Linear list r-hued coloring of K_4-minor free graphs

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Abstract

For an integer $r > 0$, and a k-list assignment L to vertices of a graph G, a linear r-hued L-coloring of a graph G is an coloring c of the vertices of G such that for every vertex v of degree $d(v)$, $c(v) \in L(v)$, the number of colors used by the neighbors of v is at least $\min\{d(v), r\}$ different colors, and such that for any two distinct colors i and j, every component of $G[c^{-1}(\{i, j\})]$ must be a union of vertex-disjoint paths. The linear list r-hued chromatic number of a graph G, $\chi^{\ell}_{L,r}(G)$, is the smallest integer k such that for any $v \in V(G)$ and every list assignment L with $|L(v)| = k$, G has a linear r-hued L-coloring. In this paper, we prove that if G is a K_4-minor

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free graph, then $\chi^{\ell}_{L,r}(G) \leq \max\{r, \lceil\Delta/2\rceil\} + \lceil\Delta/2\rceil + 2$, and every planar graph G with maximum degree Δ has $\chi^{\ell}_{L,2}(G) \leq \Delta + 7$.

Key words. Linear list r-hued coloring; K_4-minor free graph; Planar graph

1 Introduction

Throughout this paper, unless otherwise stated, $k, r > 0$ denote two integers. Graphs in this paper are simple and finite. Undefined terminologies and notations are referred to [3]. Thus for a graph G, $\Delta(G)$, $\delta(G)$, $\chi(G)$ and $\chi_L(G)$ denote the maximum degree, the minimum degree, the chromatic number and the list chromatic number of G, respectively. If $X \subseteq V(G)$, then $G[X]$ is the subgraph induced by X. For $v \in V(G)$, let $N_G(v)$ denote the set of vertices adjacent to v in G, and $d_G(v) = |N_G(v)|$. When G is understood from the context, we often use $N(v)$ and $d(v)$ for $N_G(v)$ and $d_G(v)$, respectively. For each integer $i \geq 0$, let $D_i(G)$ denote the vertices set with $d_G(v) = i$. Vertices in D_i are often called i-vertices of G.

Let G be a graph. For a mapping $c : V(G) \mapsto \{1, 2, \ldots, k\}$ and a subset $V' \subseteq V(G)$, define $c(V') = \{c(v)|v \in V'\}$. A proper k-coloring of G is a mapping $c : V(G) \mapsto \{1, 2, \ldots, k\}$ such that if $uv \in E(G)$, then $c(u) \neq c(v)$. A (k, r)-coloring is a proper k-coloring c of G such that $|c(N(v))| \geq \min\{d(v), r\}$ for any $v \in V(G)$. The *r-hued chromatic number* of G, denoted by $\chi_r(G)$, is the smallest integer k such that G has a (k, r)-coloring. When $r = 2$, $\chi_2(G)$ is also called the dynamic chromatic number of G. The study of r-hued colorings was initiated in [15]. Later, it was extended to 2-hued list colorings in [2]. It has now drawn lots of attentions, as seen in [1], [2], [5], [7], [10], [12], [17], [20], [22], [23], among others.

The concept of linear coloring was introduced by Yuster [27]. A linear k-coloring of a graph G is a proper coloring of G such that the subgraph induced by the vertices of any two color classes is a union of vertex-disjoint paths. The *linear chromatic number* $\chi^{\ell}(G)$ of G is the smallest integer k such that G has a linear k-coloring. By definition, if c is a linear coloring of G, then for each vertex v, every color occurs in $c(N_G(v))$ at most twice, and so we always have $\chi^{\ell}(G) \geq \lceil \Delta/2 \rceil + 1$. There is also a rich studies on linear colorings, as seen in [6, 8, 11, 16, 18, 24, 25, 26, 27], among other.

Esperet et al in [11] extended the notion to linear list colorings. Let G be a graph and let L be an assignment which associates with each vertex v of G a list $L(v)$ of available colors; such an L is a k-list of G if for any $v \in V(G)$, $|L(v)| \geq k$. An L-coloring is a proper coloring c such that $c(v) \in L(v)$, for every $v \in V(G)$. When such a c exists, we say that G is L-colorable. The *linear list chromatic number* of G, denoted by $\chi^{\ell}_L(G)$, is the minimum number k such that G admits a linear L-coloring for every k-list L.

Motivated by the researches mentioned above, Kong et al [13] investigated the linear list r-hued coloring of graphs. A (k, r)-coloring c of a graph G is linear if c is also a linear coloring. The *linear r-hued chromatic number* of G, denoted by $\chi^{\ell}_r(G)$, is the minimum k such that G admits a linear (k, r)-coloring. By definition, we have $\chi^{\ell}_r(G) \geq \max\{\chi^{\ell}(G), \chi_r(G)\}$. For a given assignment L of G, a *linear (L, r)-coloring* c of G is an L-coloring as well as a linear r-hued coloring. The *linear list r-hued chromatic number* of G, denoted by $\chi^{\ell}_{L,r}(G)$, is the minimum k such that for every k-list L, G always admits a linear (L, r)-coloring.

It follows from the definition that $\chi^{\ell}_{L,r}(G) \geq \max\{\lceil \Delta/2 \rceil, r\}+$

1 and $\chi_L^\ell(G) = \chi_{L,1}^\ell(G)$. For any integers s and $i > j > 0$, any linear (L, i)-coloring of G is also a linear (L, j)-coloring of G, and so $\chi_L^\ell(G) = \chi_{L,1}^\ell(G) \le \chi_{L,2}^\ell(G) \le \cdots \le \chi_{L,\Delta}^\ell(G) = \chi_{L,\Delta+1}^\ell(G) = \cdots = \chi_{L,\Delta+s}^\ell(G)$. Define

$$K(r) = \begin{cases} r+3, & \text{if } 2 \le r \le 3; \\ \lfloor 3r/2 \rfloor, & \text{if } r \ge 4. \end{cases}$$

A graph G is K_4-minor free if G does not have a subgraph contractible to K_4. The following is proved in [22].

Theorem 1.1. *(H. Song et al, Theorem 1.2 of [22]) Let G be a K_4-minor free graph with $\Delta = \Delta(G)$, and $r \ge 2$ be an integer. Then*
(i) $\chi_r(G) \le K(r)$.
(ii) $\chi_{L,r}(G) \le K(r) + 1$.

The case when $r = \Delta(G)$ of Theorem 1.1 was proved by Lih, Wang and Zhu [19]. The main purpose of this paper is to extend Theorem 1.1 to linear coloring. Define $f(\Delta, r) = \max\{r, \lceil \Delta/2 \rceil\} + \lceil \Delta/2 \rceil + 2$. The main results of this paper are the following.

Theorem 1.2. *Let G be a K_4-minor free graph with $\Delta = \Delta(G)$, and $r \ge 2$ be an integer. Then $\chi_{L,r}^\ell(G) \le f(\Delta, r)$.*

Theorem 1.3. *If G is a planar graph with $\Delta = \Delta(G)$, then $\chi_{L,2}^\ell(G) \le \Delta + 7$.*

These results will be proved in the subsequent sections.

2 Proof of Theorem 1.2

Let G be a graph with the vertex set $V = V(G)$ with $V' \subset V$ being a vertex subset. A mapping $c : V' \longrightarrow \bigcup_{v \in V'} L(v)$ is a partial (L,r)-coloring if c is a linear (L,r)-coloring of $G[V']$. For each $v \in V(G) - V'$, define $c(v) = \emptyset$, and for each vertex $v \in V(G)$, define $c_G^2(v) = \{i \in c(N_G(v)) : \text{for distinct } v_1, v_2 \in N_G(v), c(v_1) = c(v_2) = i\}$. For every vertex $v \in V'$, define

$$c[v] = \begin{cases} \{c(v)\} \cup c_G^2(v), & \text{if } |c(N_G(v))| \geq r; \\ \{c(v)\} \cup c(N(v)), & \text{otherwise.} \end{cases} \tag{1}$$

Thus, given a partial (L,r)-coloring c, $c[v]$ consists of the set of colors that cannot be used for uncolored neighbors of v. For a vertex $v \in V(G)$ which has at least one uncolored neighbor by definition, $|c_G^2(v)| \leq \lfloor (\Delta - 1)/2 \rfloor = \lceil \Delta/2 \rceil - 1$, and so by (1),

$$|c[v]| \leq \max\{r, \lceil \Delta/2 \rceil\}. \tag{2}$$

Define $S_G(u) = \{x : \text{either } d_G(x) \geq 3 \text{ with } ux \in E(G) \text{ or for some } w \in D_2(G), uw, wx \in E(G)\}$. Let $s_G(u) = |S_G(u)|$.

It is well known [9] that every K_4-minor free graph contains a vertex of degree at most two. Lih et al. [19] proved the following lemma.

Lemma 2.1. *(K. Lih, W. Wang and X. Zhu [19]). Let G be a K_4-minor free graph. Then one of the following conditions holds:*
(i) $\delta(G) \leq 1$.
(ii) There exists two adjacent 2-vertices.
(iii) There exists a vertex u with $d(u) \geq 3$ such that $s_G(u) \leq 2$.

Proof of Theorem 1.2. We argue by contradiction to prove Theorem 1.2. Assume that

$$G \text{ is a counterexample to Theorem 1.2 with } |V(G)| \text{ minimized.} \tag{3}$$

Let $\Delta = \Delta(G)$, and $k = f(\Delta, r)$. By (3), there must be a k-list L such that

$$G \text{ does not have a linear } (L, r)\text{-coloring.} \tag{4}$$

In the arguments below, we will obtain a K_4-minor free graph H by making local modifications of G such that $|V(H)| < |V(G)|$. By (3), H has a linear (L, r)-coloring c. To obtain a contradiction, we shall extend and modify c to a linear (L, r)-coloring of G.

Claim 1. $\delta(G) = 2$.

By contradiction, assume that $x \in D_1(G)$ with $N_G(x) = \{u\}$. Define $H = G - x$. Then H is also K_4-minor free and $|V(H)| < |V(G)|$. By (3), H admits a linear (L, r)-coloring c. By (1), $|c[u]| \leq \max\{r, \lceil \Delta/2 \rceil\} < k \leq |L(x)|$. Therefore, c can be extended to a linear (L, r)-coloring of G by defining $c(x) \in L(x) - c[u]$, contrary to (4). □

Claim 2. *$D_2(G)$ is an independent set.*

By contradiction, assume that for some $x, y \in D_2(G)$, $xy \in E(G)$. Denote $N_G(x) = \{u, y\}$ and $N_G(y) = \{v, x\}$. Let $H = G - x + uy$ (if $u \neq v$) or $H = G - x$ if $(u = v)$. As H is K_4-minor free with $|V(H)| < |V(G)|$, by (3), H has a linear (L, r)-coloring c with $S(c) = V(G) - \{x\}$. By (1), $c[y] = \{c(y), c(v)\}$. It follows that $|c[u] \cup c[y]| \leq |c[u]| + |c[y]| \leq \max\{r, \lceil \Delta/2 \rceil\} + 2 < k \leq |L(x)|$. Thus, c can be extend to a linear (L, r)-coloring of G by defining $c(x) \in L(x) - (c[u] \cup c[y])$, contrary to (4). □

By Lemma 2.1 and Claims 1 and 2, G contains a vertex u with $d_G(u) \geq 3$ such that $1 \leq s_G(u) \leq 2$. In the rest of the proof, we always assume that u is such a vertex. For each $x \in S_G(u)$, define

$$M_G(u, x) = N_G(x) \cap N_G(u) \cap D_2(G) \text{ and } m_G(x) = |M_G(u, x)|. \tag{5}$$

Since $d_G(u) \geq 3$ such that $s_G(u) \leq 2$, there exists a $z \in S_G(u)$ with $m_G(z) \geq 1$. Throughout the rest of this section, we assume that x is a vertex in $S_G(u)$ with $m_G(x) \geq 1$. We have the following claim.

Claim 3. $s_G(u) = 2$.

Assume that $s_G(u) = 1$, and so for some vertex x, we have $S_G(u) = \{x\}$ with $m_G(x) \geq 1$. By the definition of $S_G(u)$, $N_G(u) \subseteq (N_G(x) \cap N_G(u) \cap D_2(G)) \cup \{x\}$. It follows that $|N_G(x) \cap N_G(u) \cap D_2(G)| + |\{x\}| \geq d_G(u) \geq 3$, and so $m_G(x) = |N_G(x) \cap N_G(u) \cap D_2(G)| \geq 2$. Pick $w \in M_G(u,x)$ and define $H = G - w$. By (3), H has a linear (L,r)-coloring c. Since $m_G(x) \geq 2$, we have $c(u) \neq c(x)$ when $r \geq 2$. By (2), $|c[x]| \leq \max\{r, \lceil \Delta/2 \rceil\}$. As $N_G(u) \subseteq \{x\} \cup N_G(x)$, it follows that $|c[u] \cup c[x]| = |c(u) \cup c[x]| \leq 1 + \max\{r, \lceil \Delta/2 \rceil\} < k \leq |L(w)|$, and so c can be extend to a linear (L,r)-coloring of G by defining $c(w) \in L(w) - (c[u] \cup c[x])$, contrary to (4). □

Claim 4. *Let $w \in M_G(u,x)$ and c be a linear (L,r)-coloring of $G - w$ with $c(u) \neq c(x)$. Then* $\max\{d_G(u), d_G(x)\} \leq r$.

By contradiction and assume that $\max\{d_G(u), d_G(x)\} = d_G(u) > r$. Since $w \in M_G(u,x)$, we have $d_{G-w}(u) \geq r$, then $c(N_{G-w}(u)) \geq r$. By (1), $|c[u]| \leq \lceil \Delta/2 \rceil$, and so $|c[u] \cup c[x]| \leq |c[u]| + |c[x]| \leq \lceil \Delta/2 \rceil + \max\{r, \lceil \Delta/2 \rceil\} < k \leq |L(w)|$. As $c(u) \neq c(x)$ and $|c[u] \cup c[x]| < |L(w)|$, the partial linear (L,r)-coloring c can be extended to a linear (L,r)-coloring of G by choosing $c(w) \in L(w) - (c[u] \cup c[x])$, contrary to (4). □

Recall that $k = f(\Delta, r)$. We have the following claim.

Claim 5. $\lfloor 3r/2 \rfloor + 1 < k$.

If $\lceil \Delta/2 \rceil \leq r \leq \Delta$, then $k = \max\{r, \lceil \Delta/2 \rceil\} + \lceil \Delta/2 \rceil + 2 = r + \lceil \Delta/2 \rceil + 2$. Thus, $\lfloor 3r/2 \rfloor + 1 = r + \lfloor r/2 \rfloor + 1 \leq r + \lceil \Delta/2 \rceil + 1 <$

k. If $1 \leq r \leq \lceil \Delta/2 \rceil$, then $k = \max\{r, \lceil \Delta/2 \rceil\} + \lceil \Delta/2 \rceil + 2 = 2\lceil \Delta/2 \rceil + 2$. It follows that $\lfloor 3r/2 \rfloor + 1 \leq 3r/2 + 1 \leq 3/2\lceil \Delta/2 \rceil + 1 < k$. $\square$

By Claim 3, $s_G(u) = 2$. Let $S_G(u) = \{x, y\}$. Then by the definition of $S_G(u)$, it follows that $N_G(u) \subseteq N_G(x) \cup N_G(y) \cup \{x, y\}$. Without loss of generality, we shall always assume that $m_G(x) \geq m_G(y)$. Since $s_G(u) \geq 3$, we have $m_G(x) \geq 1$. Pick $w \in M_G(u, x)$ and define $H = G - w$. By (3), H has a linear (L, r)-coloring c. We now proceed the proof of Theorem 1.2 by a case analysis.

Case 1. $xu \in E(G)$.

As $xu \in E(H)$, $c(u) \neq c(x)$. We have $\max\{d_G(u), d_G(x)\} \leq r$ by Claim 4. Since $x \in N_G(u)$, we have $|c[u] \cup c[x]| \leq d_G(u) + d_G(x) - m_G(x) - 1$. By $m_G(x) + m_G(y) \geq d_G(u) - 2$ and by $m_G(x) \geq m_G(y)$, we conclude that $m_G(x) \geq \lceil (d_G(u) - 2)/2 \rceil = \lceil d_G(u)/2 \rceil - 1$. Hence

$$\begin{aligned} |c[u] \cup c[x]| &\leq d_G(u) + d_G(x) - m_G(x) - 1 \\ &\leq d_G(u) + d_G(x) - \lceil d_G(u)/2 \rceil \\ &\leq \lfloor d_G(u)/2 \rfloor + d_G(x) \leq \lfloor 3r/2 \rfloor \\ &< k \leq |L(w)|. \end{aligned}$$

As $c(u) \neq c(x)$, c can be extended to a linear (L, r)-coloring of G by taking $c(w) \in L(w) - (c[u] \cup c[x])$, contrary to (4). This proves Case 1.

Case 2. Both $xu \notin E(G)$ and $yu \notin E(G)$.

Since $xu, yu \notin E(G)$ and $m_G(x) \geq m_G(y)$, we conclude that $m_G(x) \geq \lceil d_G(u)/2 \rceil \geq 2$. and so there exists a $w' \in N_H(x) \cap N_H(u) \cap D_2(H)$, This implies that $c(u) \neq c(x)$. By Claim 4, we have $\max\{d_G(u), d_G(x)\} \leq r$. Since x is not adjacent to u, we have $|c[u] \cup c[x]| \leq d_G(u) + d_G(x) - m_G(x) + 1$. Hence

$$\begin{aligned}
|c[u] \cup c[x]| &\leq d_G(u) + d_G(x) - m_G(x) + 1 \\
&\leq d_G(u) + d_G(x) - \lceil d_G(u)/2 \rceil + 1 \\
&\leq \lfloor d_G(u)/2 \rfloor + d_G(x) + 1 \\
&\leq \lfloor 3r/2 \rfloor + 1 < k \leq |L(w)|.
\end{aligned}$$

As $c(u) \neq c(x)$, c can be extended to a linear (L, r)-coloring of G by choosing $c(w) \in L(w) - (c[u] \cup c[x])$, contrary to (4). This proves Case 2.

Case 3. Both $xu \notin E(G)$ and $yu \in E(G)$.

If $m_G(x) = m_G(y)$, we may interchange x and y, and so Case 3 becomes Case 1. Hence we may assume that $m_G(x) > m_G(y)$.

Case 3.1. $d_G(u)$ is odd.

Since $d_G(u)$ is odd, $m_G(x) + m_G(y) = d_G(u) - 1$ is even, and so $m_G(x) \geq m_G(y) + 2 \geq 2$.

Case 3.1.1. $m_G(x) \geq m_G(y) + 4$.

Since $m_G(x) \geq m_G(y) + 4 \geq 4$, $M_H(u, x) \neq \emptyset$, and so $c(u) \neq c(x)$. By Claim 4, we have $\max\{d_G(u), d_G(x)\} \leq r$. Hence,

$$\begin{aligned}
|c[u] \cup c[x]| &\leq d_G(u) + d_G(x) - m_G(x) + 1 \\
&\leq d_G(u) + d_G(x) - (d_G(u) + 3)/2 + 1 \\
&= \lfloor d_G(u)/2 \rfloor + d_G(x) \\
&\leq \lfloor 3r/2 \rfloor < k \leq |L(w)|.
\end{aligned}$$

As $c(u) \neq c(x)$, c can be extended to a linear (L, r)-coloring of G by choosing $c(w) \in L(w) - (c[u] \cup c[x])$, contrary to (4).

Case 3.1.2. $m_G(x) = m_G(y) + 2$.

If $m_G(x) = m_G(y) + 2 \geq 2$, then $M_H(u, x) \neq \emptyset$, and so $c(u) \neq c(x)$. By Claim 4, we have $\max\{d_G(u), d_G(x)\} \leq r$. If

$d_G(u) < r$, then

$$\begin{aligned} |c[u] \cup c[x]| &\le d_G(u) + d_G(x) - m_G(x) + 1 \\ &\le d_G(u) + d_G(x) - (d_G(u)+1)/2 + 1 \\ &= (d_G(u)+1)/2 + d_G(x) \le \lfloor r/2 \rfloor + d_G(x) \\ &\le \lfloor 3r/2 \rfloor < k \le |L(w)|. \end{aligned}$$

Thus we assume that $d_G(u) = r$. If $xy \in E(G)$, then

$$\begin{aligned} |c[u] \cup c[x]| &\le d_G(u) + d_G(x) - 1 - (m_G(x) - 1) \\ &\le d_G(u) + d_G(x) - (d_G(u)+1)/2 \\ &= (d_G(u)-1)/2 + d_G(x) \le \lfloor r/2 \rfloor + d_G(x) \\ &\le \lfloor 3r/2 \rfloor < k \le |L(w)|. \end{aligned}$$

Next we assume that $d_G(u) = r$ and $xy \notin E(G)$. In this case,

$$\begin{aligned} |c[u] \cup c[x]| &\le d_G(u) + d_G(x) - (m_G(x) - 1) \\ &\le d_G(u) + d_G(x) - (d_G(u)+1)/2 + 1 \\ &= (d_G(u)-1)/2 + d_G(x) + 1 \\ &\le \lfloor r/2 \rfloor + d_G(x) + 1 \\ &\le \lfloor 3r/2 \rfloor + 1 < k \le |L(w)|. \end{aligned}$$

In any cases, as $c(u) \neq c(x)$, c can be extended to a linear (L, r)-coloring of G by choosing $c(w) \in L(w) - (c[u] \cup c[x])$, contrary to (4). This proves Case 3.1.

Case 3.2. $d_G(u)$ is even.

Since $d_G(u)$ is even, $m_G(x) + m_G(y) = d_G(u) - 1$ is odd and $m_G(x) \ge m_G(y) + 1$.

If $m_G(x) \ge m_G(y) + 3 \ge 3$, then $m_H(x) \ge 2$, and so $c(u) \neq c(x)$. By Claim 4, we have $\max\{d_G(u), d_G(x)\} \le r$, and so $|c[u] \cup c[x]| \le d_G(u) + d_G(x) - m_G(x) + 1 \le d_G(x) + d_G(u)/2 \le \lfloor 3r/2 \rfloor + 1 < k \le |L(w)|$. As $c(u) \neq c(x)$, c can be extended to a linear (L, r)-coloring of G by choosing $c(w) \in L(w) - (c[u] \cup c[x])$, contrary to (4).

Hence $m_G(x) \leq m_G(y) + 2$. Since $m_G(x) + m_G(y)$ is odd, we may assume that $m_G(x) = m_G(y) + 1$. Since $d_G(u) \geq 4$, $m_G(y) = d_G(u)/2 - 1 \geq 1$. Choose $w' \in M_G(u, y)$ and let $H'' = G - w'$. By (3), H'' has an (L, r)-coloring c. As $uy \in E(G)$, $c(u) \neq c(y)$. By Claim 4, we have $\max\{d_G(u), d_G(y)\} \leq r$. Hence

$$\begin{aligned} |c[u] \cup c[y]| &\leq d_G(u) + d_G(y) - m_G(y) - 1 \\ &\leq d_G(u) + d_G(y) - d_G(u)/2 \\ &= d_G(u)/2 + d_G(y) \\ &\leq \lfloor 3r/2 \rfloor < k \leq |L(w)|. \end{aligned}$$

It follows from $c(u) \neq c(x)$ that c can be extended to a linear (L, r)-coloring of G by defining $c(w) \in L(w) - (c[u] \cup c[x])$, contrary to (3). This completes the proof for Case 3.2.

Since every case leads to a contradiction, this establishes the theorem and completes the proof. □

3 Proof of Theorem 1.3

The best know bound for linear chromatic number of planar graphs so far was obtained by Cai, Xie and Yang (see [4]) as shown in Theorem 3.1 below. In the proof of Theorem 3.1, the authors always count the number of available colors for uncolored vertex and never apply the technique of exchange colors. Hence the result can be extended to list version. The main purpose of this section is to extend Theorem 3.1 to linear $(L, 2)$-hued chromatic.

Theorem 3.1. *(Cai, Xie and Yang, [4]) If G is a planar graph with $\Delta = \Delta(G)$, then $\chi^{\ell}_L(G) \leq \Delta + 7$.*

The following lemma will be needed in our arguments.

Lemma 3.2. *([13]) If a graph G with minimum degree* $\delta \geq 2r-1$*, then* $\chi^{\ell}_{L,r}(G) = \chi^{\ell}_{L}(G)$.

Proof of Theorem 1.3. Let G be a planar graph, and let $k = \Delta + 7$. Suppose that L is a k-list of G. We prove the theorem by induction on $|V(G)|$, and so we may assume that G is connected. If $|V(G)| \leq \Delta+7$, then the theorem holds trivially. Hence assume that G is a planar graph with $|V(G)| \geq \Delta + 8$. As G is a planar graph, $\delta(G) \leq 5$. If $3 \leq \delta(G) \leq 5$, then $\chi^{\ell}_{L,2}(G) = \chi^{\ell}_{L}(G) \leq \Delta + 7$ by Lemma 3.2. Next we assume that $1 \leq \delta(G) \leq 2$. In the arguments below, we will first obtain a partial $(L, 2)$-coloring c of G. Then extend c to an $(L, 2)$-coloring of G to complete the inductive proof.

Case 1. $\delta(G) = 1$.

Let $v \in D_1(G)$ with $N_G(v) = \{u\}$. Since $|V(G)| \geq \Delta+8$ and since G is connected, we have $d_G(u) \geq 2$. By induction, $G - v$ has a linear $(L, 2)$-coloring c. By (1), $|c[u]| \leq \max\{2, \lceil \frac{\Delta}{2} \rceil\} < k$, and so c can be extended to a linear $(L, 2)$-coloring of G by assigning $c(v) \in L(v) - c[u]$.

Case 2. $\delta(G) = 2$.

Let $v \in D_2(G)$ with $N_G(v) = \{x, y\}$. Let $H = (G - v) + xy$ (if $xy \notin E(G)$) $H = G - v$ (if $xy \in E(G)$). By induction, H has a linear $(L, 2)$-coloring c. Thus c is a partial $(k, 2)$-coloring of G. By (1), we have $|c[x] \cup c[y]| \leq |c[x]| + |c[y]| \leq \max\{2, \lceil \frac{\Delta}{2} \rceil\} + \max\{2, \lceil \frac{\Delta}{2} \rceil\} \leq \max\{4, \Delta + 1\} < k$, and so c can be extended to a linear $(L, 2)$-coloring of G by defining $c(v) \in L(v) - (c[x] \cup c[y])$. This completes the proof of the theorem. □

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