reduction in G_0. By Lemma 3.1, $\delta(G_0') \geq \delta(G_0) \geq \kappa'(G_0) \geq 3$. Let $|V(G_0')| = n$. By Theorem 2.6(ii), $\alpha'(G_0') \geq \min\{\frac{n}{2}, \frac{n+5}{3}\}$. If $\frac{n}{2} \geq \frac{n+5}{3}$, then as $6 \geq ess'(G_0) \geq \alpha'(G_0')$, we have $\frac{n+5}{3} \leq 6$, and so $10 \leq n \leq 13$. If $\frac{n}{2} \leq \frac{n+5}{3}$, then $n \leq 10$. It follows $n \leq 13$. As G_0' is reduced and $n \leq 13$, by Theorem 2.3(i), then either G_0' is supereulerian or $G_0' \cong P(10)$. As $\alpha'(P(10)) = 5 > ess'(P(10)) = 4$, $G_0' \neq P(10)$. Hence, G_0' must be supereulerian. By Theorem 2.1(iii), G_0 is also supereulerian. This proves Lemma 3.2. □

Lemma 3.3 *Let G be a connected, essentially 3-edge-connected graph.*

(i) If G_0 is supereulerian, then G has an almost spanning closed trail.
(ii) If G_0 has a spanning trail, then G has an almost spanning trail.
(iii) If G_0 is strongly spanning trailable, then for any $e, e' \in E(G)$, G has an almost spanning $(v_e, v_{e'})$-trail.

Proof Assume G_0 is supereulerian. Let H' be a spanning Eulerian subgraph of G_0. We will construct an almost spanning closed trail H of G as follows. For each $v \in D_2(G)$ with $N_G(v) = \{u_1^v, u_2^v\}$, by the definition of G_0, $u_1^v u_2^v \in E(G_0)$. Let $H'' = H' - \cup_{v \in D_2(G)} u_1^v u_2^v$. As H' is a spanning Eulerian subgraph, for each $v \in D_2(G)$, we have $d_{H''}(u_1^v) \equiv d_{H''}(u_2^v) \pmod 2$. For each $v \in D_2(G)$, define

$$X_v = \begin{cases} K_{2,t_1}, \textit{where } |[v] - t_1| \leq 1, \text{ if } d_{H''}(u_1^v) \equiv d_{H''}(u_2^v) \equiv t_1 \equiv 1 \pmod 2, \\ K_{2,t_2}, \textit{where } |[v] - t_2| \leq 1, \text{ if } d_{H''}(u_1^v) \equiv d_{H''}(u_2^v) \equiv t_2 \equiv 0 \pmod 2, \end{cases} \tag{6}$$

where u_1^v and u_2^v are the two nonadjacent vertices of degree t_1 (if $d_{H''}(u_1^v)$ is odd) or t_2 (if $d_{H''}(u_1^v)$ is even). It follows by (6) that the subgraph $H = G[E(H'') \cup (\cup_{v \in D_2(G)} X_v)]$ is an almost spanning closed trail of G. This proves (i).

Suppose that G_0 has a spanning (w_1, w_2)-trail T. By Lemma 3.3(i), we may assume that $w_1 \neq w_2$. Let $\tilde{G}_0 = G_0 + w_1 w_2$. Then, $H' = T + w_1 w_2$ is a spanning closed trail of $\tilde{G}_0$, and so $\tilde{G}_0$ is supereulerian. Since G_0 is a contraction of G, for $i \in \{1, 2\}$, let w_i' be a vertex in the contraction preimage of w_i in G. Then by Lemma 3.3(i), $G + w_1' w_2'$ has an almost spanning closed trail T' using the edge $w_1' w_2'$, and so $T' - w_1' w_2'$ is an almost spanning trail of G. This proves (ii).

We justify Lemma 3.3(iii) by considering different possibilities of e and e'. If $e \in E(G_0)$, then let $e_1 = e$; if $e = uv$ with $u \in D_1(G) \cup D_2(G)$, then let e_1 be an edge of G_0 incident with v. Likewise, if $e' \in E(G_0)$, then let $e_2 = e'$; if $e' = u'v'$ with $u' \in D_1(G) \cup D_2(G)$, then let e_2 be an edge of G_0 incident with v'. By assumption, $G_0(e_1, e_2)$ has a spanning (v_{e_1}, v_{e_2})-trail, which can be lifted to an almost spanning (v_{e_1}, v_{e_2})-trail T' of $G(e_1, e_2)$ by using the same arguments as in the proof for Lemma 3.3(i) and by utilizing (6). By the choices of e_1 and e_2, it is routine to show that this trail T' can be adjusted to an almost spanning $(v_e, v_{e'})$-trail of G. □

Corollary 3.4 *Let G be a connected graph, G' is the reduction in G, if $G' \in \mathcal{F}'$, then G has an almost spanning trail.*

Proof Let G_0' be the core of G'. As $G' \in \mathcal{F}'$, it is routine to verify that G_0' is supereulerian. So G_0' has a spanning trail. By Theorem 2.1(iii), G_0 has a spanning trail. By Lemma 3.3(ii), then G has an almost spanning trail. □

3.1 Proof of Theorem 1.5(i)

Assume that $ess'(G) \leq 2$, then $\alpha'(G) \leq ess'(G) \leq 2$. As $G_1 = G - D_1(G)$ can be viewed as a contraction of G, $\kappa'(G_1) \leq ess'(G) \leq 2$. By Theorem 2.6(iii), G_1 is supereulerian if and only if G_1 is not isomorphic to a $K_{2,t}$, for some odd integer $t \geq 3$. Since $ess'(K_{2,t}) \geq 3$, G_1 cannot be isomorphic to a $K_{2,t}$, and so we conclude that G_1 is supereulerian. It follows by the definition of G_1 that G has an almost spanning closed trail. Therefore, we may assume that $ess'(G) \geq 3$.

By Lemma 3.1(i), G_0 is well-defined with $\kappa'(G_0) \geq 3$. As $ess'(G_0) \geq ess'(G) \geq \alpha'(G) \geq \alpha'(G_0)$, it follows by Lemma 3.2 that G_0 is supereulerian. By Lemma 3.3(i), G has an almost spanning closed trail. This completes the proof for Theorem 1.5(i). □

3.2 Proof of Theorem 1.5(ii)

To prove Theorem 1.5(ii), we need the following tools. Let $G_1 = G - D_1(G)$, and G_1' be the reduction in G_1. Assume first that $\kappa'(G_1) \geq 3$. If $\alpha'(G_1) \geq 8$, then $ess'(G_1) \geq \alpha'(G_1) - 1 \geq 8 - 1 = 7$. By Theorem 2.6(i), $F(G_1) = 0$, and so by Theorem 2.2(iii), G_1 is collapsible. Hence, $G - D_1(G)$ has a spanning trail. If $\alpha'(G_1) \leq 7$, then by Theorem 2.8(ii), G_1 is supereulerian if and only if $G_1' \notin \{P(10), P(14)\}$. As each of $P(10)$ and $P(14)$ has a spanning trail, G_1' has a spanning trail in any case. By Theorem 2.1(iii), G_1 has a spanning trail. Therefore, we assume that $\kappa'(G_1) = 2$.

By Theorem 2.8(i), if $\alpha'(G_1) \leq 3$, then G_1 is supereulerian if and only if the reduction in G_1 is not a member in $\mathcal{F}'$. If $G_1 \in \mathcal{F}'$, then by Corollary 3.4, G_1 has an almost spanning trail. Hence, we may assume that $\alpha'(G_1) \geq 4$, and so $ess'(G_1) \geq \alpha'(G_1) - 1 \geq 3$. Let G'_0 be the reduction in the G_0. By (5) and by assumption, $ess'(G'_0) \geq \alpha'(G'_0) - 1 \geq 3$. By Lemma 3.1(i), $\kappa'(G'_0) \geq 3$. If $\alpha'(G'_0) \geq 8$, then as $ess'(G'_0) \geq \alpha'(G'_0) - 1 \geq 7$, it follows by Theorem 2.6(i) that $F(G'_0) = 0$, and so by Theorem 2.2(iii) and Theorem 2.1, G_0 is collapsible. By Lemma 3.3(i), G has an almost spanning trail. Thus, we may assume $4 \leq \alpha'(G'_0) \leq 7$. Let $n = |V(G'_0)|$. By Theorem 2.6(ii), we have $\alpha'(G'_0) \geq min\{\frac{n}{2}, \frac{n+5}{3}\}$, and so

$$n = |V(G'_0)| \leq \begin{cases} 8, & \text{if } \alpha'(G'_0) = 4, \\ 10, & \text{if } \alpha'(G'_0) = 5, \\ 13, & \text{if } \alpha'(G'_0) = 6, \\ 16, & \text{if } \alpha'(G'_0) = 7. \end{cases}$$

If $|V(G'_0)| \leq 15$, by Theorem 2.4, then either G'_0 is supereulerian, whence by Theorem 2.1(iii) and Lemma 3.3(i), G has an almost spanning trail; or $G'_0 \in \{P(10), P(14)\}$, whence G'_0 has a spanning trail, and so by Theorem 2.1(iii) and Lemma 3.3 (ii), G has an almost spanning trail.

Hence, we may assume that $n = |V(G'_0)| = 16$. By Theorem 2.6(ii), we have $\alpha'(G'_0) \geq \frac{16+5}{3} = 7$. By assumption and (5), $ess'(G'_0) \geq ess'(G_0) \geq \alpha'(G'_0) - 1 \geq 6$ and $\kappa'(G'_0) \geq 3$. If $F(G'_0) \leq 2$, then by Theorem 2.2(iii), $G'_0 = K_1$ and so by Theorem 2.1 and Lemma 3.3(i), Theorem 1.5(ii) holds. Hence in the following analysis, we always assume that $n = |V(G_0)| = 16$ and $F(G'_0) \geq 3$ to find a contradiction to complete the proof.

For each integer i, let $d_i = |D_i(G'_0)|$. As $\delta(G'_0) \geq \kappa'(G'_0) \geq 3$, $d_1 = d_2 = 0$. Since $n = \sum_{j \geq 1} d_j$ and $2|E(G'_0)| = \sum_{j \geq 1} j d_j$, by Theorem 2.2(ii), we have

$$6 \leq 2F\left(G'_0\right) = d_3 - \sum_{j \geq 5} (j-4) d_j - 4,$$

which leads to

$$\begin{aligned} &10 + d_5 + 2d_6 + 3d_7 + 4d_8 + 5d_9 + \sum_{j \geq 10} (j-4) d_j \\ &\leq d_3 \leq n - d_4 - d_5 - d_6 - d_7 - d_8 - d_9 - \sum_{j \geq 10} d_j. \end{aligned} \tag{7}$$

If $d_j \geq 1$ for some $j \geq 10$, then by (7), $16 \leq d_3 \leq n - d_j \leq 15$, a contradiction. Hence, $d_j = 0$ for any $j \geq 10$. If $d_9 \geq 1$, then by (7), $15 \leq d_3 \leq 15$, forcing $d_3 = 15$, $d_9 = 1$ and $d_j = 0$ if $j \notin \{3, 9\}$. Thus, $D_3(G'_0)$ cannot be an independent set of G'_0, implying $ess'(G'_0) \leq 3 + 3 - 2 = 4$, contrary to $ess'(G'_0) \geq 6$. Hence, $d_9 = 0$. As $ess'(G_0) \geq 6$, we conclude that

$$\text{for any } j \geq 9, d_j = 0, \text{ and both } E(G[D_3(G'_0)]) = \emptyset \text{ and}$$

$$N_{G_0'}(D_3(G_0')) \subseteq \cup_{i\geq 5} D_i(G_0'). \tag{8}$$

Suppose $d_5 \geq 1$. By (7), $d_3 \geq 11$, and so there must be $3 \times 11 = 33$ edges incident with vertices $\cup_{i\geq 5} D_i(G_0')$. By (8), $d_j = 0$ for any $j \geq 9$, and so

$$\sum_{4\leq j\leq 8} d_j \geq \lceil 33/8 \rceil = 5. \tag{9}$$

By (7), we have $d_8 \leq 1$. If $d_8 = 1$, then by (7), $10 + d_5 + 2d_6 + 3d_7 + 4 \leq d_3 \leq 16 - d_4 - d_5 - d_6 - d_7 - 1$, forcing $14 \leq d_3 \leq 15 - \sum_{4\leq j\leq 7} d_j$. Hence $\sum_{4\leq j\leq 8} d_j \leq 2$, contrary to (9). This implies that $d_8 = 0$. By (7) and (8), we have

$$\text{for any } j \geq 8, d_j = 0, \text{ and } 10 + d_5 + 2d_6 + 3d_7 \leq d_3 \leq 16 - d_4 - d_5 - d_6 - d_7. \tag{10}$$

If $d_7 \geq 2$, then by (10), $16 \leq d_3 \leq 14$, a contradiction. If $d_7 = 1$, then by (10), $13 \leq d_3 \leq 15 - \sum_{4\leq j\leq 6} d_j$. It follows that $\sum_{4\leq j\leq 7} d_j \leq 3$, contrary to (9). Hence $d_7 = 0$. This, together with (10), implies that (7) now reduces to

$$\text{for any } j \geq 7, d_j = 0, \text{ and } 10 + d_5 + 2d_6 \leq d_3 \leq 16 - d_4 - d_5 - d_6. \tag{11}$$

If $d_6 \geq 3$, then by (11), $16 \leq d_3 \leq 13$, a contradiction. If $d_6 = 2$, then by (11), $14 \leq d_3 \leq 14$, whence $\sum_{4\leq j\leq 6} d_j = 2$, contrary to (9). If $d_6 = 1$, then by (11), we have $12 + d_5 \leq d_3 \leq 15 - d_4 - d_5$. Therefore, $d_4 + d_5 \leq 3$ and so $\sum_{4\leq j\leq 6} d_j = 4$, contrary to (9) again. Hence $d_6 = 0$, which further reduces (11) to

$$\text{for any } j \geq 6, d_j = 0, \text{ and } 10 + d_5 \leq d_3 \leq 16 - d_4 - d_5. \tag{12}$$

If $d_5 \geq 4$, then by (12), $14 \leq d_3 \leq 12$, a contradiction. Hence, $d_5 \leq 3$ and $d_5 = 3$ only if $d_4 = 0$. By (12), $d_4 \leq 6$ and $d_4 = 6$ only when $d_5 = 0$. As $D_3(G_0')$ is an independent set, we have $\sum_{v\in D_3(G_0')} d(v) \leq |E(G_0')| \leq \sum_{v\in V(G_0')-D_3(G_0')} d(v)$. Thus if $d_5 = 3$, then $d_3 = 13$ and $39 \leq \sum_{v\in D_3(G_0')} d(v) \leq |E(G_0')| \leq \sum_{v\in V(G_0')-D_3(G_0')} d(v) = 5d_5 \leq 15$, a contradiction; if $d_4 = 6$, then $d_3 \geq 10$ and $30 \leq \sum_{v\in D_3(G_0')} d(v) \leq |E(G_0')| \leq \sum_{v\in V(G_0')-D_3(G_0')} d(v) = 4d_6 \leq 24$, another contradiction. This, together with (12) the assumption of $d_5 \geq 1$, we must have either $d_4 \leq 5$ and $d_5 = 1$, whence by $d_3 \geq 10$, $30 \leq \sum_{v\in D_3(G_0')} d(v) \leq |E(G_0')| \leq \sum_{v\in V(G_0')-D_3(G_0')} d(v) = 4d_4 + 5d_5 \leq 25$, a contradiction; or $d_4 \leq 4$ and $1 \leq d_5 \leq 2$, whence by $d_3 \geq 10$, $30 \leq \sum_{v\in D_3(G_0')} d(v) \leq |E(G_0')| \leq \sum_{v\in V(G_0')-D_3(G_0')} d(v) = 4d_4 + 5d_5 \leq 26$, another contradiction. This indicates that we must have $d_5 = 0$.

Recall that $n = 16$, as $d_5 = 0$ and by (12), we must have $d_3 \geq 10$ and $d_4 \leq n - d_3 \leq 6$. Again by (8), both $E(G[D_3(G_0')]) = \emptyset$ and $N_{G_0'}(D_3(G_0')) \subseteq \cup_{i\geq 5} D_i(G_0')$, which implies that $30 \leq 3d_3 \leq |E(G_0')| \leq 4d_4 \leq 24$, a contradiction. This completes the proof of Theorem 1.5(ii). □

3.3 A Matching Bound for the Proof of Theorem 1.5(iii)

The main result of this subsection proves a lower bound of the matching number, which is a needed tool for our proof Theorem 1.5(iii). However, the main arguments are modifications of those in the proofs of Lemma 4.3 and Theorem 4.4 of [13]. As the conclusions are not the same, we include the proofs here for the sake of completeness.

A component H of G is an *odd component* if $|V(H)| \equiv 1 \pmod 2$. Let $o(G) = |\{Q : Q$ be an odd component of $G\}|$. Tutte [28] and Berge [2] proved the following theorem.

Theorem 3.5 (Tutte [28]; Berge [2]) *Let G be a graph with n vertices. Then, $\alpha'(G) = (n-t)/2$, if*

$$t = \max_{S \subset V(G)} \{o(G-S) - |S|\}. \tag{13}$$

The following lemma can be justified by the same argument or a slight modification in counting as those in Lemma 4.3 of [13].

Lemma 3.6 *Let G be a connected graph with $|D_1(G)| = 0$, $|D_2(G)| \le 2$ and $g(G) \ge 4$. Suppose that $S \subseteq V(G)$ is a vertex subset attaining the maximum in (13) with $|S| > 0$, $m = o(G-S)$ and that $G_1, G_2, \cdots, G_m$ are the odd components of $G-S$ satisfying $|V(G_1)| \le |V(G_2)| \le \cdots \le |V(G_m)|$. Define*

$$\begin{aligned}
&X = \{G_i : |V(G_i)| = 1, 1 \le i \le m\},\\
&Y = \{G_i : |V(G_i)| = 3, 1 \le i \le m\}, x = |X|, y = |Y|.\\
&V^* = \bigcup_{k=1}^{x+y} V(G_k),\ \ G^* = G[V^* \cup S^*] \text{ and}\\
&s^* = |S^*|,\ \ \textit{where } S^* = \{s \in S : v^*s \in E(G),\ v^* \in V^*\}.
\end{aligned} \tag{14}$$

Thus, G^* is spanned by a bipartite subgraph with (V^*, S^*) being its vertex bipartition with $|V^*| = x + 3y \ge 1$. Each of the following holds.

(i) $n \ge \sum_{i=1}^{m} |V(G_i)| + |S| \ge m|V(G_1)| + |S|$ and, if $|S| \ge 2$, then $G^* \notin \{K_1, K_2, K_{1,2}\}$.

(ii) If $x > 0$, then $s^* \ge 2$.

(iii) $m \le \frac{n+4x+2y-|S|}{5}$.

(iv) $|E(G^*)| \ge 3x + 7y - 2$.

Theorem 3.7 *Let G be a connected reduced graph with n vertices, $d_1(G) = 0$ and $d_2(G) \le 2$. Then, $\alpha'(G) \ge \min\{\frac{n-1}{2}, \frac{n+3}{3}\}$.*

Proof Let t be defined as in (13). By Theorem 3.5, we may assume that $t \ge 2$. By Theorem 2.2(i), we have $\gamma(G) < 2$ and $g(G) \ge 4$. By Lemma 2.9, we may assume that $n \ge 10$ and so $\frac{n+3}{3} < \frac{n-1}{2}$. By Theorem 3.5, to prove Theorem 3.7, it suffices to show that

$$\alpha'(G) \ge \frac{n-t}{2} \ge \frac{n+3}{3}, \text{ or equivalently },\ t \le \frac{n-6}{3}. \tag{15}$$

If $x = y = 0$, then $|V(G_1)| \geq 5$, and so by Lemma 3.6(i) that $n \geq 5m + |S|$, or $m \leq \frac{n-|S|}{5}$. It follows that

$$t = m - |S| \leq \frac{n - 6|S|}{5} \leq \frac{n-6}{5},$$

and so (15) must hold. Therefore, we may assume that $x + y > 0$, and so $|S| \geq \delta(G) \geq 2$.

If $x = 0$, then $|V(G_1)| \geq 3$, and so by Lemma 3.6(i) that $n \geq 3m + |S|$, or $m \leq \frac{n-|S|}{3}$. Thus, $|S| \geq 2$, (15) follows:

$$t = m - |S| \leq \frac{n - 4|S|}{3} \leq \frac{n-8}{3}.$$

Therefore, we may assume that $x > 0$. If $F(G^*) \leq 2$, then by Theorem 2.2(iv) and Lemma 3.6(i), and as $d_2(G) \leq 2$, we must have $G^* = K_{2,2}$ and so $x = 2$ ad $y = 0$. It follows by Lemma 3.6(iii) and by $n \geq 10$ that (15) must hold:

$$t = m - |S| \leq \frac{n + 8 - 6|S|}{5} \leq \frac{n + 8 - 12}{5} < \frac{n-6}{3}.$$

Therefore, we may assume that $F(G^*) \geq 3$, and so $y > 0$. By Lemma 3.6(iv) and Theorem 2.2(ii), $3x + 7y - 2 \leq |E(G^*)| \leq 2|V(G^*)| - 5 \leq 2(x + 3y + |S|) - 5$. This leads to $x + y \leq 2|S| - 3$ or $6|S| \geq 3(x + y + 3)$. It follows by Lemma 3.6(i) and by $y > 0$ that $n \geq x + 3y + |S| \geq \frac{3x+7y+3}{2} \geq \frac{3x-3y+3}{2}$. This, together with Lemma 3.6(ii) and $n \geq 10$, implies that

$$\begin{aligned} t = m - |S| &\leq \frac{n + 4x + 2y - 6|S|}{5} \leq \frac{n + 4x + 2y - 3(x + y + 3)}{5} \\ &= \frac{n + x - y - 9}{5} \leq \frac{n-6}{3}. \end{aligned}$$

Thus (15) always holds, and so the theorem is proved. □

Let G be a graph with $n = |V(G)|$, $\kappa'(G) \geq 2$ and $\gamma(G) \leq 2$. By Theorem 2.2(ii), $2|E(G)| = 4n - 4 - 2F(G)$. As $2|E(G)| = \sum_{i \geq 2} i d_i$ and $n = \sum_{i \geq 2} d_i$, we have

$$2F(G) + 4 + \sum_{j \geq 5} (j - 4) d_j \leq 2d_2 + d_3 \leq n + d_2 - \sum_{j \geq 4} d_j. \tag{16}$$

Corollary 3.8 *If G is a graph with $\kappa'(G) \geq 3$ and $\gamma(G) \leq 2$. If $ess'(G) \geq \alpha'(G) + 1$, then G is strongly spanning trailable.*

Proof By contradiction, we assume that for some edges $e', e'' \in E(G)$, $G(e', e'')$ does not have a spanning $(v_{e'}, v_{e''})$-trail. By Theorem 2.2(iv), we may assume that $F(G) \geq 1$. Let $n = |V(G)|$. By (16) with $\kappa'(G) \geq 3$ and $F(G) \geq 1$, we have

$$6 + d_5 + 2d_6 + 3d_7 + 4d_8 + \sum_{j\geq 9}(j-4)d_j$$
$$\leq d_3 \leq n - d_4 - d_5 - d_6 - d_7 - d_8 - \sum_{j\geq 9} d_j. \quad (17)$$

By Theorem 2.6(i), if $ess'(G) \geq 7$, then $F(G) = 0$. Hence, we may assume that $ess'(G) \leq 6$.

Assume first that $n \leq 9$, which implies that $\frac{n-1}{2} \leq \frac{n+3}{3}$. As $\alpha'(G) \leq ess'(G) - 1$ and by Theorem 3.7, we conclude that

$$n \leq 2ess'(G) - 1. \quad (18)$$

If $n \leq 7$, then construct a new graph J from $G(e', e'')$ by adding a new vertex w and two new edges $wv_{e'}$ and $wv_{e''}$. Observe that $|V(J)| \leq 10$ and, as $\kappa'(G) \geq 3$, $\kappa'(J/wv_{e'}) \geq 3$ also. It follows by Lemma 2.10 that J is collapsible, and so J has a spanning Eulerian subgraph T. But then $T - w$ is a spanning $(v_{e'}, v_{e''})$-trail of $G(e', e'')$, contrary to the assumption that $G(e', e'')$ does not have a spanning $(v_{e'}, v_{e''})$-trail. Hence, we may assume that $8 \leq n \leq 9$, and so by (18), $ess'(G) \in \{5, 6\}$. This implies that $E(G[D_3(G)]) = \emptyset$. Since $n \leq 9$ and by (17), we conclude that $d_j = 0$ for any $j \geq 7$ and $d_6 \leq 1$. As $d_3 \geq 6$, $d_4 + d_5 + d_6 = n - d_3 \leq 3$. It follows by $E(G[D_3(G)]) = \emptyset$ that $18 \leq 3d_3 \leq |E(G)| \leq 4d_4 + 5d_5 + 6d_6 \leq 5 \times 2 + 6 = 16$, a contradiction.

Hence, we may assume that $n \geq 10$, which implies that $\frac{n-1}{2} > \frac{n+3}{3}$. By Theorem 3.7 and as $\alpha'(G) \leq ess'(G) - 1$, we conclude that

$$n \leq 3(ess'(G) - 2). \quad (19)$$

Thus by (19), we must have $ess'(G) = 6$ and $n \in \{10, 11, 12\}$. By (17), for any $j \geq 10$, $d_j = 0$ and $d_9 \leq 1$. If $d_9 = 1$, then by (17), $11 \leq d_3 \leq 11$, forcing $d_3 = 11$, $d_9 = 1$ and $d_j = 0$ if $j \notin \{3, 9\}$. Thus, $D_3(G)$ cannot be an independent set of G, implying $ess'(G) \leq 3+3-2 = 4$, contrary to $ess'(G) = 6$. Hence $d_9 = 0$. If $d_7 + d_8 > 0$, then by (17), $d_7 + d_8 \leq 1$, $d_3 \geq 9$, and $d_4 + d_5 + d_6 + d_7 + d_8 \leq 12 - d_3 \leq 3$. It follows by $E(G[D_3(G)]) = \emptyset$ that $27 \leq 3d_3 \leq |E(G)| \leq 4d_4 + 5d_5 + 6d_6 + 7d_7 + 8d_8 \leq 2 \times 6 + 8 = 20$, a contradiction. This implies that $d_7 + d_8 = 0$. Thus for any $j \geq 7$, $d_j = 0$. If $d_5 + d_6 \geq 1$, by (17), we have $d_3 \geq 7$, and so there must be $3 \times 7 = 21$ edges incident with vertices $\cup_{i\geq 5} D_i(G)$. Since $d_j = 0$ for any $j \geq 7$, $d_4 + d_5 + d_6 \geq \lceil 21/6 \rceil = 4$. Hence by (17), $10 \leq d_3 \leq 12 - 4 = 8$, a contradiction. This implies that $d_5 = d_6 = 0$ also, and so a vertex in $D_3(G)$ must be adjacent to a vertex in $D_4(G)$ in G, causing a contradiction to the assumption of $ess'(G) \geq 6$. This justifies the corollary. □

3.4 Proof of Theorem 1.5(iii)

Additional lemmas are needed in our arguments to prove Theorem 1.5(iii).

Lemma 3.9 *(Lemma 2.5 of* [12]*, see also Lemma 4.2.1 of* [29]*). Let $e, e' \in E(G)$, H be a collapsible subgraph of $G(e, e')$ and v_H denote the vertex in $G(e, e')/H$ onto which H is contracted. Define*

$$v'_e = \begin{cases} v_e & \text{if } v_e \notin V(H), \\ v_H & \text{if } v_e \in V(H), \end{cases} \quad \text{and } v'_{e'} = \begin{cases} v_{e'} & \text{if } v_{e'} \notin V(H), \\ v_H & \text{if } v_{e'} \in V(H). \end{cases}$$

If $G(e, e')/H$ has a spanning $(v'_e, v'_{e'})$-trail, then $G(e, e')$ has a spanning $(v_e, v_{e'})$-trail.

Lemma 3.10 *Let $k \geq 1$ be an integer and G be a connected nontrivial graph.*

(i) (Nash–Williams [24]*, see also Yao et al., Theorem 2.4 of* [31]*) If $|E(G)| \geq k(|V(G)| - 1)$, then G contains a nontrivial subgraph H that contains k-edge-disjoint spanning trees.*
(ii) (Theorem 1.5 of [20]*) If $F(G) = 0$ and $\gamma(G) > 2$, then for any edge $e \in E(G)$, $F(G - e) = 0$.*

We start the proof of Theorem 1.5(iii). If $\alpha'(G) = 1$, then G is either spanned by a K_3 or there exists an vertex $v \in V(G)$ such that very edge of G is incident with v. Thus, it is routine to verify that for any edges $e, e' \in E(G)$, $G(e, e')$ always has a $(v_e, v_{e'})$-trail that misses only vertices in $D_1(G)$ and at most one vertex in $D_2(G)$. Therefore, we shall assume that $ess'(G) \geq \alpha'(G) + 1 \geq 3$.

Let G_0 be the core of G. By (5), $ess'(G_0) \geq ess'(G) \geq 3$. By Lemma 3.3(iii), it suffices to show that

$$\text{if } ess'(G) \geq \alpha'(G) + 1 \geq 3, \text{ then } G_0 \text{ is strongly spanning trailable.} \tag{20}$$

We shall prove (20) by contradiction, and assume that

$$G \text{ is a counterexample to (20) with } |V(G)| + |E(G)| \text{ minimized.} \tag{21}$$

Therefore, there exists a pair of distinct edges $e', e'' \in E(G_0)$ such that

$$G_0(e', e'') \text{ does not have a spanning } (v_{e'}, v_{e''})\text{-trail.} \tag{22}$$

Claim 1 *Each of the following holds.*

(i) $\kappa'(G_0) \geq 3$ and $ess'(G_0) \leq 6$.
(ii) $G_0(e', e'')$ is reduced and not collapsible.
(iii) $\gamma(G_0) \leq 2$.

By Lemma 3.1, $\kappa'(G_0) \geq 3$. If $ess'(G_0) \geq 7$, then by Theorem 2.6(i), $F(G_0) = 0$, and so by Theorem 2.2(vi), $G_0(e', e'')$ would have a spanning $(v_{e'}, v_{e''})$-trail, violating (21). Hence, (i) holds.

By Theorem 2.2(v), if $G_0(e', e'')$ is collapsible, then $G_0(e', e'')$ has a spanning $(v_{e'}, v_{e''})$-trail, contrary to the assumption. Hence, $G_0(e', e'')$ is not collapsible. Suppose that $G_0(e', e'')$ has a nontrivial collapsible subgraph H'. Then by the definition of

$G_0(e', e'')$, G_0 has a subgraph H_0 satisfying both $E(H' - \{v_e, v_{e'}\}) = E(H_0 - \{e, e'\})$ and

$$H' = \begin{cases} H_0 & \text{if } \{v_{e'}, v_{e'}\} \cap V(H') = \emptyset, \\ H_0(e') & \text{if } \{v_{e'}, v_{e''}\} \cap V(H') = \{v_{e'}\}, \\ H_0(e'') & \text{if } \{v_{e'}, v_{e''}\} \cap V(H') = \{v_{e''}\}, \\ H_0(e', e'') & \text{if } \{v_{e'}, v_{e''}\} \subseteq V(H'). \end{cases}$$

As G_0 is obtained from G via edge contractions, G contains a subgraph H such that H is the contraction preimage of H_0. Since $ess'(G/H) \geq ess'(G) \geq \alpha'(G) + 1 \geq \alpha'(G/H) + 1$, it follows by (21) that the core $(G/H)_0$ of G/H is strongly spanning trailable. By the definition of cores, $G_0/H_0 = (G/H)_0$, and so by Lemma 3.9, G_0 is also strongly spanning trailable, contrary to (21). Hence, $G_0(e, e')$ must be reduced. This proves Claim 1(ii).

To prove (iii), we assume that $\gamma(G) > 2$. Then by (2), G contains a nontrivial subgraph H with $\gamma(H) > 2$. By Claim 1(ii) and Theorem 2.2(i), $\gamma(G_0(e', e'')) < 2$ and so $\{e', e''\} \cap E(H) \neq \emptyset$. By symmetry, we assume that $e' \in E(H)$. By Lemma 3.10(ii), $F(H - e') = 0$ and so by (2), $\gamma(H - e') \geq 2$. If $e'' \notin E(H)$, then $H - e'$ is a subgraph of $G_0(e', e'')$, and so by (2), $\gamma(G_0(e', e'')) \geq \gamma(H - e') \geq 2$, contrary to the fact that $\gamma(G_0(e', e'')) < 2$. Hence, we must have $e'' \in E(H)$, and so $(H - e')(e'')$ is a subgraph of $(G_0 - e')(e'') = G_0(e', e'') - v_{e'}$.

Since $F(H - e') = 0$, it follows by definition that $\kappa'(H - e') \geq 2$, and so $F((H - e')(e'')) \leq 1$ and $\kappa'((H - e')(e'')) \geq 2$. Hence by Theorem 2.2(iii), $(H - e')(e'')$ is a nontrivial collapsible subgraph of $G_0(e', e'')$, contrary to Claim 1(ii). This justifies Claim 1(iii).

By Claim 1, $\kappa'(G_0) \geq 3$ and $\gamma(G_0) \leq 2$. By (5), we have $ess'(G_0) \geq \alpha'(G_0) + 1$. It follows from Corollary 3.8 that G_0 is strongly spanning trailable, and so $G_0(e', e'')$ has a spanning $(v_{e'}, v_{e''})$-trail, contrary to (22). This completes the proof of the theorem. □

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