Disjoint Spanning Arborescences in k-Arc-Strong Digraphs

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Abstract

For a graph G, let $\kappa'(G)$ and $\tau(G)$ denote edge-connectivity and the number of edge-disjoint spanning trees of G, respectively. Catlin et al in [Discrete Math., 309 (2009), 1033-1040] proved a characterization of $\kappa'(G)$ in terms of $\tau(G)$. In this paper, we prove a digraph version of this characterization by showing that a digraph D is k-arc-strong if and only if for any vertex v in D, D has k-arc-disjoint spanning arborescences rooted at v. We also prove a characterization of uniformly dense digraphs analogous to the characterization of uniformly

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dense undirected graphs in [Discrete Applied Math., 40 (1992) 285 - 302].

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1 Motivations of Research

We consider finite graphs and digraphs without loops but permitting multiple edges (arcs). Undefined terms and notations will follow [2] for graphs and and digraphs, and [1] specifically for digraphs. In particular, G often denotes an (undirected) graph with vertex set $V(G)$ and edge set $E(G)$, whereas D often denotes a digraph with vertex set $V(D)$ and arc set $A(D)$. Following [2], a digraph without loops and parallel arcs is called a strict digraph. For a graph G, $\kappa(G)$, $\kappa'(G)$ and $\tau(G)$ denote the connectivity, edge-connectivity and the maximum number of edge-disjoint spanning trees of G, respectively. For a digraph D, $\lambda(D)$ denotes the arc-strong connectivity of D. For any disjoint subsets $X, Y \subseteq V(D)$, define $(X, Y)_D$ to be the set of all arcs in D with tail in X and head in Y, and let

$$\partial_D^+(X) = (X, V(D) - X)_D \text{ and } \partial_D^-(X) = (V(D) - X, X)_D,$$

and let $d_D^+(X) = |\partial_D^+(X)|$ and $d_D^-(X) = |\partial_D^-(X)|$. When $X = \{v\}$, we write $d_D^+(v) = |\partial_D^+(\{v\})|$ (called the **out degree** of v in D) and $d_D^-(v) = |\partial_D^-(\{v\})|$ (called the **in degree** of v in D). A digraph D is **k-arc-strong** if for any nonempty proper subset $X \subset V(D)$, $d_D^+(X) \geq k$. Thus $\lambda(D)$ is the largest integer k such that D is k-arc-strong, and let $\mathcal{D}(k)$ denote the family of all k-arc-strong digraphs.

By the spanning tree packing theorem of Nash-Williams [25] and Tutte [28], it has been known (see [7] and [22], among others) that for any graph G, the parameters $\kappa'(G)$ and $\tau(G)$ are closely related. The following has been proved.

Theorem 1.1. *(Catlin et al, Theorem 1.1 of [7]) Let G be a connected graph and let $k \geq 1$ be an integer. Each of the following holds.*

(i) $\kappa'(G) \geq 2k$ *if and only if* $\forall X' \subseteq E(G)$ *with* $|X'| \leq k, \tau(G-X') \geq k$.

(ii) $\kappa'(G) \geq 2k+1$ *if and only if* $\forall X' \subseteq E(G)$ *with* $|X'| \leq k+1, \tau(G-X') \geq k$.

An **arborescence** is an oriented tree T such that for some $r \in V(T)$, we have $d_T^-(r) = 0$, and for any $v \in V(T) - r$, we have $d_T^-(v) = 1$. The vertex r is the root of T and T is an r-**arborescence**. Let $\tau(D)$ be the maximum number of arc-disjoint spanning arborescences in D. It is natural to investigate whether there is a similar relationship in digraphs. This is one of the motivations of this paper. One such attempt was made in Theorem 1.4 of [7]. It is unfortunately that, due to the misunderstanding of a result of Frank (mistakenly quoted as a result of Edmonds in Theorem 3.1 of [7]), the proof of Theorem 1.4 of [7] is false. Therefore, the problem of investigating the relationship between $\lambda(D)$ and $\tau(D)$ remains unanswered.

In [5], the strength $\eta(G)$ and the fractional arboricity $\gamma(G)$ of a graph G are defined and the related uniformly dense graphs are characterized. These studies have been applied in [9] to give short proofs for some results on higher-order edge toughness of a graph in [8]; and to obtained characterization of of minimal graphs whose edge-connectivity equals the spanning tree packing number in [17]. The characterizations of uniformly dense graphs have also been applied to many other studies, such as cyclic base orderings ([19]) and the problem of packing hypertrees ([18]). It is natural to consider the study on uniformly dense digraphs analogous to those characterizations in [5]. This is another motivation of the current paper.

In Section 2, we prove that $\lambda(D) \geq k$ if and only if for any vertex $v \in V(D)$, D has k-arc-disjoint spanning arborescences rooted at v, thereby obtaining a digraph result analogous to Theorem 1.1. As an application, we determine the extremal value $\min\{|A(D)|: D \in \mathcal{A}(k) \text{ and } |V(D)| = n\}$. In Section 3, we prove a characterization of uniformly dense digraphs analogous to the characterization of uniformly dense undirected graphs in [5].

2 Relationship Between k-arc-strong Connectivity and Disjoint Spanning Arborescences

We start with some digraph families with certain properties arc-disjoint spanning arborescences. This section is mainly devoted to the study of the relationship between $\mathcal{D}(k)$ and the following seemingly different families of digraphs.

Definition 2.1. *Let T be an oriented tree with a fixed vertex $r \in V(T)$. T is an* **out-arborescence rooted at** r *(or an r^+-arborescence) if $d_T^-(r) = 0$ and for any $v \in V(T) - r$, $d_T^-(v) = 1$; T is an* **in-arborescence rooted at** r *(or an r^--arborescence) if $d_T^+(r) = 0$ and for any $v \in V(T) - r$, $d_T^+(v) = 1$. In either case, r is called the root of T.*

Thus an arborescence is an out-arborescence, and an r-arborescence is an r^+-arborescence. For any function $f : V(D) \to \mathbb{R}^+$ and any subset $S \subseteq V(D)$, define $f(S) = \sum_{v \in S} f(v)$.

Definition 2.2. *Let $k \geq s \geq 1$ be integers.*
(i) Let $\mathcal{A}_1(k,s)$ be the family of digraphs such that $D \in \mathcal{A}_1(k,s)$ if and only if for any $S \subseteq V(D)$ with $|S| \leq s$, D has k arc-disjoint spanning out-arborescences whose roots are in S.
(ii) Let $\mathcal{A}_2(k,s)$ be the family of digraphs such that $D \in \mathcal{A}_2(k,s)$ if and only if for any $S \subseteq V(D)$ with $|S| \leq s$, D has k arc-disjoint spanning in-arborescences whose roots are in S.
(iii) Let $\mathcal{A}_3(k,s)$ be the family of digraphs such that $D \in \mathcal{A}_3(k,s)$ if and only if for any $u, l : V(D) \to \mathbb{R}^+$ with $u \geq l$, and for any $S \subseteq V(D)$ with $|S| \leq s$ and with $u(S) \geq k \geq l(S)$, D has k arc-disjoint spanning out-arborescences whose roots are in S in such a way that every $r \in S$ is the root of at least $l(r)$ and at most $u(r)$ of such spanning arborescences.
(iv) Let $\mathcal{A}_4(k,s)$ be the family of digraphs such that $D \in \mathcal{A}_4(k,s)$ if and only if for any $u, l : V(D) \to \mathbb{R}^+$ with $u \geq l$, and for any

$S \subseteq V(D)$ with $|S| \leq s$ and with $u(S) \geq k \geq l(S)$, D has k arc-disjoint spanning in-arborescences whose roots are in S in such a way that every $r \in S$ is the root of at least $l(r)$ and at most $u(r)$ of such spanning arborescences.

For each $i \in \{1, 2, 3, 4\}$, define

$$f_i(n, k, s) = \min\{|A(D)| : D \in \mathcal{A}_i(k, s) \text{ is strict with } |V(D)| = n\}. \tag{1}$$

For a digraph D, let D^- denote the digraph obtained from D by reversing the orientation of each arc of D. Then for any $r \in V(D) = V(D^-)$, an r^+-arborescence of D is an r^--arborescence of D^-. Therefore, $f_1(n, k, s) = f_2(n, k, s)$ and $f_3(n, k, s) = f_4(n, k, s)$. As an application of our main result in this section, we shall determine the values of these functions. The following theorem of Edmonds is very useful. Edmonds [15] proved Part (i) of Theorem 2.3, and Part (ii) of Theorem 2.3 follows immediately from Part (i) by applying Part (i) to D^-.

Theorem 2.3. *(Edmonds, [15]) Let D be a digraph and $r \in V(D)$.*
(i) Then D has k-arc-disjoint spanning r^+-arborescences if and only if for any nonempty subset $S \subseteq V(D) - r$, $d^-_D(S) \geq k$.
(ii) Then D has k-arc-disjoint spanning r^--arborescences if and only if for any nonempty subset $S \subseteq V(D) - r$, $d^+_D(S) \geq k$.

Now we can state and prove our main result in this section.

Theorem 2.4. *Let $k \geq s > 0$ be integers. The following holds.*

$$\begin{aligned}\mathcal{D}(k) &= \mathcal{A}_1(k, 1) = \mathcal{A}_2(k, 1) = \mathcal{A}_1(k, s) = \cdots = \\ &= \mathcal{A}_2(k, s) = \mathcal{A}_3(k, s) = \mathcal{A}_4(k, s).\end{aligned}$$

Proof. We first observe that by Definition 2.2, we have $\mathcal{A}_1(k, s) \subseteq \mathcal{A}_1(k, 1)$ and $\mathcal{A}_2(k, s) \subseteq \mathcal{A}_2(k, 1)$. By choosing $u = l$, we also have $\mathcal{A}_3(k, s) \subseteq \mathcal{A}_1(k, s)$ and $\mathcal{A}_4(k, s) \subseteq \mathcal{A}_2(k, s)$. It suffices to prove that $\mathcal{A}_1(k, 1) \subseteq \mathcal{A}_3(k, s)$, $\mathcal{A}_2(k, 1) \subseteq \mathcal{A}_4(k, s)$, and $\mathcal{D}(k) = \mathcal{A}_1(k, 1) = \mathcal{A}_2(k, 1)$.

Proof of $\mathcal{D}(k) = \mathcal{A}_1(k,1) = \mathcal{A}_2(k,1)$. For any $D \in \mathcal{D}(k)$, and for any $\emptyset \neq S \subseteq V(D) - r$, since $D \in \mathcal{D}(k)$, both $d_D^-(X) \geq k$ and $d_D^+(X) \geq k$, and so by Theorem 2.3, both $D \in \mathcal{A}_1(k,1)$ and $D \in \mathcal{A}_2(k,1)$. Hence $\mathcal{D}(k) \subseteq \mathcal{A}_1(k,1) \cap \mathcal{A}_2(k,1)$.

Let $D \in \mathcal{A}_1(k,1)$. For any $\emptyset \neq X \subset V(D)$, choose a vertex $r \in X$. Since $D \in \mathcal{A}_1(k,1)$, D has k-spanning r^+-arborescences. Since every spanning r^+-arborescences must have at least on arc in $\partial_D^+(X)$, we have $d_D^+(X) \geq k$. This implies $\mathcal{A}_1(k,1) \subseteq \mathcal{D}(k)$. With a similar argument, we also have $\mathcal{A}_2(k,1) \subseteq \mathcal{D}(k)$. This justifies $\mathcal{D}(k) = \mathcal{A}_1(k,1) = \mathcal{A}_2(k,1)$.

Proof of $\mathcal{A}_1(k,1) \subseteq \mathcal{A}_3(k,s)$ and $\mathcal{A}_2(k,1) \subseteq \mathcal{A}_4(k,s)$. By symmetry, it suffices to prove $\mathcal{A}_1(k,1) \subseteq \mathcal{A}_3(k,s)$.

Let $D \in \mathcal{A}_1(k,1)$. Suppose that a subset $S \subseteq V(D)$ with $|S| \leq s$, and functions $u, l : V(D) \to \mathbb{R}^+$ with $u \geq l$ and with $u(S) \geq k \geq l(S)$ are given. Let $S = \{x_1, x_2, \cdots, x_s\}$ and $K_1 = k - l(S)$. Define $a_1 = \min\{K_1, u(x_1) - l(x_1)\}$. Inductively, assume that $a_1, a_2, \cdots, a_h$ and $K_1, K_2, \cdots, K_h$ have been defined. If $h < s$, then define $K_{h+1} = K_h - a_h$ and $a_{h+1} = \min\{K_{h+1}, u(x_{h+1}) - l(x_{h+1})\}$. We claim the following.

$$\begin{cases} l(x_i) \leq l(x_i) + a_i \leq u(x_i), \text{ for each } i \text{ with } 1 \leq i \leq s, \\ \displaystyle\sum_{i=1}^{s} [l(x_i) + a_i] = k. \end{cases} \tag{2}$$

Since $u \geq l$, by the definition of the a_i's, we conclude that $a_i \geq 0$ and so $l(x_i) \leq l(x_i) + a_i \leq l(x_i) + u(x_i) - l(x_i) = u(x_i)$. Let t be the smallest integer with $1 \leq t \leq s$ such that $a_t = K_t$. If $t < s$, then by the definition of $K_{t'}$ and $a_{t'}$, for any $s \geq t' > t$, we have $K_{t'} = 0$, and so $a_{t'} = 0$. Thus the second equality in (2) follows as shown below.

$$\begin{aligned}
\sum_{i=1}^{s}[l(x_i)+a_i] &= l(S)+\sum_{i=1}^{t}a_i = l(S)+\sum_{i=1}^{t-1}a_i + K_t \\
&= l(S)+\sum_{i=1}^{t-1}a_i + (K_{t-1}-a_{t-1}) \\
&= l(S)+\sum_{i=1}^{t-2}a_i + (K_{t-2}-a_{t-2}) = \cdots = \\
&= l(S)+K_1 = l(S)+(k-l(S)) = k.
\end{aligned}$$

We now construct a new graph D' from D by adding a new vertex $r \notin V(D)$, such that for each $x_i \in S$, there is a set A_i of exactly $l(x_i)+a_i$ parallel arcs from r to x_i. By this definition and by (2), we have $d^+_{D'}(r) = \sum_{i=1}^{s}[l(x_i)+a_i] = k$ and $d^-_{D'}(r) = 0$. Since $D \in \mathcal{A}_1(k,1)$ and since $\mathcal{D}(k) = \mathcal{A}_1(k,1)$, we have $D \in \mathcal{D}(k)$. For any $X \subseteq V(D)$, if $X \neq V(D)$, then

$$d^-_{D'}(X) = d^-_D(X) + |(\{r\}, X \cap S)_{D'}|,$$

and so $d^-_{D'}(X) \geq d^-_D(X)$. As $D \in \mathcal{D}(k)$, we have $d^-_{D'}(X) \geq k$. If $X = V(D)$, then $d^-_{D'}(V(D)) = d^+_{D'}(r) = k$. It follows by Theorem 2.3 that D' has k-arc-disjoint spanning r-arborescences $T_1, T_2, \cdots, T_k$. Since $d^+_{D'}(r) = k$, every arc in $\partial^+_{D'}(r)$ is in exactly one of these T_i's. Hence for each $x_i \in S$, there are exactly $l(x_i)+a_i$ of these T_i's, each of which contains one arc parallel to (r, x_i) in A_i as the only arc entering x_i in the arborescence. It follows that by removing r from these arborescences, we obtained exactly $l(x_i)+a_i$ arc-disjoint x_i-arborescences of D. Consequently, by (2), D has k arc-disjoint spanning out-arborescences whose roots are in S in such a way that every $x_i \in S$ is the root of at least $l(x_i)$ and at most $u(x_i)$ of such spanning arborescences. As S is arbitrarily, this implies that $D \in \mathcal{A}_3(k,s)$, and so $\mathcal{A}_1(k,1) \subseteq \mathcal{A}_3(k,s)$. With a similar argument, we also have $\mathcal{A}_2(k,1) \subseteq \mathcal{A}_2(k,s)$. This completes the proof of the theorem. □

By Theorem 2.4, we can obtain the following digraph versions of

Theorem 1.1. As shown in Theorem 2.4, we have $\mathcal{D}(k) = \mathcal{A}_1(k, 1) = \mathcal{A}_2(k, 1)$, which immediately justifies Corollary 2.5

Corollary 2.5. *Let D be a digraph and let k be a positive integer. The following are equivalent.*
(i) $\lambda(D) \geq k$.
(ii) For every vertex v, D has k-arc-disjoint spanning v^+-arborescences.
(iii) For every vertex v, D has k-arc-disjoint spanning v^--arborescences.

Corollary 2.6. *Let $n > k > 0$ be integers, and D be a strict digraph on n vertices. Then $D \in \mathcal{D}(k)$ if and only if for any vertex $v \in V(D)$, there exists an arc subset X with $|X| \leq k$ such that $D - X$ has k-arc-disjoint spanning v^+-arborescences.*

Proof. Suppose $D \in \mathcal{D}(k)$. Then by Theorem 2.4, $\mathcal{D}(k) = \mathcal{A}_1(k, 1)$, and so for any vertex $v \in V(D)$, D has k-arc-disjoint spanning v^+-arborescences $T_1, T_2, ..., T_k$. By Theorem 2.9, $|A(D)| \geq nk$, and so $|A(D) - \cup_{i=1}^{k} E(T_i)| \geq kn - k(n-1) = k$. It follows that there exists a subset $X \subseteq A(D) - \cup_{i=1}^{k} E(T_i)$ with $|X| = k$, and so $D - X$ has k-arc-disjoint spanning v^+-arborescences.
Conversely, for any non empty proper subset $S \subset V(D)$ with $S \neq \emptyset$, pick a vertex $v \in S$. By assumption, there exists an arc subset $X \subset A(D)$ such that $D - X$ has k-arc-disjoint spanning arborescences. It follows that there must be at least k arcs on these arborescences going from S to $V(D) - S$, and so $|\partial_D^+(S)| \geq k$. By definition, $D \in \mathcal{D}(k)$. This proves the corollary. □

To determine values of the functions defined in (1), we need former results of Walecki and of Tillson. In Theorems 2.7 and 2.8, $m \geq 1$ is an integer.

Theorem 2.7. *(Waleski, see e.g. [3], [24] for the construction.) The edges of the complete graph on $2m+1$ vertices can be decomposed into m edge-disjoint Hamilton cycles.*

Theorem 2.8. *(Tillson [27]) For $2m \geq 8$, the arcs of the complete digraph on $2m$ vertices can be decomposed into $2m-1$ arc-disjoint directed Hamilton cycles.*

Theorem 2.9. *For each $i \in \{1, 2, 3, 4\}$, and for any integers k, n, s with $n \geq k+1$, and $n > s > 0$,*

$$f_i(n, k, s) = nk. \tag{3}$$

Proof. Note that $\mathcal{A}(k) = \mathcal{A}_1(k, 1)$. Define

$$F(n, k) = \min\{|A(D)|:D \in \mathcal{D}(k) \text{ is strict with } |V(D)| = n\}. \tag{4}$$

By Theorem 2.4, for $1 \leq i \leq 4$, we have $F(n, k) = f_i(n, k, s)$. For any $D \in D(k)$ and for any $v \in V(D)$, we have $d_D^+(v) \geq k$, and so $|A(D)| = \sum\limits_{v \in V(D)} d_D^+(v) \geq nk$. Hence $F(n, k) \geq nk$.

To show that $F(n, k) \leq nk$, for any n and any $k \leq n-1$, we shall construct a strict digraph $D \in D(k)$ such that

$$\text{for any } v \in V(D), \text{ we have } d_D^+(v) = d_D^-(v) = k. \tag{5}$$

If such a digraph D can be constructed, then $F(n, k) \leq |A(D)| = nk$, and so (3) is justified.

For each integer $n \geq 3$, let K_n denote the complete (undirected) graph on n vertices and K_n^* denote the strict complete digraph on n vertices. For each $n \geq 3$ with $n \notin \{4, 6\}$ and for each $1 \leq k \leq n-1$, we shall construct a strict digraph $D(n, k)$ on n vertices such that $D(n, k)$ has k-arc-disjoint directed Hamilton cycles. To do that, we show that K_n^* has arc-disjoint directed Hamilton cycles $C_1, C_2, \cdots C_{n-1}$. Then we define $D(n, k) = K_n^*[E(C_1) \cup E(C_2) \cup \cdots \cup E(C_k)]$, for each k with $1 \leq k \leq n-1$. Note that any Hamilton cycle C of K_n can be oriented into two arc-disjoint directed Hamilton cycles in K_n^* with opposite direction to each other. When $n \in \{4, 6\}$, we follow the similar idea to proceed the proof except when $k = n-1$. In the rest of the proofs of this theorem, if H_1 and H_2 are sub-digraphs of a digraph D, then $H_1 \cup H_2$ denotes the sub-digraph induced by the arc set $E(H_1) \cup E(H_2)$. We will construct $D(n,k)$ satisfying (5) in each of the following cases.

Case 1: $n \in \{4,6\}$.

Assume that $n \in \{4,6\}$. Then by definition, $K_n^* \in \mathcal{D}(n-1)$ and K_n^* satisfies (5) with $k = n-1$, and so we have both $D(4,3)$ and $D(6,5)$.

For $n = 4$, let $C_1^4 = (0,3,1,2)$ and $C_2^4 = (0,2,1,3)$. Then $D(4,2) = C_1^4 \cup C_2^4$ and $D(4,1) = C_1^4$.

For $n = 6$, let $C_1^6 = (0,1,2,5,4,3)$, $C_2^6 = (0,5,1,3,2,4)$, $C_3^6 = (0,3,4,5,2,1)$ and $C_4^6 = (0,4,2,3,1,5)$. Then $D(6,4) = C_1^6 \cup C_2^6 \cup C_3^6 \cup C_4^6$, $D(6,3) = C_1^6 \cup C_2^6 \cup C_3^6$, $D(6,2) = C_1^6 \cup C_2^6$, and $D(6,1) = C_1^6$.

Case 2: $n = 2s+1$ and $s \geq 1$.

By Theorem 2.7, the edge set of K_n, the undirected complete graph on n vertices, can be decomposed into s edge-disjoint Hamilton cycles. Each of these Hamilton cycles can have two opposite orientations. These give rise to a decomposition of $E(K_n^*)$ into $n-1$ arc-disjoint directed Hamilton cycles $C_1^n, C_2^n, ..., C_{n-1}^n$. Define $D(n,k) = \bigcup_{i=1}^k C_i^n$, for $k = 1, 2, \cdots, n-1$. As each C_i^n is a directed Hamilton cycle, $D(n,k)$ satisfies (5).

Case 3: $n = 2s$ and $s \geq 4$.

By Theorem 2.8, $E(K_n^*)$ can be decomposed into $n-1$ directed Hamiltonian cycles $C_1^n, C_2^n, \cdots, C_{n-1}^n$. Define $D(n,k) = \bigcup_{i=1}^k C_i^n$, for $k = 1, 2, \cdots, n-1$. As each C_i^n is a directed Hamilton cycle, $D(n,k)$ satisfies (5).

As for each $n \geq 3$ and $k \leq n-1$, a digraph $D(n,k)$ satisfying (5) can be found, by (4), we have $F(n,k) \leq kn$. This proves (3). $\square$

3 Characterization of Uniformly Dense Digraphs

For a graph G, we follow [2] to use $c(G)$ to denote the number of components of G. Then the **strength** $\eta(G)$ and **fractional arboricity**

$\gamma(G)$ of G are defined as

$$\eta(G) = \min_{X \subseteq E(G)} \left\{ \frac{|X|}{c(G-X) - c(G)} \right\},$$

and

$$\gamma(G) = \max_{X \subseteq E(G)} \left\{ \frac{|X|}{V(G[X]) - c(G[X])} \right\},$$

where the minimum and maximum are taken over all subsets X such that the corresponding denominators are not zero. In [5], it has been indicated that the well-known spanning tree packing theorem of Nash-Williams [25] and Tutte [28] can be restated as the following.

Theorem 3.1. *A nontrivial graph G has k edge-disjoint spanning trees if and only if $\eta(G) \geq k$.*

In [5], an attempt to obtain a fractional version of Theorem **??** was made, and the following characterization is proved.

Theorem 3.2. *(Theorem 6 of [5]) Let G be a connected graph. The following are equivalent.*
(i) $\gamma(G)(|V(G)| - 1) = |E(G)|$.
(ii) $\eta(G)(|V(G)| - 1) = |E(G)|$.
(iii) $\eta(G) = \gamma(G)$.
(iv) For any integers $s \geq t > 0$ with $\gamma(G) = \frac{s}{t}$, G has s spanning trees $T_1, T_2, \cdots, T_s$ such that every edge $e \in E(G)$ is in exactly t members in the multiset $\{T_1, T_2, \cdots, T_s\}$.
(v) For any integers $s \geq t > 0$ with $\eta(G) = \frac{s}{t}$, G has s spanning trees $T_1, T_2, \cdots, T_s$ such that every edge $e \in E(G)$ is in exactly t members in the multiset $\{T_1, T_2, \cdots, T_s\}$.

Any graph G satisfying Theorem 3.2(i) is called **uniformly dense**. See [5] and [23] for more background on uniformly dense graphs. The purpose of this section is to obtain the digraph version of Theorem 3.2. For a digraph D with $n = |V(D)| \geq 2$, let $\mathcal{P} = \mathcal{P}(D) =$

$\{(X_1, X_2, \cdots, X_m) \mid$ where $X_1, X_2, \cdots X_p$ are disjoint nonempty subsets of $V(D)\}$. We define

$$\eta(D) = \min_{(X_1, X_2, \cdots X_p) \in \mathcal{P}(D), p>1} \left\{ \frac{\sum_{i=1}^{p} |\partial_D^-(X_i)|}{p-1} \right\}, \tag{6}$$

and

$$\gamma(D) = \max_{(X_1, X_2, \cdots X_p) \in \mathcal{P}(D), p<n} \left\{ \frac{|A(D)| - \sum_{i=1}^{p} |\partial_D^-(X_i)|}{n-p} \right\}. \tag{7}$$

Frank proved two theorems analogues to the spanning tree packing theorem of Nash-Williams ([25]) and Tutte ([28]) and the forest covering theorem of Nash-William ([26]). A **branching** of a digraph D is a sub digraph B of D such that every weakly connected component of B is an arborescence.

Theorem 3.3. *(Frank, [21]) Let D be a nontrivial digraph. The following are equivalent.*
(i) D is has k-arc-disjoint spanning arborescences.
(ii) $\eta(D) \geq k$.

Theorem 3.4. *(Frank, [21]) Let D be a nontrivial digraph. The following are equivalent.*
(i) D has k branchings such that every arc of D is in at least one of them.
(ii) $\gamma(D) \leq k$.

Following Theorem 3.2, it is natural to define a digraph as uniformly dense if $\eta(D) = \gamma(D)$. We will apply Theorems 3.3 and 3.4 to obtain the characterization of uniformly dene digraph of Theorem 3.2, which justifies the definition of uniformly dense digraphs. We need two lemmas.

Lemma 3.5. *Let D be a digraph, and $k, t > 0$ be integers.*
(i) D has k spanning arborescences such that every $e \in A(D)$ lies in at most t of them if and only if $\eta(D) \geq \frac{k}{t}$.
(ii) D has k branchings such that every $e \in A(D)$ lies in at least t of them if and only if $\gamma(D) \leq \frac{k}{t}$.

Proof. Let D_t be the digraph obtained from D by replacing each ar $e \in A(D)$ by t parallel arcs $\{e^1, e^2, \cdots, e^t\}$, each having the same head and tails as e.

Then D_t has k arc-disjoint spanning arborescences if and only if D has k spanning arborescences such that every $e \in A(D)$ lies in at most t of them. Moreover, for each $X \subseteq (D)$, we have

$$t|\partial_D^-(X)| = |\partial_{D_t}^-(X)|.$$

By Theorem 3.3, G has k branchings such that every $e \in A(D)$ lies in at least t of them if and only if $\eta(D_t) \geq k$. By (6), $\eta(D_t) \geq k$ if and only if for any $(X_1, X_2, ..., X_p) \in \mathcal{P}(D)$ with $p > 1$,

$$k \leq \eta(D_t) = \frac{\sum_{i=1}^p |\partial_{D_t}^-(X_i)|}{p-1} = \frac{\sum_{i=1}^p t|\partial_D^-(X_i)|}{p-1}.$$

Thus $\eta(D) \geq \frac{k}{t}$, and so (i) holds. The proof for (ii) is similar and is omitted. □

Lemma 3.6. *Let D be a digraph on $|V(D)| > 1$ vertices. Define $d(D) = \dfrac{|A(D)|}{|V(D)| - 1}$, Then*

$$\eta(D) \leq d(D) \leq \gamma(D).$$

Proof. Choose $p = |V(D)| > 1$ and $X_i = \{v_i\}$ in (6), to get the first inequality. Then choose $p = 1$ and $X_1 = V(D)$ in (7) to have the second. □

Theorem 3.7. *Let D be a weakly connected nontrivial digraph on n vertices. The followings are equivalent. (A digraph satisfying any one of the following is called a* **uniformly dense** *digraph).*

(i) $|A(D)| = \eta(D)(n-1)$.

(ii) $|A(D)| = \gamma(D)(n-1)$.

(iii) $\eta(D) = \gamma(D)$.

(iv) For any integers $s \geq t > 0$ such that $\gamma(D) = \frac{s}{t}$. there exists an integer $t > 0$ such that D has a family $\mathcal{F} = \{T_1, T_2, \cdots, T_s\}$ of

spanning arborescences such that every $e \in A(D)$ lies in exactly t members in $\mathcal{F}$.
(v) For any integers $s \geq t > 0$ such that $\eta(D) = \frac{s}{t}$. there exists an integer $t > 0$ such that D has a family $\mathcal{F} = \{T_1, T_2, \cdots, T_s\}$ of spanning arborescences such that every $e \in A(D)$ lies in exactly t members in $\mathcal{F}$.

Proof. By Lemma 3.6, (iii) implies (i) and (ii). It remains to show that each of (i) and (ii) implies (iv), and (iv) implies (iii).

(i) $\Longrightarrow$ (iv). (The proof for (ii) $\Longrightarrow$ (iv) is similar, and will be omitted.) For any integers $s \geq t > 0$ with $s = t\eta(D)$, by Lemma 3.5 (i). D has a family $\mathcal{F}$ of s spanning arborescences such that every arc in D lies in at most t members of $\mathcal{F}$. It follows by (i) that

$$t(\eta(D)(n-1) = s(n-1) \leq t|A(D)| \leq t\eta(D)(n-1).$$

This forces that every arc of D is in exactly t members of $\mathcal{F}$.

(iv) $\Longrightarrow$ (iii). Suppose that D has a family $\mathcal{F}$ of s spanning arborescences with the property that every $e \in A(D)$ lies in exactly t members in $\mathcal{F}$. Then by Lemma 3.5 and by Lemma 3.6,

$$\gamma(D) \leq \frac{s}{t} \leq \eta(D) \leq \gamma(D).$$

Hence we must have (iii). □

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