A Discharging Method to Find Subgraphs Having Two Edge-disjoint Spanning Trees

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Abstract

We introduce the discharging method in the study of subgraphs with two edge-disjoint spanning trees. As an application, we present a short proof for the theorem that every 3-connected, essentially 10-connected line graph is hamiltonian connected.

Keywords: hamiltonian connected, line graph, edge-disjoint spanning trees

1 Introduction

We use [1] for terminology and notations not defined here, and consider finite graphs without loops. A graph G is trivial if it contains no edges. A vertex (edge) cut X is essential if $G - X$ has at least two non-trivial components. For an integer $k > 0$, a graph G is essentially k-connected if G does not have an essential vertex cut X with $|X| < k$, and G is essentially k-edge-connected if G does not have an essential edge cut X with $|X| < k$. For $v \in V(G)$, $N_G(v)$, $d_G(v)$ and $\tau(G)$ represent the neighborhood of v, the degree of G, and the maximum number of edge-disjoint trees in G, respectively. For an integer $i > 0$, we define $D_i(G) = \{v \in V(G) | d_G(v) = i\}$

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and $D_{\geq i}(G) = \bigcup_{k \geq i} D_k(G)$. For subsets A, B of $V(G)$ with $A \cap B = \emptyset$, we denote $E(A,B) = \{ab \in E(G) | a \in A, b \in B\}$ and $e(A,B) = |E(A,B)|$.

Let $X \subseteq E(G)$ be an edge subset. The contraction G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. A subset D of the vertex set $V(G)$ is a dominating set if every edge has at least one end-vertex in D. Let $e_1, e_2 \in E(G)$. We use "(e_1, e_2)-trail" to denote a trail having the end-edges e_1 and e_2. An (e_1, e_2)-trail is called dominating if each edge of G is incident with at least one internal vertex of the trail. An (e_1, e_2)-trail is called spanning if it contains all the vertices of G. A graph is dominating trailable if for each pair of e_1 and e_2 of edges of G there exists a dominating trail with end-edges e_1 and e_2. Similarly, one can define the spanning trailable graphs.

Theorem 1.1 *(Nash-Williams [7]) Let G be a graph. If* $|E(G)| \geq k(|V(G)|-1)$, *then G has a nontrivial subgraph H such that* $\tau(H) \geq k$.

Theorem 1.2 *Let G be a graph, and let H be a subgraph of G.*
(i) (Catlin and Lai, Theorem 4 of [2]) Suppose that $\tau(G) \geq 2$. *For any* $e_1, e_2 \in E(G)$, *G has a spanning* (e_1, e_2)*-trail if and only if* $\{e_1, e_2\}$ *is not an essential edge cut of G.*
(ii) (Liu et al., Lemma 2.1 of [5]) If $\tau(H) \geq 2$ *and* $\tau(G/H) \geq 2$, *then* $\tau(G) \geq 2$.

The line graph of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have at least one vertex in common. From the definition of a line graph, a subset $X \subseteq V(L(G))$ is a vertex cut of $L(G)$ if and only if X is an essential edge cut of G. In 1986, Thomassen proposed the following conjecture.

Conjecture 1.3 *(Thomassen [10]) Every 4-connected line graph is hamiltonian.*

A subgraph of G that isomorphic to a $K_{1,2}$ or a 2-cycle is called a P_2-subgraph of G, where a 2-cycle is a graph consisting of two edges sharing two end-vertices. An edge cut X of G is a P_2-edge cut of G if at least two components of $G - X$ contain P_2-subgraphs. By the definition of a line graph, if $L(G)$ is not a complete graph, then $L(G)$ is essentially k-connected if and only if G does not have a P_2-edge cut with size less than

k. In 2006, Lai et al. considered the hamiltonicity of 3-connected line graphs and showed that the high essential connectivity of a 3-connected line graph can guarantee the existence of a hamiltonian cycle as follows.

Theorem 1.4 *(Lai, Shao, Wu and Zhou [3]) Every 3-connected, essentially 11-connected line graph is hamiltonian.*

Conjecture 1.5 *(Lai, Shao, Wu and Zhou [3]) Every 3-connected, essentially 4-connected line graph is hamiltonian.*

Yang et al in [12] found an infinite family of 3-connected, essentially 4-connected non-hamiltonian. Thus the question becomes to find the smallest integer k such that every 3-connected, essentially k-connected line graph is hamiltonian, and we know now that $k \geq 5$. Recently, Li and Yang in [4] used spanning trees packing theorem of Nash-Williams [6] and Tutte [11] to improve Theorem 1.4 by directly counting the number of edges between partitioned vertex subsets.

Theorem 1.6 *(Li and Yang [4]) Every 3-connected, essentially 10-connected line graph is hamiltonian connected.*

In this paper, we utilize the discharge method to prove the Theorem 1.7 below, which will be applied to yield an alternative proof of Theorem 1.6.

Theorem 1.7 *Let G be a connected graph with* $\delta(G) \geq 3$. *If for any* P_2*-subgraph H in G, we have* $e(V(H), V(G) - V(H)) \geq 10$, *then G has a nontrivial subgraph T with* $\tau(T) \geq 2$.

Theorem 1.7 will be proved in Section 2. An alternative proof of Theorem 1.6 as an application of Theorem 1.7 will be presented in the last section.

2 Proof of Theorem 1.7

Throughout this section, we assume that G is a graph satisfying the hypothesis of Theorem 1.6 . We define the initial charge at v as $ch_1(v) = d_G(v)$. The recharging rules are defined as follows.

(R1) Let $v \in D_i(G) (i = 5, 6, 7)$. If $|N_G(v) \cap D_3(G)| \leq 1$, we define the charge of v as

$$ch_2(v) := \begin{cases} ch_1(v) - 1, & \text{if } |N_G(v) \cap D_3(G)| = 1 \\ ch_1(v), & \text{if } |N_G(v) \cap D_3(G)| = 0 \end{cases}.$$

If $w \in D_3(G) \cap N_G(v)$, we define the discharge of w as

$$ch_2(w) := ch_1(w) + 1.$$

(R2) Let $v \in D_i(G) (i \geq 8)$, we define the charge of v as

$$ch_2(v) := 4.$$

For every $w \in N_G(v)$, we define the discharge of w as

$$ch_2(w) := ch_1(w) + \frac{d_G(v) - 4}{d_G(v)}.$$

As $d_G(v) \geq 8$, we have $\frac{d_G(v)-4}{d_G(v)} \geq \frac{1}{2}$.

Proof of Theorem 1.7. For $v \in D_i(G), i \geq 4$, we have $ch_2(v) \geq 4$ by (R1) and (R2). So we only need to consider $v \in D_3(G)$.

Case 1. $N_G(v) \cap D_3(G) = \emptyset$.

If $N_G(v) \cap (D_5(G) \cup D_6(G) \cup D_7(G)) \neq \emptyset$, by (R1), we have $ch_2(v) = 4$. Otherwise, $N_G(v) \cap (D_5(G) \cup D_6(G) \cup D_7(G)) = \emptyset$. Since $e(V(H), V(G) - V(H)) \geq 10$, $|N_G(v) \cap D_4(G)| \leq 1$, Thus $|N_G(v) \cap D_{\geq 8}(G)| \geq 2$. By (R2), we have

$$ch_2(v) \geq 3 + \frac{1}{2} + \frac{1}{2} = 4.$$

Case 2. $N_G(v) \cap D_3(G) \neq \emptyset$.

Since $e(V(H), V(G) - V(H)) \geq 10$, $|N_G(v) \cap D_3(G)| = 1$. Let $N_G(v) \cap D_3(G) = \{u\}$. Notice that $e(V(H), V(G) - V(H)) \geq 10$. For any $w \in (N_G(v) \cup N_G(u) - \{u, v\})$, $d_G(w) \geq 8$. By (R2), we have

$$ch_2(v) \geq 3 + \frac{1}{2} + \frac{1}{2} = 4, \text{ and } ch_2(u) \geq 3 + \frac{1}{2} + \frac{1}{2} = 4.$$

Therefore,

$$2|E(G)| = \sum_{v \in V(G)} d_G(v) = \sum_{v \in V(G)} ch_2(v) \geq 4|V(G)|,$$

and so $|E(G)| \geq 2|V(G)|$. By Theorem 1.1, G has a nontrivial subgraph T such that $\tau(T) \geq 2$. ∎

3 An Application of Hamiltonian-connected line graphs

Let G be a connected, essentially 3-edge-connected graph such that $L(G)$ is not a complete graph. The core of the graph G, denoted by G_0, is obtained from G by deleting all the vertices of degree 1 and contracting exactly one edge xy or yz for each path xyz in G with $d_G(y) = 2$.

Lemma 3.1 *(Shao [9]) Let G be a connected, essentially 3-edge-connected graph. Then the core G_0 of G satisfies the following.*
(i) G_0 is uniquely defined and the edge connectivity of G_0 is at least 3.
(ii) If G_0 is spanning trailable, then $L(G)$ is hamiltonian connected.

Proof of Theorem 1.6. Let $L(G)$ be a 3-connected, essentially 10-connected line graph. To prove $L(G)$ is hamiltonian connected, by Theorem 1.2(i) and Lemma 3.1, it suffices to prove that $\tau(G_0) \geq 2$, where G_0 is the core of G. By contradiction, assume that $\tau(G_0) < 2$. We choose $L(G)$ such that $|V(G_0)|$ is minimized, this is subject to $L(G)$ being 3-connected and essentially 10-connected and $\tau(G_0) < 2$. By Theorem 1.2(ii),

$$G_0 \text{ does not have a nontrivial subgraph } T \text{ such that } \tau(T) \geq 2. \quad (1)$$

Thus, if H is a P_2-subgraph in G_0, then $H = K_{1,2}$.

Claim 1. Let H be a P_2-subgraph in G_0. Then $e(V(H), V(G) - V(H)) \geq 10$.

Assume that $e(V(H), V(G) - V(H)) \leq 9$. Let $H = K_{1,2}$ with $E(H) = \{xy, yz\}$. Then $X = E(V(L), V(G) - V(L))$ is an edge cut in G_0. Let $H, H_1, H_2, \cdots, H_k$ be components of $G_0 - X$. Then each $H_i (i = 1, \cdots, k)$ does not contain a P_2-subgraph, which follows from the connectivity assumption. Thus each H_i is either a single vertex or a single edge.

Assume that $E(H_1) = \{uw\}$. As $d_{G_0}(u) \geq 3$ and $d_{G_0}(w) \geq 3$, we have $|N_{G_0}(u) \cap \{x, y, z\}| \geq 2$ and $|N_{G_0}(w) \cap \{x, y, z\}| \geq 2$. By Theorem 2.1 and G_0 has no subgraph T such that $\tau(T) \geq 2$, $d_{G_0}(u) = 3$ and $d_{G_0}(w) = 3$, and the subgraph induced by $\{x, y, z\} \cup \{u, w\}$ is one of graphs in Figure 1. If $k = 1$, since each of the graphs in Figure 1 contains a vertex of degree 2, we would have a contradiction with the fact that $\delta(G_0) \geq 3$. So $k \geq 2$. Notice that $H, H_1, H_2, \cdots, H_k$ are components of $G_0 - X$. We have either $V(H_2) \cap (N_{G_0}(y) \cup N_{G_0}(z)) \neq \emptyset$ or $V(H_2) \cap (N_{G_0}(y) \cup N_{G_0}(x)) \neq \emptyset$. Without loss of generality, we assume that $V(H_2) \cap (N_{G_0}(y) \cup N_{G_0}(z)) \neq \emptyset$.

Consider the new edge cut $X' = E(\{x,u,w\}, V(G_0) - \{x,u,w\})$. Then X' is a P_2-edge-cut, and so $|X'| \geq 10$. As $e(\{x,u,w\},\{y,z\}) \in \{3,4\}$, we have $|N_{G_0}(x) \cap (V(H_2) \cup \cdots \cup V(H_k))| \geq 6$. Therefore, $|X| \geq 6+4 = 10$, a contradiction. So each H_i is a single vertex.

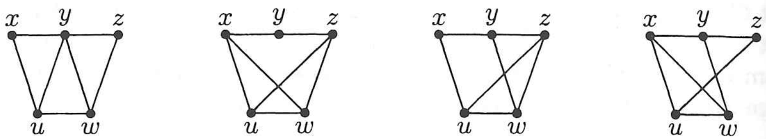

Figure 1.

Let $V(H_i) = \{a_i\}$. Since G_0 is 3-edge-connected, we have $a_i x, a_i y, a_i z \in E(G)$ and $k \geq 2$. Thus the subgraph induced by $V(H) \cup \{a_1, a_2\}$ in G_0 contains 2 edge-disjoint spanning trees, contrary to (1). Therefore, Claim 1 holds.

By Claim 1 and Theorem 1.7, G_0 has a nontrivial subgraph T such that $\tau(T) \geq 2$, contrary to (1). ∎

Using the cl^M-closure introduced by Ryjáček and Vrána on claw-free graphs (Theorem 9, [8]), we have the following.

Corollary 3.2 *Every 3-connected, essentially 10-connected claw-free graph is hamiltonian connected.*

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