On r-hued coloring of graphs with maximum average degree less than 3*

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Abstract

For an integer $r > 0$, an r-hued coloring of a graph G is a proper coloring c such that for any vertex v of degree $d(v)$, there are at least $\min\{d(v), r\}$ kinds of colors present at its neighborhood. The *r-hued chromatic number* of G, $\chi_r(G)$, is the least number of colors used to r-hued color G. We prove that, when $r \geq 17$, every graph with maximum average degree less than three can have an r-hued coloring using $r+2$ colors. The result also holds for list r-hued coloring. Our result is optimal and extends the result in [Discrete Mathematics 317(2014)19-32].

Keywords: (k, r)-coloring; r-hued coloring; maximum average degree

1 Introduction

Graphs in this paper are simple and finite. Undefined terminologies and notations are referred to [1]. Thus for a graph G, $\Delta(G)$, $\delta(G)$, and $\chi(G)$ denote the maximum degree, the minimum degree, and chromatic number of G, respectively. For $v \in V(G)$, let $N_G(v)$ denote the set of vertices adjacent to v in G, $N_G[v] = N_G(v) \bigcup \{v\}$ and $d_G(v) = |N_G(v)|$. When G

*The research of Jianliang Wu is partially supported by NNSF of China (11271006, 11631014) and Shandong Province Natural Science Foundation of China (ZR2018MA001). The research of Hong-Jian Lai is partially supported by NNSF of China (11771039, 11771443).

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can be understood from the context, we often use $N(v)$ and $d(v)$ for $N_G(v)$ and $d_G(v)$, respectively.

For a given integer $r > 0$, an r-hued coloring of a graph G is a proper coloring of G such that for any vertex v with degree $d(v)$, there are at least $\min\{d(v), r\}$ kinds of colors present at its neighborhood. If the color set used for an r-hued coloring is $\{1, 2, \ldots, k\}$ (denoted by $\bar{k}$ for simplicity), the r-hued coloring is also called as a (k, r)-coloring. The *r-hued chromatic number* of G, denoted by $\chi_r(G)$, is the smallest k such that G has a (k, r)-coloring. The initial study of r-hued colorings was in [8] and [7], where $\chi_2(G)$ was called *the dynamic chromatic number* of G.

By the definition of $\chi_r(G)$, it follows immediately that $\chi(G) = \chi_1(G)$, and $\chi_\Delta(G) = \chi(G^2)$, where G^2 is the square graph of G. Thus r-hued coloring is a generalization of the classical vertex coloring. A list assignment L of a graph G associates a set $L(v)$ of colors with each vertex $v \in V(G)$. An (L, r)-coloring of G is an r-hued coloring c such that for any vertex v, $c(v) \in L(v)$. The *list r-hued chromatic number* of G, denoted as $\chi_{L,r}(G)$, is the least integer k such that for every list assignment L with $|L(v)| = k$ for every $v \in V(G)$, G has an (L, r)-coloring. When $\chi_{L,r}(G) \leq k$, then we also say that G admits a list (k, r)-coloring. By definition, $\chi_{L,r}(G) \geq \chi_r(G)$ holds for any graph G.

The r-hued chromatic numbers of some classes of graphs are known. For example, the result on complete graphs, cycles, trees and complete bipartite graphs can be found in [6]. In [7], an analogue of Brooks Theorem for χ_2 is proved. It is shown in [4] that $\chi_2(G) \leq 5$ holds for any planar graph G. In [9], it is shown that any K_4-minor free graph G satisfies $\chi_r(G) \leq r + 3$ when $2 \leq r \leq 3$ and $\chi_r(G) \leq \lfloor 3r/2 \rfloor + 1$ when $r \geq 4$.

Note that any graph G with $\Delta(G) \geq r$ satisfies $\chi_r(G) \geq r + 1$. Indeed, if we consider a vertex of maximum degree and its neighbors, they form a set in which at least $r + 1$ colors are needed for an r-hued coloring of G. It is therefore natural to ask when this lower bound is reached. For this purpose, researchers have studied conditions for a graph G to have its r-hued chromatic number being close to $r + 1$. The girth of a graph G, denoted $g(G)$, is the length of a shortest cycle in G.

Theorem 1.1. [10] *Any planar graph G with girth at least 6 satisfies $\chi_r(G) \leq r + 5$ when $r \geq 3$*

The average degree of a graph G, denoted $ad(G)$, is $\frac{\sum_{v \in V(G)} d_G(v)}{|V(G)|}$. The maximum average degree of a graph G, denoted $mad(G)$, is the maximum of $ad(H)$ on every subgraph H of G. Song et al. [12] used maximum average degree as a measure of sparseness and proved the following results:

Theorem 1.2. [12] *For any graph G, $\chi_r(G) \leq r+1$ in each of the following cases:*

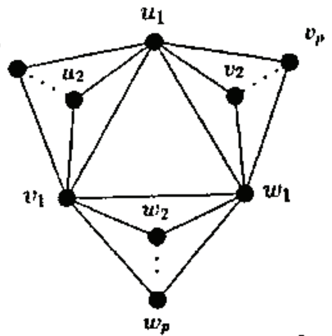

Figure 1: a graph G_p with $r = 2p$, $mad(G_p) = 4 - \frac{2}{p}$ and $\chi_r(G_p) = \frac{3r}{2}$

(1) $r \geq 5$ *and* $mad(G) < \frac{12}{5}$;
(2) $r \geq 6$ *and* $mad(G) < \frac{5}{2}$;
(3) $r \geq 8$ *and* $mad(G) < \frac{18}{7}$.

The results of Bonamy et al. in [2] motivate the following: investigate the value $h(N)$ for real number N such that any graph G with $mad(G) < N$ satisfies $\chi_r(G) \leq r + h(N)$ when r is sufficiently large. We know that $N \leq 4$ due to the family of graphs depicted in Fig. 1 (called Shannons triangle), which are of increasing r, of maximum average degree < 4 and that need $\frac{3r}{2}$ colors to be r-hued colored. Song et al. [11] prove the following.

Theorem 1.3. [11] *There exists a function h such that for a small enough $\epsilon > 0$, every graph with $mad(G) < 4 - \epsilon$ satisfies $\chi_r(G) \leq r + h(\epsilon)$.*

In this paper, we have proved the following result.

Theorem 1.4. *When $r \geq 17$, any graph G with maximum average degree $mad(G) < 3$ admits a list $(r + 2, r)$-coloring.*

It is optimal in a sense due to the graphs depicted in Fig. 2. Since 2-distance coloring is the special case of r-hued coloring when $r = \Delta$, Theorem 1.4 extends the following result.

Theorem 1.5. [3] *Every graph G with $\Delta(G) \geq 17$ and $mad(G) < 3$ admits a list 2-distance $(\Delta(G) + 2)$-coloring.*

2 Notations and terminology

For each vertex v of graph G, to count the number of vertices in $N_G[v]$ which can affect the color choice of its neighbors, we define the modified degree $d'(v)$ of v as following.

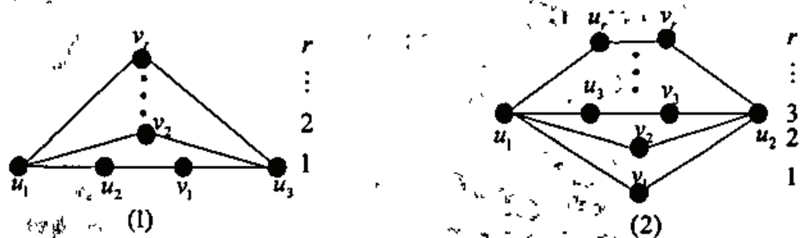

Figure 2: (1) a graph G_r with $r < 7$, $mad(G_r) = 3 - \frac{7-r}{r+2}$ and $\chi_r(G_r) = r+3$;

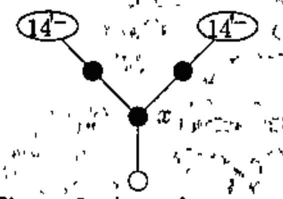

Figure 3: A weak vertex

$$d'(v) = \begin{cases} d(v), & \text{if } d(v) \leq r; \\ 1, & \text{if } d(v) \geq r+1. \end{cases} \quad (1)$$

A vertex of degree k, at least k, at most k in G, is called **k-vertex, k^+-vertex, k^--vertex** respectively. In the figures illustrated in this paper, we use a solid vertex to denote a vertex which is only adjacent to the vertices shown in the figures, and use a hollow vertex to denote a vertex which might be adjacent to other vertices not shown in the figures. A symbol inside a hollow vertex represents the degree or modified degree (using superscript $'$) of the vertex .

For vertices $x, y \in V(G)$, a path $xa_1 \ldots a_p y$ with $d_G(a_i) = 2, 1 \leq i \leq p$, is called a p-link of G. If a p-link $xa_1 \ldots a_p y$ exists, then we say that x and y are p-linked.

A vertex x is *weak* when it is of degree 3 and 1-linked to two vertices of modified degree at most 14, or twice 1-linked to a vertex of modified degree at most 14 (see Fig. 3). A weak vertex is represented with a w label inside ($\bar{w}$ if it is not weak).

A vertex x is *special* when it is one of the following types (see Fig. 4):
Type (S_1): a vertex of degree 2 adjacent to another vertex of degree 2;
Type (S_2): a vertex of degree 2 that is adjacent to a vertex of degree 3 which is adjacent to another vertex of degree 2 and to a vertex of modified degree at most 7;
Type (S_3): a weak vertex 1-linked to another weak vertex.

A vertex $v \in V(G)$ is *positive* if $d_G(v) \geq 4$ and $N_G(v)$ contains a special vertex. Let $v, w \in V(G)$, if there exist $v_1, v_2 \in N_G(v)$ and $w_1, w_2 \in N_G(w)$

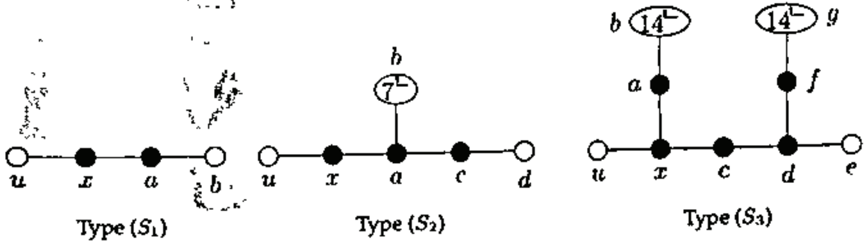

Figure 4: Special vertices x

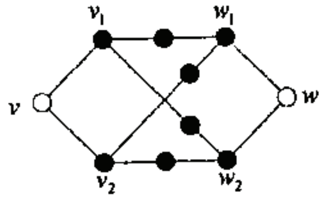

Figure 5: A lock with locked vertices v and w

such that for $i, j \in \{1,2\}$, v_i and w_j are 1-linked with $d_G(v_1) = d_G(v_2) = d_G(w_1) = d_G(w_2) = 3$, then each of v and w is a locked vertex. See Fig. 5 for an illustration. The configuration depicted in Fig. 5 is called a lock.

Let G be a graph with $V = V(G)$, and let $V' \subsetneq V$ be a vertex subset. As in [1], $G[V']$ is the subgraph of G induced by V'. A mapping $c : V' \mapsto \bar{k}$ is a **partial (k,r)-coloring** of G if c is a (k,r)-coloring of $G[V']$. Given a partial (k,r)-coloring c on $V' \subset V(G)$, for each $v \in V - V'$, define $\{c(v)\} = \emptyset$; and for every vertex $v \in V$, we extend the definition of $c(N_G(v))$ by setting $c(N_G(v)) = \cup_{z \in N_G(v)} \{c(z)\}$, and define

$$c[v] = \begin{cases} \{c(v)\}, & \text{if } |c(N_G(v))| \geq r; \\ \{c(v)\} \cup c(N_G(v)), & \text{otherwise.} \end{cases} \tag{2}$$

We have the following observation.

Observation 2.1. *Let c be a partial (k,r)-coloring of G. For any uncolored vertex u under c, and for any $v \in N_G(u)$, by the definition of $c[v]$, we have $|c[v]| \leq \min\{d_G(v), r\}$ and $\bigcup_{v \in N_G(u)} c[v]$ represents the colors that cannot be used as $c(u)$. Furthermore, if v has only one uncolored neighbor, then $|c[v]| \leq d'(v)$.*

3 Proof of Theorem 1.4

We shall argue by contradiction to prove Theorem 1.4, and assume that for some $r \geq 17$, there exists a graph G with $mad(G) < 3$ such that for some list assignment L with $|L(v)| = r+2$ for every $v \in V(G)$ and G does not have an (L,r)-coloring. Throughout the rest of this section, we assume that

$$G \text{ is a counterexample to Theorem 1.4 such that } |V(G)| + |E(G)| \text{ is minimized.} \tag{3}$$

For any $S \subset V(G) \cup E(G)$, we use L to denote the restriction of L on $G - S$. By (3), $|V(G)| \geq r+3$, and for any non-empty proper subset $S \subset V(G) \cup E(G)$, $G - S$ has an (L,r)-coloring. In the following two subsections, we first investigate the structure of this minimum counterexample G, and then use charge and discharge method to obtain a contradiction to complete the proof.

3.1 The structure of the minimal counterexample

Lemma 3.1. *Let x be a special vertex with the notation of Fig. 4.*

(i) *If x is Type (S_1), then u and b are r-vertices;*

(ii) *If x is Type (S_2), then u and d are r-vertices;*

(iii) *If x is Type (S_3), then u and e are r-vertices.*

Proof. By contradiction and symmetry, we may assume that u is not an r-vertex, and so $d'(u) \leq r-1$.

(i) By (3), $G - \{x, a\}$ has an (L,r)-coloring c. As $|c[b] \bigcup \{c(u)\}| \leq r+1$, $|c[u] \bigcup \{c(b)\}| \leq d'(u) + 1 \leq r$, we can extend c to be an (L,r)-coloring of G by setting $c(a) = \alpha \in L(a) - (c[b] \bigcup \{c(u)\})$, $c(x) \in L(x) - (c[u] \bigcup \{\alpha, c(b)\})$, contrary to (3). This proves (i).

(ii) By (3), $G - \{x\}$ has an (L,r)-coloring c. Let c_0 denote the restriction of c to $V(G) - \{x, a\}$. Since $|c_0[u] \bigcup c_0[a]| \leq d'(u) + 2 \leq r+1$, we can extend c_0 to c_1 by taking $c_1(x) \in L(x) - (c_0[u] \bigcup c_0[a])$. Thus c_1 is an (L,r)-coloring of $V(G) - \{a\}$ with $c_1(x), c_1(b), c_1(c)$ being distinct. Now we have $|c_1[x] \bigcup c_1[b] \bigcup c_1[c]| \leq d'(x) + d'(b) + d'(c) \leq 2+7+2 < r+1$, and so c_1 can be extended to an (L,r)-coloring c_2 of G by defining $c_2(a) \in L(a) - (c_1[x] \bigcup c_1[b] \bigcup c_1[c])$, contrary to (3). This proves (ii).

(iii) By (3), $G - \{x\}$ has an (L,r)-coloring c. Let c_0 denote the restriction of c to $V(G) - \{x, a, c\}$. For $|c_0[u] \bigcup c_0[a] \bigcup c_0[c]| \leq d'(u) + 1 + 1 \leq r+1$, we can extend c_0 to c_1 by taking $c_1(x) \in L(x) - (c_0[u] \bigcup c_0[a] \bigcup c_0[c])$. Thus c_1 is an (L,r)-coloring of $V(G) - \{a, c\}$ satisfying $c_1(x) \neq c_1(b)$ and $c_1(x) \neq c_1(d)$. Now we have $|c_1[x] \bigcup c_1[b]| \leq 2 + d'(b) \leq 2 + 14 < r+1$, and so c_1 can be extended to an (L,r)-coloring c_2 of $G - \{c\}$ by defining $c_2(a) \in L(a) - (c_1[x] \bigcup c_1[b])$. For $|c_2[x] \bigcup c_2[d]| \leq d'(x) + d'(d) \leq 3 + 3 < r+1$, and so c_2 can be extended to an (L,r)-coloring c_3 of G by defining $c_3(c) \in L(c) - (c_2[x] \bigcup c_2[d])$, contrary to (3). This proves (iii) and completes the proof of Lemma 3 1. □

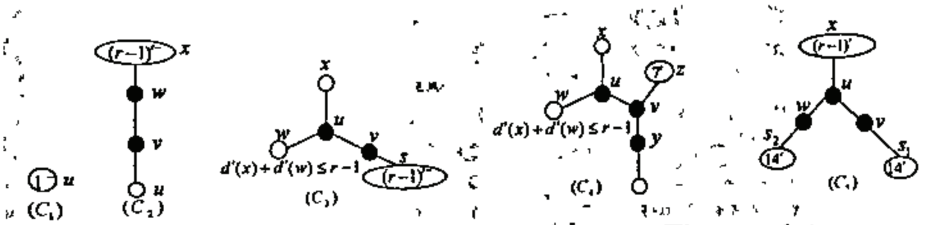

Figure 6: Forbidden configurations $C_1 - C_5$ for Theorem 1.4.

Lemma 3.2. *Each of the following holds.*
(i) *G is connected and $\delta(G) \geq 2$.*
(ii) *G has no vertex u which is 2-linked to a vertex x via a path $uvwx$ with $d'(x) \leq r-1$.*
(iii) *G has no 3-vertex u with $N_G(u) = \{x, w, v\}$, $d_G(v) = 2$ and $d'(x) + d'(w) \leq r-1$, that is 1-linked to a vertex s via a path uvs with $d'(s) \leq r-1$.*
(iv) *G has no 3-vertex u with $N_G(u) = \{x, w, v\}$, such that $d'(x) + d'(w) \leq r-1$ and v has exactly three neighbors u, y, z with $d'(z) \leq 7$ and $d_G(y) = 2$.*
(v) *G has no 3-vertex u with $N_G(u) = \{x, w, v\}$, such that $d'(x) \leq r-1$, and u is 1-linked through v (resp. through w) to a vertex s_1 (resp s_2) of modified degree at most 14. (Note u is a weak vertex).*
(vi) *G has no 4-vertex u with $N_G(u) = \{v, w, x, y\}$, such that $d'(w) \leq 7, d'(x) \leq 3, d'(y) \leq 3$, and u is 1-linked through v to a vertex of modified degree at most 14.*
(vii) *G has no 4-vertex u with $N_G(u) = \{v, w, x, y\}$, such that $d'(x)+d'(y) \leq r-1$, and u is 1-linked through v (resp. through w) to a vertex s_1 (resp s_2) of modified degree at most 14.*
(viii) *G has no 5-vertex u with $N_G(u) = \{v, w, x, y, z\}$, such that $d'(w) \leq 7, d'(x) \leq 3, d'(y) \leq 3, d_G(z) = 2$, and u is 1-linked through v to a vertex of modified degree at most 7.*
(ix) *G has no 6-vertex u with $N_G(u) = \{v, w, x, y, z, t\}$, such that $d'(w) \leq 7, d'(x) \leq 3, d'(y) \leq 3, d_G(z) = d_G(t) = 2$, and u is 1-linked through v to a vertex of modified degree at most 7.*
(x) *G has no 7-vertex u with $N_G(u) = \{v, w_1, \ldots, w_6\}$, such that $d'(v) \leq 7$ and u is 1-linked through w_i, $(1 \leq i \leq 6)$ to a vertex of modified degree at most 3.*
(xi) *G has no r-vertex u, that has three neighbors v, w, x such that x is a special vertex, $d_G(v) = d_G(w) = 3$, v, w are both 1-linked to a same 3-vertex y, and v (resp. w) is 1-linked through z_1 (resp. z_4) to a vertex y_1 (resp. y_2) which is of modified degree at most 14 and distinct from y. (Note that v, w are weak vertices).*

Proof. (i) This follows from (3).
(ii) By contradiction, we assume that G has such a vertex u as illustrat-

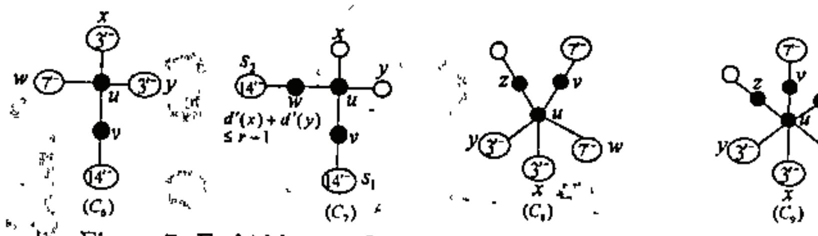

Figure 7: Forbidden configurations $C_6 - C_9$ for Theorem 1.4.

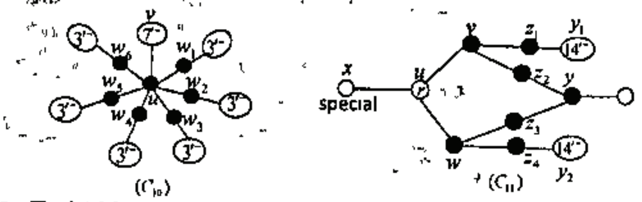

Figure 8: Forbidden configurations C_{10} and C_{11} for Theorem 1.4.

ed in (C_2) of Fig. 6. By (3), $G - \{v, w\}$ has an (L, r)-coloring c. As $|c[u] \bigcup \{c(x)\}| \leq d'(u) + 1 \leq r + 1$, we can extend c to c_1 by letting $c_1(v) \in L(v) - (c[u] \bigcup \{c(x)\})$. Thus c_1 is a partial (L, r)-coloring with $c_1(v) \neq c_1(x)$. As $d_G(v) = d_G(w) = 2$, we have $|c_1[v] \bigcup c_1[x]| \leq 2 + d'(x) \leq r + 1$, which allows c_1 be further extended to an (L, r)-coloring c_2 of G by choosing $c_2(w) \in L(w) - (c_1[v] \bigcup c_1[x])$, contrary to (3). This proves (ii).

(iii) By contradiction, we assume G has such a vertex u as illustrated in (C_3) of Fig. 6. By (3), $G - \{v\}$ has an (L, r)-coloring c. Let c_0 denote the restriction of c to $V(G) - \{v, u\}$. For $|c_0[u] \bigcup c_0[s]| \leq 2 + d'(s) \leq r + 1$, then we can extend c_0 to c_1 by taking $c_1(v) \in L(v) - (c_0[u] \bigcup c_0[s])$. This results in an (L, r)-coloring c_1 of $V(G) - \{u\}$ with $c_1(v), c_1(w), c_1(x)$ being distinct. Now we have $|c_1[x] \bigcup c_1[w] \bigcup c_1[v]| \leq d'(x) + d'(w) + d(v) \leq r - 1 + 2 = r + 1$, and so c_1 can be extended to an (L, r)-coloring c_2 of G by defining $c_2(u) \in L(u) - (c_1[x] \bigcup c_1[w] \bigcup c_1[v])$, contrary to (3). This proves (iii)

(iv) By contradiction, we assume G have such a vertex u as illustrated in (C_4) of Fig. 6. By (3), $G - uv$ has an (L, r)-coloring c. Let c_0 denote the restriction of c to $V(G) - \{v, u\}$. For $|c_0[x] \bigcup c_0[w] \bigcup c_0[v]| \leq d'(x) + d'(w) + 2 \leq r - 1 + 2 = r + 1$, then we can extend c_0 to c_1 by taking $c_1(u) \in L(u) - (c_0[x] \bigcup c_0[w] \bigcup c_0[v])$. c_1 is an (L, r)-coloring of $V(G) - \{v\}$ with $c_1(u), c_1(y), c_1(z)$ being distinct. Now we have $|c_1[u] \bigcup c_1[y] \bigcup c_1[z]| \leq d'(u) + d'(y) + d'(z) \leq 3 + 2 + 7 < r + 1$, and so c_1 can be extended to an (L, r)-coloring c_2 of G by defining $c_2(v) \in L(v) - (c_1[u] \bigcup c_1[y] \bigcup c_1[z])$, contrary to (3). This proves (iv).

(v) By contradiction, we assume G have such a vertex u as illustrated in (C_5) of Fig. 6. By (3), $G-\{u,v,w\}$ has an (L,r)-coloring c_0. For $|c_0[x]\bigcup c_0[w]\bigcup c_0[v]| \leq d'(x)+1+1 \leq r+1$, then we can extend c_0 to an (L,r)-coloring c_1 of $G-\{v,w\}$ by taking $c_1(u) \in L(u)-(c_0[x]\bigcup c_0[w]\bigcup c_0[v])$. Since $|c_1[s_i]\bigcup c_1[u]| \leq 14+2 < r+1$ for $i \in \{1,2\}$, c_1 can be extended to an (L,r)-coloring c_2 of G by defining $c_2(v)=\alpha \in L(v)-(c_1[s_1]\bigcup c_1[u])$ and $c_2(w) \in L(w)-(c_1[s_2]\bigcup c_1[u]\bigcup\{\alpha\})$, contrary to (3). This proves (v).

(vi) By contradiction, we assume G has such a vertex u as illustrated in (C_6) of Fig. 7. By (3), $G-\{v\}$ has an (L,r)-coloring c. Let c_0 denote the restriction of c to $V(G)-\{v,u\}$. Since $|\bigcup_{s\in N(v)} c_0[s]| \leq 14+3 \leq r+1$, we can extend c_0 to c_1 by letting $c_1(v) \in L(v)-(\bigcup_{s\in N(v)} c_0[s])$. Since $|\bigcup_{s\in N(u)} c_1[s]| \leq 7+3+3+2 \leq r+1$, we can extend c_1 to an (L,r)-coloring c_2 of G by letting $c_2(u) \in L(u)-(\bigcup_{s\in N(u)} c_1[s])$, contrary to (3). This proves (vi).

(vii) By contradiction, we assume G has such a vertex u as illustrated in (C_7) of Fig. 7. By (3), $G-\{v,w\}$ has an (L,r)-coloring c. Let c_0 denote the restriction of c to $V(G)-\{v,w,u\}$. As $|\bigcup_{s\in N(u)} c_0[s]| \leq d'(x)+d'(y)+1+1 \leq r+1$, we can extend c_0 to c_1 by letting $c_1(u) \in L(u)-(\bigcup_{s\in N(u)} c_0[s])$. Since $|c_1[s_i]\bigcup c_1[u]| \leq 14+3 < r+1$ for $i \in \{1,2\}$, c_1 can be extended to an (L,r)-coloring c_2 of G by defining $c_2(v)=\alpha \in L(v)-(c_1[s_1]\bigcup c_1[u])$ and $c_2(w) \in L(w)-(c_1[s_2]\bigcup c_1[u]\bigcup\{\alpha\})$, contrary to (3). This proves (vii).

(viii) By contradiction, we assume G has such a vertex $u$as illustrated in (C_8) of Fig. 7. By (3), $G-\{v\}$ has an (L,r)-coloring c. Let c_0 denote the restriction of c to $V(G)-\{v,u\}$. As $|\bigcup_{s\in N(u)} c_0[s]| \leq 7+3+3+2+1 \leq r+1$, we can extend c_0 to c_1 by letting $c_1(u) \in L(u)-(\bigcup_{s\in N(u)} c_0[s])$. As $|\bigcup_{s\in N(v)} c_1[s]| \leq 7+5 \leq r+1$, we can extend c_1 to an (L,r)-coloring c_2 of G by letting $c_2(v) \in L(v)-(\bigcup_{s\in N(v)} c_1[s])$, contrary to (3). This proves (viii).

(ix) By contradiction, we assume G has such a vertex u as illustrated in (C_9) of Fig. 7. By (3), $G-\{v\}$ has an (L,r)-coloring c. Let c_0 denote the restriction of c to $V(G)-\{v,u\}$. As $|\bigcup_{s\in N(u)} c_0[s]| \leq 7+3+3+2+2+1 \leq r+1$, we can extend c_0 to c_1 by letting $c_1(u) \in L(u)-(\bigcup_{s\in N(u)} c_0[s])$. As $|\bigcup_{s\in N(v)} c_1[s]| \leq 7+6 \leq r+1$, we can extend c_1 to an (L,r)-coloring c_2 of G by letting $c_2(v) \in L(v)-(\bigcup_{s\in N(v)} c_1[s])$, contrary to (3). This proves (ix).

(x) By contradiction, we assume G has such a vertex u as illustrated in (C_{10}) of Fig. 8. By (3), $G-\{u,w_1,\dots,w_6\}$ has an (L,r)-coloring c. As $|\bigcup_{s\in N(u)} c[s]| \leq 7+6 \leq r+1$, we can extend c to c_0 by letting $c_0(u) \in L(u)-(\bigcup_{s\in N(u)} c[s])$. For $1 \leq i \leq 6$, $|\bigcup_{s\in N(w_i)} c_{i-1}[s]| \leq 3+i+1 \leq r+1$, we can extend c_{i-1} to c_i by letting $c_i(w_i) \in L(w_i)-(\bigcup_{s\in N(w_i)} c_{i-1}[s])$. c_6 is an (L,r)-coloring of G, contrary to (3). This proves (x).

(xi) By contradiction, we assume G has such a vertex u as illustrated in

(C_{11}) of Fig. 8. Suppose x is of Type $(S_1), (S_2)$ or (S_3) of special vertices with the notation of Fig. 4. Note that some vertices may coincide between Fig. 4 and Fig. 8.

Let

$$A = \begin{cases} \{a\} & \text{if } x \text{ is of Type } (S_1); \\ \{a, c\} & \text{if } x \text{ is of Type } (S_2) \text{ or } (S_3). \end{cases} \tag{4}$$

By (3), $G - (\{v, w, x, y, z_1, \dots, z_4\} \bigcup A)$ has an (L, r)-coloring c_0. If x is of Type (S_1) (resp. (S_2)), $|\bigcup_{s \in N(a)} c_0[s]| \le r+1$ (resp. $|\bigcup_{s \in N(c)} c_0[s]| \le r+1$), we can extend c_0 to c_1 by letting $c_1(a) \in L(a) - (\bigcup_{s \in N(a)} c_0[s])$ (resp. $c_1(c) \in L(a) - (\bigcup_{s \in N(c)} c_0[s])$). For the three types, $|\bigcup_{s \in N(x)} c_1[s]| \le (r-2)+2 = r$. Furthermore, $|\bigcup_{s \in N(y)} c_1[s]| \le r$, $|\bigcup_{s \in N(v)} c_1[s]| \le (r-2)+1 = r-1$, and $|\bigcup_{s \in N(w)} c_1[s]| \le (r-2)+1 = r-1$. It is to say, x, y have at least 2 available colors and v, w have at least 3 available colors under c_1. Denote the lists of available colors of v, w, x, y by $A(v), A(w), A(x), A(y)$ respectively.

Now we will extend c_1 to an (L, r)-coloring c_2 where v, w, x, y will be colored. Suppose $A(x) \cap A(y) \ne \phi$. Choose $k \in A(x) \cap A(y)$, then let $c_2(x) = c_2(y) = k$. Both v and w have at least 2 available colors left after x, y are colored. $c_2(v)$ and $c_2(w)$ can be determined.

Suppose $A(x) \cap A(y) = \phi$. So $A(v) \backslash A(x) = A(v) \backslash A(y) = \phi$ can not happen, w.l.o.g, assume that $A(v) \backslash A(x) \ne \phi$. Choose $c_2(v) \in A(v) \backslash A(x)$, $c_2(y) \in A(y) - \{c_2(v)\}$, $c_2(w) \in A(w) - \{c_2(v), c_2(y)\}$. For $|A(x) - \{c_2(v), c_2(y)\}| = |A(x)| \ge 2$, choose $c_2(x) \in A(x) - \{c_2(v), c_2(y), c_2(w)\} = A(x) - \{c_2(w)\}$.

If x is of Type (S_2) (resp. (S_3)), vertex a (resp. vertices a, c) has at most 11 constraints (resp. 17, 6), so we can color them. The vertices z_i have at most $17 (\le r+1)$ constraints, so we can color them. Thus the coloring c_2 can be extended to an (L, r)-coloring of G, contrary to (3). This proves (xi) and completes the proof of Lemma 3.2. □

Lemma 3.3. *Each positive vertex is an r-vertex and each special vertex is adjacent to exactly one positive vertex.*

Proof. It follows from Lemma 3.1 and the definitions of positive vertex and special vertex. □

Let $H(G)$ be the subgraph of G induced by the edges incident to at least a special vertex. We prove several lemmas on the properties of special vertices and of the graph $H(G)$

Lemma 3.4. *Each cycle of $H(G)$ with an odd number of special vertices contains a subpath $s_1 v_1 s_2 v_2 s_3$ where s_1, s_2, s_3 are special vertices of type (S_3) and v_1, v_2 are 2-vertices of G.*

Proof. Let C be a cycle of $H(G)$ with an odd number of special vertices. For G is simple and all edges of C have to be incident to a special vertex, C can not contain only one special vertex. So C contains at least three special vertices.

Case 1. C does not contain positive vertex. For a special vertex of type (S_1) or (S_2) is a 2-vertex and has a positive neighbor (by Lemma 3.1), every special vertex of C must be of type (S_3). Let s_1, s_2, s_3 be three special vertices of C appearing consecutively along C. A special vertex of type (S_3) is of degree 3, adjacent to two 2-vertices and a positive vertex. Let v_1, v_2 be the two neighbors of s_2 on C. By Lemma 3.1, v_1 and v_2 must be 2-vertices of G and not be special vertices. $s_1v_1s_2v_2s_3$ is the required path.

Case 2. C contains at least one positive vertex. Let $p_1, \dots, p_l$ be the set of positive vertices of C appearing in this order along C while walking in a chosen direction. In the context, the subscripts are understood modulo l. Let Q_i, $1 \le i \le l$, be the subpath of C between p_i and p_{i+1} (in the same chosen direction along C). (Note that if $l = 1$, $Q_1 = C$.) As C contains an odd number of special vertices, there exists an i such that Q_i contains an odd number of special vertices. If Q_i contains just one special vertex v, then Q_i is not a cycle. Since $H(G)$ contains only edges incident to special vertices, then Q_i has length 2 and v is adjacent to two different positive vertices, a contradiction to Lemma 3.3. So Q_i contains at least 3 special vertices. Let s_1, s_2, s_3 be three special vertices of Q_i appearing consecutively along Q_i.

If one of the s_j is of Type (S_1), let x be such a vertex. With the notation of Fig. 4, Q_i must be the path $uxab$ where $\{u, b\} = \{p_i, p_{i+1}\}$, x, a are the two special vertices of Q_i, a contradiction.

If one of the s_j is of Type (S_2), let x be such a vertex. With the notation of Fig. 4, vertex x is of degree 2, so its two neighbors u, a are on C, with u a positive vertex and a a vertex of degree 3. Vertex a is not adjacent to r-vertices, so by Lemma 3.3 a is not a special vertex. Let c be the neighbor of a on C that is distinct from x. As all the edges of $H(G)$ are incident to special vertices and c is a special vertex which is adjacent to a 3-vertex, c is a special vertex of Type (S_2) and can play the role of c of Fig. 4. Let $d \in N_G(c)\backslash\{a\}$, d is a positive vertex. So Q_i is the path $uxacd$ and contains just two special vertices, a contradiction.

So s_1, s_2, s_3 are all of Type (S_3). A special vertex of Type (S_3) is of degree 3, adjacent to two vertices of degree 2 and to a positive vertex. Let $N_C(s_2) = \{v_1, v_2\}$, so v_1 and v_2 are of degree 2 that are not special vertices. As $H(G)$ contains only edges incident to special vertices, $s_1v_1s_2v_2s_3$ is the required path. □

Lemma 3.5. *$H(G)$ does not contain a 2-connected subgraph of size at least* *three with exactly two special vertices.*

Proof. Suppose by contradiction that $H(G)$ contains a 2-connected subgraph C of size ≥ 3 that has exactly two special vertices $S = \{x, y\}$. We divide the proof based on the type of x. In the following, we use the notation of Fig. 4.

Case 1. x is of type (S_1). So $d_G(x) = 2$. Let $N_G(x) = \{u, a\}$, $N_G(a) = \{x, b\}$. By Lemma 3.1, u and b are r-vertices and not special vertices. For C is 2-connected, then u a, b are all in C. Because C is a subgraph of $H(G)$ and has exactly two special vertices, then $y = a$ and $u = b$. So u is a cut vertex of G. By (3) and the fact u is an r-vertex, $G - xy$ has an (L, r)-coloring c which is also an (L, r)-coloring of G. This contradicts to (3).

Case 2. x is of type (S_2). So $d_G(x) = 2$. Let $N_G(x) = \{u, a\}$, $N_G(a) = \{x, b, c\}, N_G(c) = \{a, d\}$ where $d'(b) \leq 7$. By Lemma 3.1, u and d are r-vertices and not special vertices. For a has no neighbor of degree r, by Lemma 3.3 a is not a special vertex. As C is 2-connected subgraph of $H(G)$, then u and a are all in C and all their neighbors in C must be special vertices. Note that C contains exactly two special vertices, then $y \in N_C(u) \cap N_C(a)$. Since $d_G(a) = 3$, y is also of type (S_2).Without loss of generality, assume that $y = c$.

By (3), $G - \{x, y, a\}$ has an (L, r)-coloring c_0. As $|c_0[u] \cup c_0[a]| = |c_0[u] \cup \{c_0(b)\}| \leq r$, we extend c_0 to c_1 by letting $c_1(x) \in L(x) - (c_0[a] \cup c_0[u])$. For $c_1(x) \in c_1[u] \cap c_1[a]$, then $|c_1[u] \cup c_1[a]| \leq r + 1$, and we extend c_1 to c_2 by letting $c_2(y) \in L(y) - (c_1[a] \cup c_1[u])$. Now we have $|\bigcup_{s \in N_G(a)} c_2[s]| \leq 2 + 2 + 7 \leq r + 1$, and so c_2 can be extended to an (L, r)-coloring c_3 of G by letting $c_3(a) \in L(a) - (\bigcup_{s \in N_G(a)} c_2[s])$, contrary to (3).

Case 3. x is of type (S_3). So $d_G(x) = 3$. Let $N_G(x) = \{u, a, c\}$, $N_G(a) = \{x, b\}, N_G(c) = \{x, d\}$ where $d'(b), d'(d) \leq 14$. Since C is 2-connected, at least two of $\{u, a, c\}$ are in C, let $Y = N_C(x)$. Assume by symmetry that either $\{u, c\} \subseteq Y$ or $\{a, c\} \subseteq Y$. By Lemma 3.1, u is r-vertex and not special vertex. By Lemma 3.3, neither a nor c are special vertices. It is, none of Y is a special vertex, so all the neighbors of Y in C are special vertices. Since C contains exactly two special vertices, y is adjacent to every vertex of Y. Since y is adjacent to at least one 2-vertex in $\{a, c\}$ that is not a special vertex, then y is of type (S_3).

By (3), $G - \{x, a, c\}$ has an (L, r)-coloring c_0. When $\{u, c\} \subseteq Y$, Then $y = d$ and $c_0(d) \in c_0[u]$; When $\{a, c\} \subseteq Y$, $b = d = y$. In both cases, $|\bigcup_{s \in N_G(x)} c_0[s]| = |c_0[u] \cup \{c_0(b), c_0(d)\}| = |c_0[u] \cup \{c_0(b)\}| \leq r + 1$. we can extend c_0 to c_1 by letting $c_1(x) \in L(x) - (\bigcup_{s \in N_G(x)} c_0[s])$. Since $d'(x) + d'(b) \leq 17$ and $d'(x) + d'(d) \leq 17$, c_1 can be easily extended to an (L, r)-coloring of G, contrary to (3). □

Lemma 3.6. *Every 2-connected subgraph of $H(G)$ that contains exactly three special vertices is a cycle.*

Proof. Suppose by contradiction that $H(G)$ contains a 2-connected subgraph C that is not a cycle and has exactly three special vertices $S = \{s_1, s_2, s_3\}$.

We first show that C contains a cycle C' with $S \subseteq C' \subsetneq C$. As C is 2-connected, by Menger's Theorem there exist two internally vertex-disjoint paths Q, Q' between s_1, s_2. Let C' be the cycle $Q \cup Q'$. If it does not contain s_3, then C' is a 2-connected subgraph of size at least three with exactly two special vertices, a contradiction to Lemma 3 5.

By Lemma 3.4, cycle C' contains a subpath $x_1v_1x_2v_2x_3$ where $\{x_1, x_2, x_3\} = \{s_1, s_2, s_3\}$, all s_i's are of Type (S_3) and v_1, v_2 are 2-vertices of G. Vertices x_1, x_3 are special vertices of Type (S_3), they are of degree 3 and only adjacent to positive vertices and to vertices of degree 2, so they are not adjacent. The graph $H(G)$ contains only edges incident to special vertices, so there exists a vertex y of C' adjacent to x_1, x_3, and $x_1v_1x_2v_2x_3yx_1$ is the cycle C'. If C' has some chords in $H(G)$, then $H(G)$ contains a cycle with only two special vertices, a contradiction to Lemma 3.5. So C' is an induced cycle of $H(G)$ and so C' has strictly less vertices than C. Let y' be a vertex of C distinct from $x_1, v_1, x_2, v_2, x_3, y$. Vertex y' is not a special vertex, C is 2-connected and $H(G)$ contains only edges incident to special vertices, so y' is adjacent to at least two vertices in S. Then $H(G)$ contains a cycle with only two special vertices, a contradiction to Lemma 3.5. □

We need the following lemma from Erdős et al [5]:

Lemma 3.7. [5] *If G is a 2-connected graph that is neither a clique nor an odd cycle, and L is a list assignment on the vertices of G such that $\forall u \in V(G), |L(u)| \geq d(u)$, then G is L-colorable.*

The following lemmas 3.8 and 3.9 can be obtained in a similar way as in [3]. We include their proofs in the appendix for completeness.

Lemma 3.8. [3] *Every 2-connected subgraph of $H(G)$ of size at least three is either a cycle with an odd number of special vertices or a subgraph of a lock of $H(G)$.*

A *cactus* is a connected graph in which any two cycles have at most one vertex in common.

Lemma 3.9. [3] *Every connected component of $H(G)$ is either a cactus where each cycle has an odd number of special vertices or a lock.*

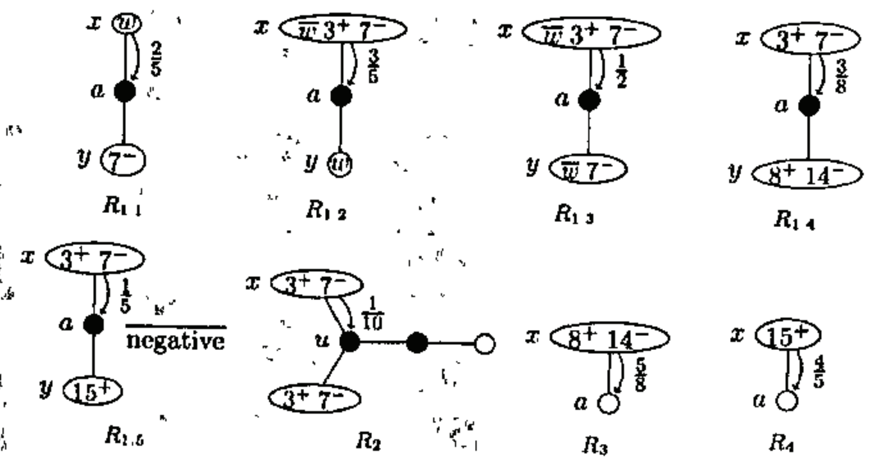

Figure 9: Discharging rules $R_{1,i}$, R_2, R_3 and R_4.

3.2 Discharging

A *negative* vertex is a special vertex of type (S_1) or (S_2), or a vertex of degree 2 adjacent to two special vertices of type (S_3). In this case we say that the negative vertex is of type (N_1), (N_2) or (N_3) respectively.

We assign the initial charges to the vertices of G equals its degree. Let R_1, R_2, R_3, R_4, and R_g ("g" stands for "global") be five discharging rules defined below (see Fig. 9). We also use a so-called common pot denoted by p_0 which is empty at the beginning, receives weight from some vertices and gives weight to some others.

We will describe the discharging rules from the point of view of the givers. Any 2^--vertices can only be receivers. Consider any 3^+-vertex x of G and the common pot p_0.

(R_1) $3 \le d_G(x) \le 7$. x has a 2-neighbor a and $N_G(a) = \{x, y\}$.

$(R_{1.1})$ If x is weak and $d_G(y) \le 7$, then x gives $\frac{2}{5}$ to a.

$(R_{1.2})$ If x is not weak and y is weak, then x gives $\frac{3}{5}$ to a.

$(R_{1.3})$ If x, y are not weak and $d_G(y) \le 7$, then x gives $\frac{1}{2}$ to a.

$(R_{1.4})$ If $8 \le d_G(y) \le 14$, then x gives $\frac{3}{8}$ to a.

$(R_{1.5})$ If $d_G(y) \ge 15$ and a is not negative, then x gives $\frac{1}{5}$ to a.

(R_2) $3 \le d_G(x) \le 7$ and x is adjacent to a 3-vertex u that is also adjacent to a 2-vertex and a 7^--vertex. Then x gives $\frac{1}{10}$ to u.

(R_3) $8 \le d_G(x) \le 14$. Then x gives $\frac{5}{8}$ to each of its neighbors.

(R_4) $d_G(x) \ge 15$. Then x gives $\frac{4}{5}$ to each of its neighbors.

(R_g) Every positive vertex gives an additional $\frac{2}{5}$ to the common pot p_0; p_0 gives $\frac{1}{5}$ to each negative vertex.

For any vertex of $V(G) \cup \{p_0\}$, let $w(x)$ be the final charge after all discharging process are finished on it.

Lemma 3.10. $w(p_0) \geq 0$.

Proof. $\forall X \subseteq V(G)$, let $n(X)$ be the number of negative vertices in X and $p(X)$ be the number of positive vertices in X. To prove the lemma, we just to show that each component C of $H(G)$ satisfies $p(C) \geq \lceil \frac{n(C)}{2} \rceil$.

Let C be a component of $H(G)$. By Lemma 3.9, C is a catus where each a cycle has an odd number of special vertices or a lock. If C is a lock, then $n(C) = 4$ and $p(C) = 2$, so $p(C) \geq \lceil \frac{n(C)}{2} \rceil$. So we assume that C is a catus where each a cycle has an odd number of special vertices.

Claim 3.11. *Every connected subgraph C' of C, whose pendant vertices are positive vertices, whose special vertices have their positive neighbors in C', whose negative vertices of Type (N_3) are adjacent to their two neighbors in C', satisfies $p(C') \geq \lceil \frac{n(C')}{2} \rceil$.*

Proof. Suppose by contradiction that C' is a counterexample with minimum number of vertices.

Suppose first that C' has a pendant vertex u. Let x be the neighbor of the positive vertex u in C'. As $H(G)$ contains only edges incident to special vertices, x is a special vertex which has degree at least 2 in C'. We consider different cases according to the Type of x and its number of neighbors in C'.

Case 1. x is of Type (S_1). Then let a be the neighbor of x distinct from u. We have $a \in C'$ and a is a special vertex of Type (S_1) and the positive neighbor b of a is also in C' by the assumption. Let $C'' = C - \{u, x, a\}$. C'' also satisfies the condition of the claim and with less vertices, by minimality, $p(C'') \geq \lceil \frac{n(C'')}{2} \rceil$. For $n(C'') = n(C') - 2$ and $p(C'') = p(C') - 1$, then $p(C') \geq \lceil \frac{n(C')}{2} \rceil$.

Case 2. x is of Type (S_2). Then let a be the neighbor of x distinct from u and a is of degree 3 in G. Let b, c be the two neighbors of a, distinct from x. Since a is not positive, a can not be a pendant vertex of C', at least one of b, c is in C'. We assume w.l.o.g. that c is in C'. As $H(G)$ contains only edges incident to special vertices and C' is a subgraph of $H(G)$, c is also a special vertex of Type (S_2). If b is also in C', let $C'' = C - \{u, x\}$, we have $n(C'') = n(C') - 1$ and $p(C'') = p(C') - 1$; If b is not in C', let $C'' = C - \{u, x, a, c\}$, we have $n(C'') = n(C') - 2$ and $p(C'') = p(C') - 1$. Whether b is in C' or not, C'' also satisfies the condition of the claim and with less vertices, by minimality, $p(C'') \geq \lceil \frac{n(C'')}{2} \rceil$. We always have $p(C') \geq \lceil \frac{n(C')}{2} \rceil$.

Case 3. x is of Type (S_3) and has two neighbors in C'. Then let c be the neighbor of x distinct from u that is in C'. c is of degree 2 in G and is not positive, its other neighbor in C' denoted by d is also in C'. d must be a special vertex of Type (S_3) and c be a negative vertex of Type (N_3). Let $N_G(d) = \{c, e, f\}$ where e is the positive neighbor of d and $d_G(f) = 2$ (see Fig. 4). By the assumption, e must be in C'. If f is also in C', let $C'' = C - \{u, x, c\}$; If f is not in C', let $C'' = C - \{u, x, c, d\}$. In both cases, we have $n(C'') = n(C') - 1$ and $p(C'') = p(C') - 1$, and C'' satisfies the condition of the claim and with less vertices, by minimality, $p(C'') \geq \lceil \frac{n(C'')}{2} \rceil$. We always have $p(C') \geq \lceil \frac{n(C')}{2} \rceil$.

Case 4. x is of Type (S_3) and has three neighbors in C'. Let $N_G(x) = \{u, a, c\}$, a, c are in C' and of degree 2 in both G and C'. Let $N_G(a) = \{x, b\}$ and $N_G(c) = \{x, d\}$, and b, d are also in C'. a, c are not special vertices, so b, d must be special vertices of Type (S_3). Let h (resp. e) denote the positive number of b (resp. d). If b and d both have their three neighbors in C', let $C'' = C - \{u, x, a, c\}$; If d has their three neighbors in C' not b, let $C'' = C - \{u, x, c, a, b\}$; If b has their three neighbors in C' not d, let $C'' = C - \{u, x, a, c, d\}$; If none of b and d has their three neighbors in C', let $C'' = C - \{u, x, a, c, b, d\}$. In the four cases, we have $n(C'') = n(C') - 2$ and $p(C'') = p(C') - 1$. If C'' is connected, C'' satisfies the condition of the claim and with less vertices than C'; If C'' is not connected, each component of C'' satisfies the condition of the claim and with less vertices than C'. By minimality on each component of C'', we have $p(C'') \geq \lceil \frac{n(C'')}{2} \rceil$. So we have $p(C') \geq \lceil \frac{n(C')}{2} \rceil$.

Now we assume that C' has no pendant vertex. Suppose C' is a single vertex v. Then by the assumption, v is neither a special vertex nor a negative vertex of Type (N_3). So $p(C') \geq \lceil \frac{n(C')}{2} \rceil = 0$. If C' is a subgraph of a lock of $H(G)$, we can verify that $2 = p(C') \geq \lceil \frac{n(C')}{2} \rceil$, where $n(C') \leq 4$. Now we can assume that C' is not a single vertex, not a subgraph of a lock of $H(G)$. So C' is a cactus, contains no pendant vertex, contains a cycle C'' of size at least 3 such that $C''' = C' \backslash C''$ is connected (note that we may have $C' = C''$ and C''' empty). Cycle C'' is a cycle of C, by Lemma 3.9, C'' has an odd number of special vertices. Let S be the set of special vertices of C'', $s = |S|$. By Lemma 3.4, cycle C'' contains a subpath $s_1 v_1 s_2 v_2 s_3$ where s_1, s_2, s_3 are special vertices of Type (S_3) and v_1, v_2 are of degree 2 in G. By assumption, s_2 has its positive neighbor z in C'. For C'' is a cycle in which s_2 has two neighbors v_1, v_2, then z is not in C''. So the only vertex of C'' that has some neighbors in $C' \backslash C''$ is s_2. So all the positive vertices that are adjacent to $S \backslash \{s_2\}$ are vertices of C' and thus of C''. A positive vertex of C'' has at most two special neighbors in C'', so $p(C'') \geq \lceil \frac{s-1}{2} \rceil$. A special vertex of Type (S_1) or (S_2) is a negative vertex of Type (N_1) or (N_2). A negative vertex of Type (N_3) must have its two special neighbor in C''. v_1, v_2 are negative vertices of Type (N_3), so $s > n(C'')$ and $p(C'') \geq \lceil \frac{s-1}{2} \rceil \geq \lceil \frac{n(C'')}{2} \rceil$. Now C''' satisfies the condition of the claim and with less vertices than C', by minimality $p(C''') \geq \lceil \frac{n(C''')}{2} \rceil$. So finally, $p(C') = p(C'') + p(C''') \geq \lceil \frac{n(C'')}{2} \rceil + \lceil \frac{n(C''')}{2} \rceil \geq \lceil \frac{n(C'')+n(C''')}{2} \rceil = \lceil \frac{n(C')}{2} \rceil$. □

Let C' be the graph obtained from C by removing all pendant vertices that are not positive vertices. For all special vertices and negative vertices are of degree 2 or 3 and have their incident edges in $H(G)$, then a pendant vertex of C must be not special, not negative, only be a 2-vertex a of G incident to a special vertex of Type (S_3) (with notations of Fig. 4). So C' is a connected subgraph of C, whose pendant vertices are positive vertices, whose special vertices have their positive neighbors in C', whose negative vertices of Type (S_3) are adjacent to their two neighbors in C'. So $n(C') = n(C)$ and $p(C') = p(C)$. By the claim above, we have that $p(C') \geq \lceil \frac{n(C')}{2} \rceil$ which induce that $p(C) \geq \lceil \frac{n(C)}{2} \rceil$. This completes the proof of the lemma. □

Lemma 3.12. *For any vertex $u \in V(G)$, $w(u) \geq 3$.*

Proof. Let u be a vertex of G. By Lemma 3.2(i), G has no vertices of degree 0 or 1. So $d_G(u) \geq 2$. We divide our rest proof into 9 cases depending on $d_G(u)$.

Case 1. $d_G(u) = 2$.

Subcase 1.1 u is adjacent to a 2-vertex u_2. So u is a special vertex of Type (S_1) and also a negative vertex of Type (N_1). By lemma 3 1, u is adjacent to a r-vertex, by R_4 it receive $\frac{4}{5}$ from the r-neighbor. As a negative vertex, u also receives $\frac{1}{5}$ from the common pot by R_g. So $w(u) = d_G(u) + \frac{4}{5} + \frac{1}{5} = 3$.

Subcase 1.2 Both neighbors v_1 and v_2 of u are of degree at least 3.

Subcase 1.2.1 u has two weak neighbors. So u is a negative vertex of Type (N_3) It receives $\frac{2}{5}$ from its each weak neighbor by $R_{1.1}$ and get additional $\frac{1}{5}$ from the common pot by R_g. So $w(u) = d_G(u) + 2 \times \frac{2}{5} + \frac{1}{5} = 3$.

Subcase 1.2.2 u has one weak neighbor w and one non-weak neighbor v. Note that $d_G(v) \geq 3$. When $3 \leq d_G(v) \leq 7$, u receives $\frac{2}{5}$ from w by $R_{1\,1}$ and $\frac{3}{5}$ from v by $R_{1\,2}$, so $w(u) = d_G(u) + \frac{2}{5} + \frac{3}{5} = 3$; When $8 \leq d_G(v) \leq 14$, u receives $\frac{3}{8}$ from w by $R_{1\,4}$ and $\frac{5}{8}$ from v by R_3, so $w(u) = d_G(u) + \frac{3}{8} + \frac{5}{8} = 3$; When $d_G(v) \geq 15$, u receives $\frac{1}{5}$ from w by $R_{1.5}$ or from the common pot p_0 by R_g and $\frac{4}{5}$ from v by R_4. So $w(u) = d_G(u) + \frac{4}{5} + \frac{1}{5} = 3$.

Subcase 1.2.3 u has two non-weak neighbors v and v'. Assume that $d_G(v) \leq d_G(v')$.

Note that $d_G(v') \geq d_G(v) \geq 3$. When $3 \leq d_G(v) \leq 7$ and $3 \leq d_G(v') \leq 7$, u receives $\frac{1}{2}$ from each neighbor by $R_{1\,3}$, so $w(u) = d_G(u) + \frac{1}{2} + \frac{1}{2} = 3$, When $3 \leq d_G(v) \leq 7$ and $8 \leq d_G(v') \leq 14$, u receives $\frac{3}{8}$ from v by $R_{1\,4}$

and $\frac{5}{8}$ from v' by R_3, so $w(u) = d_G(u) + \frac{3}{8} + \frac{5}{8} = 3$; When $3 \le d_G(v) \le 7$ and $d_G(v') \ge 15$, u receives $\frac{1}{5}$ from v by $R_{1.5}$ if it is non-negative or from the common pot p_0 by R_g if it is negative, and $\frac{4}{5}$ from v by R_4, so $w(u) = d_G(u) + \frac{4}{5} + \frac{1}{5} = 3$; When $8 \le d(v) \le d(v')$, u receives at least $\frac{5}{8}$ from each of its neighbors by R_3 or R_4. So $w(u) = d_G(u) + 2 \times \frac{5}{8} \ge 3$.

Case 2. $d_G(u) = 3$.

Subcase 2.1 u has three neighbors y_1, y_2 and y_3 of degree 2. By Lemma 3.2(iii), for $1 \le i \le 3$, u is 1-linked (through a path uy_iz_i) to a r-vertex z_i. So y_1, y_2 and y_3 are negative vertices of Type (N_2), no rule applies to u. It is, $w(u) = d_G(u) = 3$.

Subcase 2.2 u has exactly two neighbors y_1 and y_2 of degree 2, for $1 \le i \le 2$, u is 1-linked (through a path uy_iz_i) to z_i. Let x be the third neighbor of u, $d_G(x) \ge 3$.

If $d'(x) \ge r - 2$, then $d_G(x) \ge d'(x) \ge r - 2 \ge 15$. u can receive $\frac{4}{5}$ from x by R_4 and give nothing to x. When u is weak, vertex u gives at most $\frac{2}{5}$ $(= \max\{\frac{2}{5}, \frac{3}{8}, \frac{1}{5}\})$ to each of y_1, y_2 by $R_{1.1}$, $R_{1.4}$ or $R_{1.5}$. So $w(u) \ge d_G(u) + \frac{4}{5} - 2 \times \frac{2}{5} = 3$. When u is non-weak, without loss of generality, suppose $d'(z_1) \ge 15$. So $d_G(z_1) \ge d'(z_1) \ge 15$. Vertex u gives at most $\frac{1}{5}$ to y_1 by $R_{1.5}$, u gives at most $\frac{3}{5}$ $(= \max\{\frac{3}{5}, \frac{1}{2}, \frac{3}{8}, \frac{1}{5}\})$ to y_2 by $R_{1.2}$, $R_{1.3}$, $R_{1.4}$ or $R_{1.5}$. So $w(u) \ge d_G(u) + \frac{4}{5} - \frac{1}{5} - \frac{3}{5} = 3$.

If $d'(x) \le r - 3$, by Lemma 3.2(iii), for $1 \le i \le 2$, z_i is an r-vertex. When $d'(x) \le 7$, y_1 and y_2 are negative vertices of Type (N_2), vertex u gives nothing to each of y_1 and y_2. By Lemma 3.2(iv), rule R_2 can not apply from u to x. So $w(u) = d_G(u) = 3$; When $d'(x) \ge 8$, vertex u gives $\frac{1}{5}$ to each of y_1 and y_2 by $R_{1.5}$. Vertex x gives at least $\frac{5}{8}$ $(= \min\{\frac{5}{8}, \frac{4}{5}\})$ to u by R_3 or R_4. So $w(u) \ge d_G(u) + \frac{5}{8} - 2 \times \frac{1}{5} \ge 3$.

Subcase 2.3 u has exactly one neighbors y of degree 2. Let z be the neighbor of y distinct from u. Let w and x be the other neighbors of u, where $d_G(w) \ge d_G(x) \ge 3$.

When $3 \le d_G(w) \le 7$, by Lemma 3.2(iii), z must be a r-vertex and y is not negative. So u gives $\frac{1}{5}$ to y by $R_{1.5}$. By Lemma 3.3(4), rule R_2 can not apply from u to x or w, so vertex u gives nothing to x and w. On the contrary, u receive $\frac{1}{10}$ from both w and x by R_2. So $w(u) = d_G(u) + 2 \times \frac{1}{10} - \frac{1}{5} = 3$; When $8 \le d_G(w) \le 14$, by Lemma 3.2(iv), rule R_2 can not apply from u to x, so vertex u gives nothing to x by R_2. u receive $\frac{5}{8}$ from w by R_3 and gives at most $\frac{3}{5}$ to y by R_1. So $w(u) \ge d_G(u) + \frac{5}{8} - \frac{3}{5} \ge 3$; When $d_G(w) \ge 15$, u receive $\frac{4}{5}$ from w by R_4, gives at most $\frac{3}{5}$ to y by R_1 and at most $\frac{1}{10}$ to x by R_2. So $w(u) \ge d_G(u) + \frac{4}{5} - \frac{3}{5} - \frac{1}{10} \ge 3$.

Subcase 2.4 All the neighbors of u have degree at least 3 and at most 7. By Lemma 3.2(iv), rule R_2 can not apply from u to any one neighbor of u. So $w(u) = d_G(u) = 3$.

Subcase 2.5 u has no neighbors of degree 2 and have at least one neighbor of degree at least 8. u receives at least $\frac{5}{8}$ $(= \min\{\frac{5}{8}, \frac{4}{5}\})$ from one of its neighbor R_3 or R_4 and gives at most $\frac{1}{10}$ to each of its two other neighbors by R_2. So $w(u) \geq d_G(u) + \frac{5}{8} - 2 \times \frac{1}{10} \geq 3$.

Case 3. $d_G(u) = 4$.

Subcase 3.1 u have at least 3 neighbors y_1, y_2 and y_3 of degree 2. It is, for $1 \leq i \leq 3$, u is 1-linked (through a path uy_iz_i) to a vertex z_i, assume that $d_G(z_1) \geq d_G(z_2) \geq d_G(z_3)$. Let x be the neighbor of u distinct from y_1, y_2, y_3. When $d_G(z_2) \leq 14$, by Lemma 3.2(vii), $d'(x) \geq r-2$, so $d_G(x) \geq d'(x) \geq r-2 \geq 15$. u receive $\frac{4}{5}$ from x by R_4 and gives at most $\frac{3}{5}$ to each y_i by $R_{1.2}, R_{1.3}, R_{1.4}$ or $R_{1.5}$, for $1 \leq i \leq 3$. So $w(u) \geq d_G(u) + \frac{4}{5} - 3 \times \frac{3}{5} = 3$; When $d_G(z_2) \geq 15$ and $d_G(z_3) \leq 14$, by Lemma 3.2(vi), $d'(x) \geq 8$, so $d_G(x) \geq d'(x) \geq 8$. u receive at least $\frac{5}{8}$ from x by R_3 or R_4, gives $\frac{1}{5}$ to each of y_1 and y_2 by $R_{1.5}$ and gives at most $\frac{3}{5}$ to y_3 by $R_{1.2}, R_{1.3}$ or $R_{1.4}$. So $w(u) \geq d_G(u) + \frac{5}{8} - 2 \times \frac{1}{5} - \frac{3}{5} \geq 3$; When $d_G(z_3) \geq 15$, by Lemma 3.2(vi), none of $\{R_{1.2}, R_{1.3}, R_{1.4}\}$ applies from u to each of its neighbors. u gives at most $\frac{1}{5}$ to each of them by $R_{1.5}$ or R_2. So $w(u) \geq d_G(u) - 4 \times \frac{1}{5} \geq 3$.

Subcase 3.2 u have exactly two neighbors y_1, y_2 of degree 2. It is, for $i = 1, 2$, u is 1-linked (through a path uy_iz_i) to a vertex z_i, assume that $d_G(z_1) \geq d_G(z_2)$. Let w, x be the other neighbors of u distinct from y_1, y_2, assume that $d_G(w) \geq d_G(x) \geq 3$. When $d_G(z_1) \leq 14$, by Lemma 3.2(vii), $d'(w) + d'(x) \geq r$. It is easy to know that $d_G(w) \geq 8$. u receives at least $\frac{5}{8}$ from w by R_3 or R_4, gives at most $\frac{3}{5}$ to each of y_1 and y_2 by $R_{1.2}, R_{1.3}$ or $R_{1.4}$, and gives at most $\frac{1}{10}$ to x by R_2. So $w(u) \geq d_G(u) + \frac{5}{8} - 2 \times \frac{3}{5} - \frac{1}{10} \geq 3$, When $d_G(z_1) \geq 15$, u gives $\frac{1}{5}$ to y_1 by $R_{1.5}$, gives at most $\frac{3}{5}$ to y_2 by $R_{1.2}, R_{1.3}, R_{1.4}$ or $R_{1.5}$, and gives at most $\frac{1}{10}$ to each of w, x by R_2. So $w(u) \geq d_G(u) - \frac{1}{5} - \frac{3}{5} - 2 \times \frac{1}{10} = 3$.

Subcase 3.3 u have at most one neighbor of degree 2. u gives at most $\frac{3}{5}$ by $R_{1.i}$ $(2 \leq i \leq 5)$ if u has a 2-neighbor, and gives at most $\frac{1}{10}$ to each of its 3^+-neighbors by R_2. So $w(u) \geq 4 - \frac{3}{5} - 3 \times \frac{1}{10} \geq 3$.

Case 4. $d_G(u) = 5$.

Subcase 4.1 u have at least four neighbors y_1, y_2, y_3 and y_4 of degree 2. It is, for $1 \leq i \leq 4$, u is 1-linked (through a path uy_iz_i) to a vertex z_i, assume that $d_G(z_1) \geq d_G(z_2) \geq d_G(z_3) \geq d_G(z_4)$. Let x be the neighbor of u distinct from y_1, y_2, y_3, y_4. When $d_G(z_4) \leq 7$, by Lemma 3.2(viii), $d'(x) \geq 8$, so $d_G(x) \geq d'(x) \geq 8$. u receive at least $\frac{5}{8}$ from x by R_3 or R_4 and gives at most $\frac{3}{5}$ to each y_i (for $1 \leq i \leq 4$) by $R_{1.2}, R_{1.3}, R_{1.4}$ or $R_{1.5}$. So $w(u) \geq d_G(u) + \frac{5}{8} - 4 \times \frac{3}{5} \geq 3$; When $d_G(z_4) \geq 8$, by Lemma 3.2(viii), neither $R_{1,2}$ nor $R_{1,3}$ applies from u to each of its neighbors. u gives at most $\frac{3}{8}$ to each of them by $R_{1.4}$, $R_{1.5}$ or R_2. So $w(u) \geq d_G(u) - 5 \times \frac{3}{8} \geq 3$.

Subcase 4.2 u have at most three neighbors of degree 2. u gives at most $3 \times \frac{3}{5}$ by $R_{1.i}$, gives at most $2 \times \frac{1}{10}$ by R_2. So $w(u) \geq d_G(u) - 3 \times \frac{3}{5} - 2 \times \frac{1}{10} = 3$.

Case 5. $d_G(u) = 6$.

Subcase 5.1 u have at least five neighbors of degree 2. Choose $y_1, \dots, y_5$ be five 2-neighbors of u such that the other neighbor (denoted by z_1) of y_1 has the minimum degree as possible. Let x be the neighbor of u distinct from $y_1, \dots, y_5$. By Lemma 3 2(ix), either $d'(x) \geq 8$ or $d_G(z_1) \geq 8$. When $d_G(z_1) \geq 8$, u gives at most $\frac{3}{8}$ to each of its neighbors by $R_{1.4}$, $R_{1.5}$ or R_2, so $w(u) \geq d_G(u) - 6 \times \frac{3}{8} \geq 3$; When $d'(x) \geq 8$, then $d_G(x) \geq 8$ and u receive at least $\frac{5}{8}$ from x by R_3 or R_4, gives at most $\frac{3}{5}$ to each of its 2-neighbors by R_1, so $w(u) \geq d_G(u) + \frac{5}{8} - 5 \times \frac{3}{5} \geq 3$.

Subcase 5.2 u have at most four neighbors of degree 2. u gives at most $4 \times \frac{3}{5}$ by $R_{1.*}$, gives at most $2 \times \frac{1}{10}$ by R_2. So $w(u) \geq d_G(u) - 4 \times \frac{3}{5} - 2 \times \frac{1}{10} \geq 3$.

Case 6. $d_G(u) = 7$.

Subcase 6.1 u have at least six neighbors of degree 2 which is also adjacent to a 3^--neighbor. By Lemma 3.2(x), u must have a 8^+-neighbor. u receive at least $\frac{5}{8}$ from its 8^+-neighbor by R_3 or R_4 and gives at most $\frac{3}{5}$ to each of its 2-neighbors by $R_{1.2}, R_{1.3}, R_{1.4}$ or $R_{1.5}$. So $w(u) \geq d_G(u) + \frac{5}{8} - 6 \times \frac{3}{5} \geq 3$.

Subcase 6.2 u have at most five neighbors of degree 2 which is also adjacent to a 3^--neighbor. u gives at most $5 \times \frac{3}{5}$ by $R_{1.2}$, gives at most $2 \times \frac{1}{2}$ by $R_{1.3}, R_{1.4}, R_{1.5}$ or R_2. So $w(u) \geq d_G(u) - 5 \times \frac{3}{5} - 2 \times \frac{1}{2} = 3$.

Case 7. $8 \leq d_G(u) \leq 14$. R_3 applies from u to each of its neighbors. $w(u) \geq d_G(u) - d_G(u) \times \frac{5}{8} \geq 3$.

Case 8. $15 \leq d_G(u) < r$. R_4 applies from u to each of its neighbors. $w(u) \geq d_G(u) - d_G(u) \times \frac{4}{5} \geq 3$.

Case 9. $d_G(u) \geq r$. R_4 applies from u to each of its neighbors, R_g also applies to u when it is positive. $w(u) \geq d_G(u) - d_G(u) \times \frac{4}{5} - \frac{2}{5} \geq 3$. □

Proof of Theorem 1.4. By Lemma 3.10 and 3.12, every vertex u of G has a final charge $w(u)$ at least 3. It follows that

$$3 \leq \frac{\sum_{u \in V(G)} w(u)}{|V(G)|} \leq \frac{\sum_{u \in V(G)} d_G(u)}{|V(G)|} \leq mad(G) < 3.$$

This contradiction establishes Theorem 1.4. □

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Appendix

Proof of Lemma 3.8. Suppose by contradiction that $H(G)$ contains a 2-connected subgraph C of size ≥ 3 that is not a cycle with an odd number of special vertices nor a subgraph of a lock of $H(G)$. Let $S = \{s_1, \ldots, s_p\}$ be the special vertices of C. By Lemma 3.5 and 3.6, $p \geq 4$. Let $\mathfrak{S}$ be the graph with $V(\mathfrak{S}) = S$ where there is an edge between s_i and s_j if and only if they are adjacent or have a common neighbor in G. If s_i and s_j have a common neighbor v, by definition of special vertex (see Fig. 4) and Lemma 3.3, v is either r-vertex or of degree at most 3.

Calim 1. $\mathfrak{S}$ is not a clique of size at least 4.

Proof of Calim 1. Suppose, by contradiction that $\mathfrak{S}$ is a clique with $p \geq 4$. Given a special vertex x, we say that a special vertex x', distinct from x, satisfies the property P_x if it is either adjacent to x in G or has a non-positive common neighbor with x in G. At most two vertices can satisfy P_x (vertex a of Fig. 4 if x is of Type (S_1), vertices b, c if x is of Type (S_2), vertices b, d if x is of Type (S_3)). Note that if x' satisfies P_x, then x satisfies $P_{x'}$.

Firstly we show that there exist two special vertices in S that do not have a positive common neighbor in G If not, that every pair of vertices of S has a positive common neighbor. By Lemma 3.3, every special vertex has at most one positive neighbor, so all the vertices of S are adjacent to the same positive vertex v. As C is 2-connected, there is a path Q in $C\backslash\{v\}$ between s_1, s_2. Let s_i be the first special vertex, distinct from s_1, appearing along Q while starting from s_1 (maybe $i = 2$ if there is no special vertex in the interior of Q). Let Q' be the subpath of Q between s_1 and s_i (maybe $Q = Q'$). Then $Q' \cup \{v\}$ forms a 2-connected subgraph of size ≥ 3 with exactly two special vertices, a contradiction to Lemma 3.5. So there exist two special vertices x, x' in S that do not have a positive common neighbor in G. Since S is a clique of $\mathfrak{S}$, vertices x, x' are adjacent or have a common non-positive neighbor, so x satisfies $P_{x'}$ (and x' satisfies P_x).

Secondly we show that every vertex of $S\backslash\{x, x'\}$ satisfies either P_x or $P_{x'}$. Suppose there exists a special vertex $y \in S$ that satisfies neither P_x nor $P_{x'}$. Since S is a clique of $\mathfrak{S}$, vertex y has a common positive neighbor z with x and z' with x'. Since x and x' have no positive common neighbor, z and z' are distinct. Thus y has two positive neighbors, a contradiction to lemma 3.3.

If two vertices y, y' of $S\backslash\{x, x'\}$ satisfy P_x, then at least three vertices, x', y, y' verify P_x, a contradiction to the fact that at most two vertices can satisfy P_x . So there is at most one vertex of $S\backslash\{x, x'\}$ satisfying P_x and similarly at most one satisfying $P_{x'}$. So $p = 4$ and we can assume, w.l.o.g., that $S = \{x, x', y, y'\}$, where vertex y satisfies P_x and not $P_{x'}$ and vertex y' satisfies $P_{x'}$ and not P_x. Thus x, x' must be of type (S_3) and x has a common positive neighbor z with y' and similarly x' has a common positive neighbor z' with y. Since x, x' do not have a common positive neighbor, z and z' are distinct. Vertices y, y' have at most one positive neighbor, thus they do not have a common positive neighbor. Since S is a clique of $\mathfrak{S}$, y satisfies $P_{y'}$. Let $(y_1, y_2, y_3, y_4) = (x, x', y', y)$ (subscripts are understood modulo 4).

Suppose there exists $i \in \{1, 2, 3, 4\}$ such that y_i, y_{i+1} are adjacent in G. Two special vertices can be adjacent only if they are of Type (S_1). So y_i, y_{i+1} are of Type (S_1) and of degree two. Then y_i is only adjacent to y_{i+1} and to a positive vertex in $\{z, z'\}$. If y_i is adjacent to y_{i-1}, then $y_{i-1} = y_{i+1}$, a contradiction. If y_i is not adjacent to y_{i-1}, then y_i have a common non-positive neighbor with y_{i-1} which contradicts the fact that the other neighbor than y_{i+1} of y_i must be an r-vertex. So y_i, y_{i+1} are not adjacent in G for any $1 \leq i \leq 4$. Let w_i be a non-positive common neighbor of y_i, y_{i+1}.

Suppose there exists $i \in \{1, 2, 3, 4\}$ such that $d_G(y_i) = 2$, then y_i must be of Type (S_2) and so do y_{i-1}, y_{i+1}. Here $w_{i-1} = w_i$, $N_G(w_i) = \{y_{i-1}, y_i, y_{i+1}\}$ and $d_G(y_{i-1}) = d_G(y_i) = d_G(y_{i+1}) = 2$. The similar analysis replacing i with $i + 1$ leads to the result $w_i = w_{i+1}$ and $N_G(w_{i+1}) = \{y_i, y_{i+1}, y_{i+2}\}$, a contradiction. So $d_G(y_i) \geq 3$ for any $1 \leq i \leq 4$.

Then for any $1 \leq i \leq 4$, y_i are of Type (S_3), $d_G(y_i) = 3$, and $d_G(w_i) = 2$. So $y_1, \ldots, y_4, w_1, \ldots w_4, z, z'$ induce a lock. All the edges incident to $S =$

$\{y_1, \ldots, y_4\} = \{s_1, \ldots, s_4\}$ belong to a lock, contradicting the definition of C. □

By Lemma 3.5, $\mathfrak{S}$ is not a edge. If $\mathfrak{S}$ is a triangle, c has exactly 3 special vertices, and by Lemma 3.6, it is a cycle, contradicts to the definition of C. Combined with Claim 1, $\mathfrak{S}$ is not a clique.

For C is 2-connected, then $\mathfrak{S}$ is connected. Suppose, by contradiction, $\mathfrak{S}$ is not 2-connected. Then there exist three special vertices s, s', s'' of S such that s', s'' appear in two different components of $\mathfrak{S} - s$. As C is 2-connected, there exists a path Q between s' and s'' in $C - s$. Q contains only edges incident to special vertices, so in $\mathfrak{S}$ it corresponds to a path $\mathfrak{S}$ between s' and s'', a contradiction. So $\mathfrak{S}$ is 2-connected.

Suppose, by contradiction, that $\mathfrak{S}$ is an odd cycle with ≥ 5 vertices. For C is 2-connected subgraph of $H(G)$ that is not a cycle, there exists a vertex x with $d_C(x) \geq 3$. If x is not a special vertex, then x has at least 3 special neighbors in C which form a clique of size at least 3 in $\mathfrak{S}$, a contradiction. So x is a special vertex of Type (S_3) (see Fig. 4) and its three neighbors in G are also in C. For C is 2-connected, then either $d_{\mathfrak{S}}(x) \geq 3$ which contradicts to $\mathfrak{S}$ being a cycle, or C has a cycle with exactly two special vertices, contradicts to Lemma 3.5. So $\mathfrak{S}$ is not an odd cycle.

By (3), $G' = G \backslash (S \cup \{v \in N_G(S) | d_G(v) \leq 3\})$ has an (L, r)-coloring. Next we will extend it to be an (L, r)-coloring of G. For every vertex x of S, it has exactly a neighbor left in G' which is r-vertex of G, and the number of constraints on it is at most $r + 2 - d_{\mathfrak{S}}(x)$ (see Fig. 4). So the number of available colors of any vertex x in S is at least $\mathfrak{S}(x)$. For $\mathfrak{S}$ is neither a clique nor an odd cycle, apply lemma 3.7 to $\mathfrak{S}$. So we can color S. Every vertex of $\{v \in N_G(S) | d_G(v) \leq 3\}) \backslash S$ has at most 17 constraints, hence we can extend the coloring to the whole graph, a contradiction. □

Proof of Lemma 3.9. All the edges of a lock are incident to special vertices of Type (S_3), so all edges of a lock appear in $H(G)$. The only vertices of a lock that can have neighbors outside a lock are locked vertices (vertices v, w in Fig. 5). By Lemma 3.2(xi), a locked vertex is incident to only two special vertices which are also in the same lock. A lock is a connected component of $H(G)$.

Let C be a connected component of $H(G)$ that is not a lock, by lemma 3.8, the every block of C is a cycle with odd number of special vertices. So C is a cactus where each cycle has an odd number of special vertices. □