Supereulerian digraphs with forbidden induced subdigraphs containing short dipaths

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Abstract

A digraph $D$ is supereulerian if $D$ has a spanning eulerian subdigraph. We investigate forbidden induced subdigraph conditions for a strong digraph to be supereulerian. Let $P_k$ denote the dipath on $k$ vertices. For $k \in \{2, 3, 4\}$, we determine the smallest integer $h_k$ such that if a strong strict digraph $D$ containing a subdigraph $H$ isomorphic to $P_k$ always satisfies $|A(D(V(H)))| \geq h_k$, then $D$ is supereulerian. For $k \geq 5$, we show that $k^2 - 4k + 8 \leq h_k \leq k(k - 1)$.

Key words. Strong arc connectivity, eulerian digraphs, supereulerian digraphs, forbidden induced subdigraphs

1 Introduction

We consider finite graphs and digraphs. Undefined terms and notations will follow [7] and [6]. We use $(u, v)$ to represent an arc oriented from a vertex $u$.

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to a vertex $v$. As in [7], a digraph $D$ is strict if $D$ has no loops and if for any pair of distinct vertices $u, v \in V(D)$, there is at most one arc in $D$ oriented from $u$ to $v$. Throughout out this paper, we only consider strict digraphs. We use $D \cong D'$ to mean that the two digraphs $D$ and $D'$ are isomorphic. For an integer $n > 0$, we use $K_n^*$ to denote the complete digraph on $n$ vertices. Hence for every pair of distinct vertices $u, v \in V(K_n^*)$, there is exactly one arc $(u, v)$ in $A(K_n^*)$. For a digraph $D$, the underlying graph of $D$, denoted by $G(D)$, is obtained from $D$ by erasing the orientations of all arcs of $D$. A digraph $D$ is weakly connected if $G(D)$ is connected.

Following [4], for a digraph $D$ with $X, Y \subseteq V(D)$, define

$$(X, Y)_D = \{(x, y) \in A(D) : x \in X, y \in Y\}.$$ When $Y = V(D) - X$, we define

$$\partial^+_D(X) = (X, V(D) - X)_D$$

and

$$\partial^-_D(X) = (V(D) - X, X)_D.$$ 

For a vertex $v \in V(D)$, $d^+_D(v) = |\partial^+_D(\{v\})|$ and $d^-_D(v) = |\partial^-_D(\{v\})|$ are the out-degree and the in-degree of $v$ in $D$, respectively. Finally, we define the following notations: $\delta^+(D) = \min\{d^+_D(v) : v \in V(D)\}$ and $\delta^-(D) = \min\{d^-_D(v) : v \in V(D)\}$. Let $N^+_D(v) = \{u \in V(D) - v : (u, v) \in A(D)\}$ and $N^-_D(v) = \{u \in V(D) - v : (v, u) \in A(D)\}$ denote the out-neighbourhood and in-neighbourhood of $v$ in $D$, respectively. We call the vertices in $N^+_D(v)$, $N^-_D(v)$ the out-neighbours, in-neighbours of $v$. When the digraph $D$ is understood from the context, we often omit the subscript $D$. For an integer $k > 0$, let $P_k$ denote a path (or a dipath) on $k$ vertices.

Boesch, Suffel, and Tindell [5] in 1977 proposed the supereulerian problem, which seeks to characterize graphs that have spanning Eulerian subgraphs. Pulleyblank [16] later in 1979 proved that determining whether a graph is supereulerian is NP-complete. Since then, there have been lots of researches on this topic. For the literature of supereulerian graphs, see Catlin’s first survey [8] on the topic and its updates [9] and [15].

It is natural to study supereulerian digraphs. A digraph $D$ is eulerian if $D$ is weakly connected and for every $v \in V(D)$, $d^+_D(v) = d^-_D(v)$; and is supereulerian if $D$ contains a spanning eulerian subdigraph. The main
problem is to determine supereulerian digraphs. Some earlier studies were
done by Gutin [11, 12], and recent developments can be found in [1, 2, 3,
5, 13, 14], among others.

Forbidden induced subgraph conditions have been a widely investigated
topic. Given a graph $K$, a graph $G$ is said to be $K$-free if $G$ does not have
an induced subgraph isomorphic to $K$. This is equivalent to saying that
if $G$ has a subgraph $H$ isomorphic to $K$, then $|E(G[V(H)])| \geq |E(H)| + 1$.
Sufficient $K_{1,3}$-free conditions for hamiltonicity have been intensively
studied, as seen in [10]. For a vertex $w$ of $G$, let

$$M_2(w) = G\{x \in V(G) : 1 \leq d_G(w, x) \leq 1\}.$$ 

For $w \in V(G)$, let $N_2(w)$ be the subgraph induced by the set of edges
$uw$, such that either $u$ or $v$ is adjacent to $w$. A vertex $w$ of $G$ is $N^1$-locally
connected ($N_2$-locally connected, respectively) if $M_2(w)$ ($N_2(w)$,
respectively) is connected. If every vertex of $G$ is $N^1$-locally connected
($N_2$-locally connected, respectively), then $G$ is $N^1$-locally connected
($N_2$-locally connected, respectively). Recently, Saito and Xiong proved
the following.

**Theorem 1.1** (Saito and Xiong, [17]) Let $H$ be a connected graph of order
at least three, $P_k$ be an undirected path on $k$ vertices, and $G$ be a connected
$N^1$-locally connected graph. Each of the following holds.

(i) Every $2$-edge connected $H$-free graph is supereulerian if and only if $H$
is $K_{1,2}$.

(ii) Every $N^1$-locally connected $H$-free graph is supereulerian if and only if
$H$ is either $K_{1,2}$ or $K_{1,3}$

(iii) If $G$ is $P_3$-free, then $G$ is supereulerian, if $G$ is $P_6$-free, then $G$ is
supereulerian or the Petersen graph.

These motivates the current study on forbidden induced subdigraph
sufficient conditions for supereulerian digraphs. Throughout the rest of
the paper, for an integer $k \geq 2$, $P_k$ denotes the dipath on $k$ vertices. A
subdigraph $H$ of a digraph $D$ is a $P_k$-subdigraph if $H$ is isomorphic to $P_k$.
If $D$ does not have an induced $P_k$, then for any $P_k$-subdigraph $H$ of $D$, we
must have $|A(D[V(H)])| \geq k$. 

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Definition 1.1 For integers \( h \geq k \geq 2 \), we define \( \mathcal{F}(P_k, h) \) to be the family of all strict digraphs such that \( D \in \mathcal{F}(P_k, h) \) if and only if \( D \) is strong and satisfies both of the following.

(i) \( D \) contains at least one dipath \( P_k \) with \( |A(D[V(P_k)])| = h \), and

(ii) for any dipath \( P_k \) in \( D \), \( |A(D[V(P_k)])| \geq h \).

If \( D \in \mathcal{F}(P_k, h) \), then we also call \( D \) a \( \mathcal{F}(P_k, h) \)-digraph. It is known (for example, Corollary 3.1 of [2]) that every strong \( \mathcal{F}(P_k, k^2 - k) \) digraph is supereulerian. Thus it is of interest to determine the smallest \( h_k \) such that every strong strict digraph in \( \mathcal{F}(P_k, h_k) \) is supereulerian. A digraph \( D \) is transitive if for every triple of distinct vertices \( x, y, z \in V(D) \) with \( (x, y), (y, z) \in A(D) \), then \( (x, z) \in A(D) \). Thus if \( D \) is a transitive digraph, then \( D \in \mathcal{F}(P_3, 3) \), and so \( \mathcal{F}(P_k, h) \) digraphs also represent some of the well studied families of digraphs.

The main purpose of this research is to investigate, for smaller values of \( k \) with \( k \leq h \leq k(k - 1) \), the behavior of graphs in \( \mathcal{F}(P_k, h) \) and to determine the value of \( h_k \). We show that \( h_2 = 2 \), \( h_3 = 5 \), \( h_4 = 8 \), and for \( k \geq 5 \), we show that \( k^2 - 4k + 8 \leq h_k \leq k(k - 1) \). Our results are presented in the subsequent sections.

2 Supereulerian digraphs in \( \mathcal{F}(P_2, h) \cup \mathcal{F}(P_3, h) \)

In this section, we investigate the supereulerianity of digraphs in \( \mathcal{F}(P_2, h) \) with \( h = 2 \) and in \( \mathcal{F}(P_3, h) \) with \( 3 \leq h \leq 6 \). We need a necessary condition for a digraph to be supereulerian. Let \( D \) be a digraph and \( U \subseteq V(D) \). Let \( t_0(U) \) be the smallest integer \( t \) such that \( D[U] \) has a collection of arc disjoint directed trails \( T_1, T_2, \ldots, T_t \) with \( U = \cup_{i=1}^{t} V(T_i) \). For any subset \( A \subseteq V(D) \) \( - U \), define \( B := V(D) - U - A \), and

\[
h(U, A) := \min\{\delta_+^{-1}(A), \delta_-^{-1}(A)\} + \min\{|(U, B)_D|, |(B, U)_D|\} - t_0(U).
\]

Then we have the following proposition.

Proposition 2.1 (Hong, Lai and Liu, Proposition 2.1 of [18]) If \( D \) has a spanning eulerian subdigraph, then for any \( U \subseteq V(D) \), we have \( h(U, A) \geq 0 \).
Digraphs in $F(P_3, h)$ with $3 \leq h \leq 4$ are not necessarily supereulerian, as can be seen in the example below.

![Diagram](image)

**Figure 1** The digraph $D = D(\alpha, \beta, k, \ell)$.

**Example 2.1** Let $\alpha, \beta, k > 0$ be integers with $\alpha, \beta \geq k + 1$, and let $A$ and $B$ be two disjoint set of vertices with $|A| = \alpha$ and $|B| = \beta$. Let $\ell \geq \alpha \beta + 1$ be an integer, and $U$ be a set of vertices disjoint from $A \cup B$ with $|U| = \ell$. We construct a digraph $D = D(\alpha, \beta, k, \ell)$ such that $V(D) = A \cup B \cup U$ and the arcs of $D$ are given as required in (D1) and (D2) below. (See Figure 1).

(D1) $D[A \cup B] \cong K_{\alpha+\beta}$ is a complete digraph.

(D2) For every vertex $u \in U$, and for every $v \in A$, $(u, v) \in A(D)$ and for every $w \in B$, $(w, u) \in A(D)$. Thus for any $u \in U$, we have $N^+_D(u) = A$ and $N^-_D(u) = B$. No two vertices in $U$ are adjacent.

Direct computation yields

$$h(U, A) = |\partial^+_D(A)| + |(U, B)_D| - e_0(U) = \alpha \beta - |U| < 0,$$

and so by Proposition 2.1, any $D = D(\alpha, \beta, k, \ell)$ is nonsupereulerian. By Definition 1.1, $D \in F(P_3, 4)$. Thus Example 2.1 indicates that $F(P_3, 4)$ contains infinitely many nonsupereulerian digraphs.

An arc $(u, v)$ of $D$ is symmetric in $D$ if both $(u, v), (v, u) \in A(D)$. A digraph $D$ is symmetric if $|V(D)| = 1$ or if $|A(D)| > 0$ and every arc of $D$ is symmetric. A digraph $D \neq K_1$ is symmetrically connected, if for every $u, v \in V(D)$, $D$ contains a symmetric $(u, v)$-dipath.

**Theorem 2.1** ([2]) If $D$ is symmetrically connected, then $D$ is supereulerian.
Observe that if $D \in \mathcal{F}(P_3,2) \cup \mathcal{F}(P_3,5)$, then $D$ is symmetrically connected. Thus by Theorem 2.1, every digraph in $\mathcal{F}(P_3,2) \cup \mathcal{F}(P_3,5)$ is supereulerian. Hence we have our conclusions in this section.

Proposition 2.2 Let $D$ be a digraph.
(i) Every digraph in $\mathcal{F}(P_3,2) \cup \mathcal{F}(P_3,5)$ is supereulerian.
(ii) Not every digraph in $\mathcal{F}(P_3,3) \cup \mathcal{F}(P_3,4)$ is supereulerian.

3 Supereulerian digraphs in $\mathcal{F}(P_4, h)$

Throughout this section, $k > 0$ denotes an integer. In this section, we will study the supereulerianicity of digraphs in $\mathcal{F}(P_4, h)$, and determine the smallest value of $\eta_4$ such that every digraph in $\mathcal{F}(P_4, \eta_4)$ is supereulerian.

We start with some terminology and definitions. For a digraph $D$ and a subdigraph $S$ of $D$, an $(x, y)$-dipath $P$ is called an $(S, S)$-dipath if $V(P) \cap V(S) = \{x, y\}$. An $(S, S)$-dipath $P$ is shortest if for some $x, y \in V(S)$, $P$ is an $(x, y)$-dipath and $P$ is shortest among all $(x, y)$-dipath in $D - A(S)$.

Definition 3.1 Let $D$ be a digraph. Suppose that $S$ is an eulerian subdigraph of $D$ with $|V(S)|$ maximised. A shortest $(S, S)$-dipath $H$ with $|V(H)| = k + 2 \geq 3$ is called a $k$-handle of $S$ in $D$.

The following is a necessary condition for a digraph to be supereulerian.

Lemma 3.1 (K.A. Alsatami et al, Lemma 2 of [3]) A digraph $D$ is non-supereulerian if there exist vertex-disjoint subdigraphs $(A, B_1, \ldots, B_m)$ of $D$, for some integer $m > 0$, satisfying each of the following.
(i) $N^{-}(B_i) \subseteq V(A)$, $\forall i \in \{1, 2, \ldots, m\}$.
(ii) $|\partial^{-}(A)| \leq m - 1$.

In the rest of this section, we examine the supereulerian membership of digraphs in $\mathcal{F}(P_4, h)$, for each value $h$ with $4 \leq h \leq 12 = |\mathbb{A}(K_4^+)|$, and to determine the value of $\eta_4$ such that every digraph in $\mathcal{F}(P_4, \eta_4)$ is supereulerian.

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Proposition 3.1 Let \( D \) be a digraph. There exists an infinite family of nonsupereulerian digraphs in \( F(P_4, 7) \). More precisely, there are infinitely many \( 1 \)-handles in \( F(P_4, 7) \).

Proof. Let \( M = xy \) be a symmetric dipath, \( Q = xy \) be a dipath and \( H_i = xuy, 1 \leq i \leq \ell \) be dipaths. Define \( D_\ell = M \cup Q \cup \left( \bigcup_{i=1}^{\ell} H_i \right) \cup \{(x,y)\} \), as depicted in Figure 2. It is routine to verify that \( D_\ell \in F(P_4, 7) \). By Lemma 3.1 with \( A = D([x]), B_1 = D([u]) \) and \( B_2 = D([v]) \), we conclude that \( D_\ell \) is nonsupereulerian. By Definition 3.1, \( D_\ell \) is a \( 1 \)-handle.

We make the following observation for nonsupereulerian strong digraph:

Observation 3.1 Suppose that \( D \) is a nonsupereulerian strong digraph. Let \( S \) be a maximal eulerian subdigraph of \( D \) and let \( H = xu_1...u_ky \) be a \( k \)-handle of \( S \) and \( Q = xu_1...u_ky \) be a shortest \((x,y)\)-dipath in \( S \) with \( k + s \) minimized. We have the following observations.

(A) If for some \( i \) with \( 1 \leq i \leq k \), we have \( \{(u_i,x),(y,u_i)\} \cap A(D) \neq \emptyset \), then either \( S \cup \{(x,u_1),(u_1,u_2),...,(u_{i-1},u_i),(u_i,x)\} \) or \( S \cup \{(u_i,y),(u_i,u_{i+1}),...,(u_{k-1},u_k),(u_k,y)\} \) would violate the maximality of \( S \). Hence for any \( i \) with \( 1 \leq i \leq k \), we have \( \{(u_i,x),(y,u_i)\} \cap A(D) = \emptyset \).

(B) If for some \( i \) with \( 1 \leq i \leq k - 1 \), we have \( \{(x,u_{i+1}),(u_i,y)\} \cap A(D) \neq \emptyset \), then \( H' = xu_{i+1}...u_ky \) or \( H'' = xu_1...u_ky \) is a shorter \((S,S)\)-dipath,
contrary to the fact that $H$ is a shortest $(S, S)$-dipath, stated in Definition 3.1. Hence for any $1 \leq i \leq k - 1$, we have $\{(x, u_{i+1}), (u_i, y)\} \cap A(D) = \emptyset$.

(C) By Definition 3.1, $H$ is a shortest $(S, S)$-dipath. The minimality of $k + s$ implies that for every $i, j$ with $1 \leq i \leq s, 1 \leq j \leq k$, we have $\{(v_i, u_j), (u_j, v_i)\} \cap A(D) = \emptyset$, and for every $j$ with $i + 2 \leq j$, we have $(u_i, u_j) \notin A(D)$.

Theorem 3.1 Each of the following holds.

(i) Every digraph $D$ in $\mathcal{F}(P_4, 8)$ is supersimerian.

(ii) $h_4 = 8$.

Proof. As (ii) follows from (i) and Proposition 3.1, it suffices to prove (i). Assume that $D \in \mathcal{F}(P_4, 8)$. By contradiction, we assume that $D$ is a nonsupersimerian digraph. Let $S$ be a maximal eulerian subdigraph of $D$ and let $H = xu_1 \ldots u_k y$ be a $k$-handle of $S$ and, for some integer $s \geq 1$, let $Q = xu_1 \ldots u_i y$ be a shortest $(x, y)$-dipath in $S$ such that

$$k + s \text{ is minimized.}$$

(1)

We consider three cases.

Case 1. $k \geq 3$.

In this case, $P' = xu_1 u_2 u_3$ is a $P_4$ in $D$. By Observation 3.1 ((A) and (B)), we conclude that $\{(u_1, x), (x, u_2), (u_2, x), (x, u_3), (u_3, x)\} \cap A(D) = \emptyset$. It follows that $|A(D[V(P')]| < 8$, contrary to the assumption that $D \in \mathcal{F}(P_4, 8)$.

Case 2. $k = 2$.

In this case, $P'' = xu_1 u_2 y$ is a $P_4$ in $D$. By Observation 3.1 ((A) and (B)), we conclude that $\{(u_1, x), (x, u_2), (u_2, x), (y, u_2), (y, u_1), (u_1, y)\} \cap A(D) = \emptyset$. It follows that $|A(D[V(P'')]| < 8$, contrary to the assumption that $D \in \mathcal{F}(P_4, 8)$.

Case 3. $k = 1$.

Claim 1. $(y, x) \notin A(D)$.

By contradiction, we assume that $(y, x) \in A(D)$. If $(y, x) \notin A(S)$, then $SU(A(H)) \cup \{(y, x)\}$ is violation to the maximality of $S$. Hence $(y, x) \in A(S)$.
Since $P^{(3)} = u_1 y v x_1$ is a $P_4$ in $D$, by Observation 3.1 ((A) and (C)), we conclude that $\{(u_1, x), (y, u_1), (u_1, v_1), (v_1, u_1)\} \cap A(D) = \emptyset$. Since $D \in F(P_4, 8)$, we have $|A(D)[V(P^{(3)})]| \geq 8$, and so $\{(y, v_1), (v_1, x)\} \subset A(D)$.

Suppose first that $s = 1$. If $\{(y, v_1), (v_1, x)\} \cap A(S) = \emptyset$, then $S \cup \{(y, u_1), (v_1, x), (x, u_1), (u_1, y)\}$ is a violation to the maximality of $S$. Hence $\{(y, v_1), (v_1, x)\} \cap A(S) \neq \emptyset$, whence $S - \{(x, v_1), (v_1, y)\} + \{(x, u_1), (u_1, y)\}$ is a violation to the maximality of $S$. In either case, a contradiction obtains and so we must have $s \geq 2$. Note that $P^{(4)} = y u_1 v x_1 v_2$ is a $P_4$ in $D$. By Observation 3.1 ((A) and (C)), we conclude that $\{(y, u_1), (u_1, v_1), (v_1, u_1), (u_1, v_2), (v_2, u_1)\} \cap A(D) = \emptyset$. It follows that $|A(D)[V(P^{(4)})]| < 8$, contrary to the assumption that $D \in F(P_4, 8)$. This justifies Claim 1.

Claim 2. For any $i$ with $1 \leq i \leq s$, $(y, v_i) \notin A(D)$.

By contradiction, we assume that $(y, u_i) \in A(D)$ for some $i$ with $1 \leq i \leq s$. Then $P^{(5)} = x u_1 y v_i$ is a $P_4$ in $D$. Since $D \in F(P_4, 8)$, we must have $|A(D)[V(P^{(5)})]| \geq 8$. By Observation 3.1 ((A) and (C)) and Claim 1, $\{(u_1, x), (y, u_1), (u_1, v_1), (v_1, u_1), (y, x)\} \cap A(D) = \emptyset$. It follows that $|A(D)[V(P^{(5)})]| < 8$, contrary to the assumption that $D \in F(P_4, 8)$. This justifies Claim 2.

By Claims 1 and 2, we have,

$$\{(y, u_1), (y, u_2), \ldots, (y, u_s), (y, x)\} \cap A(D) = \emptyset. \quad (2)$$

Claim 3. For any $z \in V(D)$, $(y, z) \notin A(D)$.

By contradiction, we assume that for some $z \in V(D)$, we have $(y, z) \in A(D)$. Then $P^{(6)} = x u_1 y z$ is a $P_4$ in $D$. Since $D \in F(P_4, 8)$, we have $|A(D)[V(P^{(6)})]| \geq 8$.

By Observation 3.1 (A) and by (2), we have $\{(u_1, x), (y, u_1), (y, x)\} \cap A(D) = \emptyset$. It follows that $|\{(z, u_1), (u_1, x)\} \cap A(D)| \geq 1$.

Suppose first that $\{(z, u_1), (u_1, x)\} \subseteq A(D)$. If $z \notin V(S)$, then $S \cup \{(y, z), (z, u_1), (u_1, y)\}$ violates the maximality of $S$. If $z \in V(S)$, then $S \cup \{(z, u_1), (u_1, z)\}$ violates the maximality of $S$. We have $|\{(z, u_1), (u_1, z)\} \cap A(D)| = 1$. Since $D \in F(P_4, 8)$, therefore, we have $|A(D)[V(P^{(6)})]| \geq 8$.

Thus $\{(z, y), (z, x), (x, z), (x, y), (u_1, y), (y, x)\} \subseteq A(D)$. Since
\[ \{(z, u_2), (u_1, z)\} \cap A(D) = \emptyset \], we consider just two subcases.

Subcase 3.1. \( (z, u_1) \in A(D) \) but \( (u_1, z) \notin A(D) \).

If \( z \notin V(S) \), then \( S \cup \{(y, z), (z, u_1), (u_1, y)\} \) violates the maximality of \( S \). Hence we assume that \( z \in V(S) \). The dipath \( P'(7) = v_2yzu_1 \) is a \( P_4 \) in \( D \). By Observation 3.1((A) and (C)) and by (2), we have \( \{(u_1, z), (y, u_1), (u_1, v_2), (y, u_1), (y, v_2)\} \cap A(D) = \emptyset \). It follows that \( |A(D[V(P'(7)])]| < 8 \), contrary to the assumption that \( D \in F(P_4, 8) \). This contradiction indicates that Subcase 3.1 does not occur.

Subcase 3.2. \( (u_1, z) \in A(D) \) but \( (z, u_1) \notin A(D) \).

Note that \( P(9) = u_1zv_1 \). Since \( D \in F(P_4, 8) \), we have
\[ |A(D[V(P(9)])]| \geq 8. \]

By Observation 3.1((A) and (C)) and by (2), \( \{(u_1, x), (u_1, v_1), (u_1, u_1), (z, u_1)\} \cap A(D) = \emptyset \), and so \( (z, v_1) \in A(D) \).

Recall that \( Q \) is a shortest \((x, y)\)-dipath in \( S \) as defined in (1). The dipath \( P(9) = u_1zv_1v_2 \) is a \( P_4 \) in \( D \), where \( v_2 = y \) when \( Q = xv_1y \). Since \( D \in F(P_4, 8) \), we have \( |A(D[V(P(9)])]| \geq 8 \). By Observation 3.1 (C) and by (2), \( \{(u_2, u_1), (v_1, u_1), (v_2, u_1), (z, u_1)\} \cap A(D) = \emptyset \), where \( (u_1, v_2) \notin A(D) \) when \( v_2 \neq y \) and \( (y, v_1) \notin A(D) \) when \( v_2 = y \). These imply that \( |A(D[V(P(9)])]| < 8 \), contrary to the assumption that \( D \in F(P_4, 8) \).

4 Lower bound of \( h_k \)

For a given integer \( k > 1 \), let \( h_k \) denote the smallest integer such that every strong strict digraph in \( F(P_k, h_k) \) is supereulerian. It is known that \( k < h_k \leq k(k - 1) \). In this final remark section, we would present a lower bound of \( h_k \) for \( k \geq 5 \), as stated below.
Proposition 4.1 For $k \geq 5$, $h_k \geq k^2 - 4k + 8$.

Proof. For each integer $k \geq 5$, we shall show that there exists an infinite family of nonsupereulerian strong digraphs in $\mathcal{F}(P_k, k^2 - 4k + 7)$.

Let $M'$ be a complete digraph isomorphic to $K^*_{k-1}$ with vertex set \( \{z, u_1, u_2, \ldots, u_{k-3}, y\} \) and set $z_0 = y$. Let $M$ be the digraph obtained from $M'$ by deleting all the arcs $(z_j, z)$ from $M'$, where $0 \leq j \leq k - 3$. Let $Q = xuy$ be a dipath and $H_i = xuv_i, 1 \leq i \leq \ell$ be dipaths. Define $D_\ell = M \cup Q \cup \left( \bigcup_{i=1}^{\ell} H_i \right)$, as depicted in Figure 3. It is routine to verify that $D_\ell \in \mathcal{F}(P_k, k^2 - 4k + 7)$. By Lemma 3.1 with $A = D([x])$, $B_1 = D([u])$, $B_2 = D([v])$, we conclude that $D_\ell$ is nonsupereulerian.

We conclude this section with the following conjecture.

Conjecture 4.1 For every integer $k \geq 5$, $h_k = k^2 - 4k + 8$.

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