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Sensitivity of *r*-hued colouring of graphs

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ABSTRACT

A proper k-colouring c of a graph G is a (k, r)-colouring if for every vertex v with degree $d(v)$ there are at least $\min\{d(v), r\}$ different colours present in the neighbourhood of v. The *r-hued chromatic number* of G, $\chi_r(G)$, is the least integer k such that G has a (k, r)-colouring. We show that, for any $r \geq 2$, there exist infinitely many graphs G with the property that G contains a subgraph H satisfying $\chi_r(H) > \chi_r(G)$. We also determine, for any graph G and any $e \in E(G)$ or $v \in V(G)$, the best possible upper and lower bounds of $\chi_r(G) - \chi_r(G - e)$, and those of $\chi_r(G) - \chi_r(G - v)$. We also study the structure of the graphs reaching the optimal bounds.

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1. Introduction

Graphs in this paper are simple and finite. Undefined terminologies and notations are referred to [1]. Thus $\Delta(G)$, $\delta(G)$ and $\chi(G)$ denote the maximum degree, the minimum degree and the chromatic number of a graph G, respectively. For $v \in V(G)$, let $N_G(v)$ be the set of vertices adjacent to v in G, $N_G[v] = N_G(v) \cup \{v\}$, and $d_G(v) = |N_G(v)|$. When G is understood from the context, the subscript G is often omitted in these notations. When no confusion on G arises, we will often use Δ for $\Delta(G)$. A cycle of length n is often denoted by C_n. A complete graph with n vertices is denoted by K_n.

Definition 1.1: For an integer $k > 0$, let $[k] = \{1, 2, \ldots, k\}$. Given a graph G, if $c : V(G) \mapsto [k]$ is a mapping, and if $V' \subseteq V(G)$, then define $c(V') = \{c(v) | v \in V'\}$. The mapping c is a *k-colouring* of G if it satisfies

(C1) $c(u) \neq c(v)$ for every edge $uv \in E(G)$;
For an integer $r > 0$ and a vertex $v \in V(G)$, we say that v satisfies (C2) under c if the following condition also holds:
(C2) $|c(N_G(v))| \geq \min\{d_G(v), r\}$.
A k-colouring $c : V(G) \mapsto [k]$ is a (k, r)*-colouring* of a graph G if (C2) holds for every vertex $v \in V(G)$.

The *r-hued chromatic number* of G, denoted by $\chi_r(G)$, is the smallest integer k such that G has a (k, r)-colouring. The study of this parameter was initiated in [9,15] where 2-hued colouring is called as dynamic colouring and the 2-hued chromatic number is called the dynamic chromatic number.

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In [10], dynamic colouring is generalized to r-hued colouring, where the name used is rth order conditional colouring and the r-hued chromatic number is called the rth order conditional dynamic chromatic number. As suggested by referees, this was renamed the r-hued colouring the first time in [16], and subsequently used in [17–19], among others. An alternative name, r-dynamic colouring, is also being used as in [5,6], among others. In [11], it was shown that (3,2)-colourability remains NP-complete even when restricted to planar bipartite graphs with maximum degree at most 3 and with arbitrarily high girth. This is quite different from the well-known result that the classical 3-colorablity is polynomially solvable for graphs with maximum degree at most 3. There have been studies on upper bounds of r-hued chromatic number for general graphs [4,13], for graphs excluding a specific minor [16], and for the particular cases of $r=2$ [3,7,8] and $r = \Delta$ [2,12].

Studying critical graphs is common in investigations on graph colouring. As seen in [1], every graph G contains a χ_1-critical subgraph, since $\chi_1(H) \leq \chi_1(G)$ holds for any subgraph $H \subseteq G$. However, it is shown in [14] that when $r=2$, there exist some graph G and a subgraph H of G with $\chi_r(H) > \chi_r(G)$. As these examples can be routinely extended for any integer r with $r \geq 2$, it is thus of interest to investigate the sensitivity problem on how the value of χ_r changes when an existing edge is removed or a new edge is added to a graph, or an existing vertex is removed or a new vertex is added to a graph. Miao et al in [14] investigated this problem for the special case when $r=2$.

Theorem 1.2 (Theorem 2.1 of [14]): *Each of the following holds.*

(i) *Let G be a connected graph with $|V(G)| \geq 3$. Then for every edge $e \in E(G)$,*

$$\chi_2(G) - 2 \leq \chi_2(G-e) \leq \chi_2(G) + 2. \tag{1}$$

(ii) *There exists a graph G such that $\chi_2(G-e) = \chi_2(G) + 2$ for at least one edge $e \in E(G)$.*
(iii) *If a connected graph G is such that $\chi_2(G-e) = \chi_2(G) - 2$ for at least one edge e in G, then $G = C_5$.*

Theorem 1.3 (Theorem 2.2 of [14]): *Let G be a connected graph with $|V(G)| \geq 2$. If G does not contain a subdivision of $K_{3,3}$, then $\chi_2(G-e) \leq \chi_2(G) + 1$ for every $e \in E(G)$.*

This motivates our current research. In this paper, we investigate the difference between the r-hued chromatic number of a graph G and that of $G-e$, $G-v$ respectively, for an edge $e \in E(G)$ or a vertex $v \in V(G)$; and the difference between $\chi_r(G)$ and $\chi_r(G+e)$, for an edge $e \notin E(G)$ but both of whose end vertices are in G. In particular, we prove the following.

Theorem 1.4: *Let $r \geq 2$ be an integer, and G be a connected graph with $|V(G)| \geq 2$.*

(i) *For any $e \in E(G)$, $\chi_r(G) - 2 \leq \chi_r(G-e) \leq \chi_r(G) + 2$.*
(ii) *For every graph G there is an edge $e \in E(G)$ such that $\chi_r(G-e) \neq \chi_r(G) + 2$.*
(iii) *If $\chi_r(G-e) = \chi_r(G) + 2$ for some $e \in E(G)$, then G must contain a subdivision of $K_{3,3}$.*

Theorem 1.5: *Let $r \geq 2$ be an integer, G be a graph with $|V(G)| \geq 2$, and u,v be any pair of nonadjacent vertices in $V(G)$.*

(i) *If u,v are in the same component of G, then $\chi_r(G) - 2 \leq \chi_r(G+uv) \leq \chi_r(G) + 2$.*
(ii) *If u,v are in different components of G, then $\chi_r(G) - 1 \leq \chi_r(G+uv) \leq \chi_r(G) + 1$.*

Both bounds in Theorem 1.4(i) are best possible in the sense that there are infinitely many graphs achieving the lower bound and infinitely many graphs achieving the upper bound. In Theorem 1.5, each bound in (i) and (ii) is best possible in the sense that there are infinitely many graphs achieving the lower bound and infinitely many graphs achieving the upper bound.

Section 2 is devoted to comparing $\chi_r(G)$ and $\chi_r(G-e)$ and investigating the graphs achieving the upper or lower bounds. The comparison between $\chi_r(G)$ and $\chi_r(G-v)$, and that between $\chi_r(G)$ and $\chi_r(G+e)$ are studied in Sections 3 and 4, respectively. Discussions for future study problems are addressed in the last section.

Throughout the rest of this paper, we will use the following notation. Let G be a graph, k,r be positive integers such that $k \geq \chi(G)$ and let $c : V(G) \mapsto [k]$ be a proper k-colouring of G. We define $B_r(G,c) = \{v \in V(G) : v$ does not satisfy (C2) in Definition 1.1$\}$. Thus c is a (k,r)-colouring if and only if $B_r(G,c) = \emptyset$. Let $V' \subseteq V(G)$ be a vertex subset of a graph G. As in [1], $G[V']$ is the subgraph of G induced by V'. A mapping $c : V' \mapsto [k]$ is a *partial* (k,r)*-colouring* of G if c is a (k,r)-colouring of $G[V']$. The subset V', denoted by $S(c)$, is the *support* of c. If c_1, c_2 are two partial (k,r)-colourings of G such that $S(c_1) \subseteq S(c_2)$ and such that for any $v \in S(c_1)$, $c_1(v) = c_2(v)$, then we say that c_2 is an *extension* of c_1.

2. Sensitivity of edge deletions

In [14], it is shown that there exist infinitely many graphs G with an edge $e \in E(G)$ such that $\chi_2(G) < \chi_2(G-e)$. The goal in this section is to prove Theorem 1.4, as well as to investigate the problem of whether there exist graphs G' and G'' such that for every $e \in E(G')$, $\chi_r(G'-e) = \chi_r(G') + 2$, and that for every $e \in E(G'')$, $\chi_r(G''-e) = \chi_r(G'') - 2$; and the properties of such graphs if they exist.

2.1. Proof of Theorem 1.4(i)

By Proposition 2.1 of [10], we have the following observation, stated as Lemma 2.1 below.

Lemma 2.1: *Let $r \geq 2$ be an integer and G be a connected graph. Then $\chi_r(G) \leq 2$ if and only if $G \in \{K_1, K_2\}$.*

Lemma 2.2: *Let $r \geq 2$ be an integer and G be a connected graph. Let $e = u_1u_2 \in E(G)$ and $k = \chi_r(G)$. If $c : V(G) \mapsto [k]$ is a (k,r)-colouring of G (also viewed as a proper k-colouring of $G-e$), then each of the following holds.*

(i) $B_r(G-e,c) \subseteq \{u_1,u_2\}$.
(ii) *For some $i \in \{1,2\}$, if $u_i \in B_r(G-e,c)$, then $d_G(u_i) > r$ and $|c(N_{G-e}(u_i))| = r-1$. Moreover, u_{3-i} is the only vertex in $N_G(u_i)$ coloured with $c(u_{3-i})$.*
(iii) $\chi_r(G-e) \leq \chi_r(G) + |B_r(G-e,c)|$.

Proof: By definition, c is also a k-colouring of $G-e$ and every vertex $z \in V(G-\{u_1,u_2\})$ satisfies (C2) under c, and so (i) holds. Let $u_i \in \{u_1,u_2\} \cap B_r(G,c)$. If $d_G(u_i) \leq r$, then $|c(N_G(u_i))| = |N_G(u_i)|$, and so $|c(N_{G-e}(u_i))| = |N_{G-e}(u_i)|$, contrary to the assumption that $u_i \in B_r(G,c)$. Hence we must have $d_G(u_i) > r$ and $|c(N_{G-e}(u_i))| = r-1$. If there exists another vertex $w \in N_G(u_i)$ with $c(w) = c(u_{3-i})$, then since c is a (k,r)-colouring of G, we have $|c(N_G(u_i))| \geq r$. Since $c(w) = c(u_{3-i})$, we have $|c(N_{G-e}(u_i))| = |c(N_G(u_i) - u_{3-i})| = |c(N_G(u_i))| \geq r$, contrary to the assumption that $u_i \in B_r(G,c)$. Thus (ii) must hold.

We are to prove (iii). By Lemma 2.2(ii), for each $u_i \in B_r(G,c)$, there exist $u_i', u_i'' \in N_{G-e}(u_i)$ with $c(u_i') = c(u_i'')$. Define

$$c'(z) = \begin{cases} c(z), & \text{if } z \notin \{u_i''\} \text{ for each } u_i \in B_r(G,c) \\ k+i, & \text{if } z = u_i'' \text{ and } u_i \in B_r(G,c). \end{cases}$$

Since every vertex $z \in V(G-\{u_1,u_2\})$ satisfies (C2) under c, it follows by Definition 1.1 that c' is a $(k+|B_r(G,c)|, r)$-colouring of $G-e$ or can be easily transformed into a $(k+|B_r(G,c)|, r)$-colouring of $G-e$ when $B_r(G,c) = \{u_2\}$. This completes the proof of the lemma. ■

The theorem below implies Theorem 1.4(i).

Theorem 2.3: *Let $r > 0$ be a positive integer and let G be a connected graph with $|V(G)| \geq 3$. Each of the following holds.*

(i) *For any edge $e \in E(G)$, $\chi_r(G) - 2 \leq \chi_r(G-e) \leq \chi_r(G) + 2$.*
(ii) *For any integer $r \geq 2$, there exist infinitely many graphs G with $\Delta(G) \geq r$ such that $\chi_r(G-e) = \chi_r(G) + 2$ for at least one edge $e \in E(G)$.*
(iii) *For any $r \geq 2$, there exists a graph G with an edge $e \in E(G)$ such that $\chi_r(G-e) = \chi_r(G) - 2$.*

Proof: (i) By Lemma 2.2, we have $\chi_r(G-e) \leq \chi_r(G) + 2$, and so it suffices to prove that $\chi_r(G) - 2 \leq \chi_r(G-e)$. Let $e = u_1u_2 \in E(G)$, $k_1 = \chi_r(G-e)$ and $c : V(G-e) \mapsto [k_1]$ be an (k_1, r)-colouring of $G-e$. Define

$$c_1(z) = \begin{cases} c(z), & \text{if } z \notin \{u_1, u_2\} \\ k_1 + i, & \text{if } z = u_i, 1 \leq i \leq 2 \end{cases}$$

As c is a (k_1, r)-colouring of $G-e$, it follows by definition that c_1 is a (k_1+2, r)-colouring of G, and so $\chi_r(G) \leq \chi_r(G-e) + 2$.

(ii) For any integer $\ell \geq 2r$, and a vector $(n_1, n_2, \ldots, n_\ell)$ with $n_i \geq 2$, $1 \leq i \leq \ell$, we will construct a graph $G = G(\ell, r)$ on $n = \sum_{i=1}^{\ell} n_i + 2$ vertices with the desired property described in Theorem 2.3(ii). Let H denote a complete ℓ-partite graph with a vertex set partition $V_1, V_2, \ldots, V_\ell$, such that $|V_i| = n_i \geq 2$ for each i with $1 \leq i \leq \ell$. Let u and v be two additional vertices not in $V(H)$. Obtain a new graph $G = G(\ell, r)$ from H and $\{u, v\}$ by joining u to all vertices in $(\cup_{i=1}^{r-1} V_i) \cup \{v\}$ and joining v to all vertices in $\cup_{i=r}^{2r-2} V_i$. It is routine to verify that $\chi_r(G) = \chi(G) = l$ and that $\chi_r(G-uv) = l+2$, since there exist two sets V_i, V_j with $1 \leq i \leq r-1$ and $r \leq j \leq 2r-2$ which must be coloured with at least two colours.

(iii) Let n, n_1, n_2, n_3, r be positive integers and n_4 be an integer satisfying $0 \leq n_4 < n_2$, $r = n_1 + n_2 + n_3 - 1$ and $n = \sum_{i=1}^{4} n_i + 2$. For $1 \leq i \leq 4$, let $H_i = K_{n_i}$ with $V_i = V(H_i)$ be disjoint cliques, and u,v be two additional vertices not in $\cup_{i=1}^{4} V_i$. Obtain a new graph $G = G(r)$ from the disjoint union $\cup_{i=1}^{4} H_i$ and $\{u, v\}$ by joining u to all vertices in $V_1 \cup V_4 \cup \{v\}$, joining each vertex in V_1 to every vertex in V_2, joining each vertex in V_2 to every vertex in V_3, and joining v to every vertex in $V_3 \cup V_4$. Since $\Delta(G) = r$ and any pair of vertices in $V_1 \cup V_2 \cup V_3 \cup \{u, v\}$ are at distance at most 2 in G, $\chi_r(G) \geq n_1 + n_2 + n_3 + 2 = r + 3$; If V_4 is not empty, the distance from every vertex in V_4 to every vertex in V_2 is 3 in both G and $G-uv$. It is routine to verify that $\chi_r(G) = n_1 + n_2 + n_3 + 2 = r + 3$ and $\chi_r(G-uv) = n_1 + n_2 + n_3 = r + 1$. ■

2.2. Graphs with $\chi_r(G-e) = \chi_r(G) - 2$ for every $e \in e(G)$

Using the fact that the Peterson graph is edge transitive, it is routine to show that if $G = P(10)$ is the Petersen graph, then $\chi_r(G-e) = \chi_r(G) - 2$ holds for every edge $e \in E(G)$ and $r \geq 3$. This subsection is devoted to investigating graphs with the property that for every edge $e \in E(G)$, $\chi_r(G-e) = \chi_r(G) - 2$. We start with a lemma.

Lemma 2.4: *Let $r \geq 2$ be an integer and G be a connected graph such that $\chi_r(G-e) = \chi_r(G) - 2$ for an edge $e = uv \in E(G)$. Let $k = \chi_r(G-e)$ and c be a (k, r)-colouring of $G-e$. Each of the following holds.*

(i) $\max\{d_G(u), d_G(v)\} \leq r$.
(ii) $c(u) \neq c(v)$, $c(v) \in c(N_{G-e}(u))$ *and* $c(u) \in c(N_{G-e}(v))$.

Proof: Without loss of generality, we assume that $d_G(u) \leq d_G(v)$. By contradiction, we assume that Lemma 2.4 fails. If either $d_{G-e}(v) \geq r$ or $c(u) = c(v)$, let

$$c_1(z) = \begin{cases} c(z), & \text{if } z \neq v \\ k+1, & \text{if } z = v \end{cases} \tag{2}$$

By definition, c_1 is a proper $(k+1)$-colouring of G with $B_r(G, c_1) \subseteq \{v\}$. When either $d_{G-e}(v) \geq r$ or $c(u) = c(v)$, v also satisfies (C2) in Definition 1.1. This implies that $v \notin B_r(G, c_1)$, and so c_1 is a $(k+1, r)$-colouring of G, contrary to the assumption that $\chi_r(G-e) = \chi_r(G) - 2$. Hence we must have (i) and $c(u) \neq c(v)$.

Now suppose that $c(u) \notin c(N_{G-e}(v))$. Define c_1 as in (2). By definition, c_1 is a proper $(k+1)$-colouring of G with $B_r(G, c_1) \subseteq \{v\}$. Since c is a (k, r)-colouring of $G{-}e$, it follows by Definition 1.1 that $|c(N_{G-e}(v))| \geq \min\{r, d_{G-e}(v)\}$. Since $c(u) \notin c(N_{G-e}(v))$, we have $|c_1(N_G(v))| \geq |c(N_{G-e}(v))| + 1 \geq \min\{r, d_G(v)\}$. This implies that $v \notin B_r(G, c_1)$, and so c_1 is a $(k+1, r)$-colouring of G, contrary to the assumption that $\chi_r(G-e) = \chi_r(G) - 2$. Thus we must have $c(u) \in c(N_{G-e}(v))$, and so Lemma 2.4(ii) follows by symmetry. ■

Theorem 2.5: *Let $r > 1$ be an integer and G be a connected graph such that $\chi_r(G-e) = \chi_r(G) - 2$ for every edge $e \in E(G)$. Each of the following holds.*

(i) *$\Delta(G) \leq r$, and any edge must lie in a 5-cycle.*
(ii) *If $r = 2$, then $G \cong C_5$.*
(iii) *If $\chi_r(G) \geq (r-1)r + 2$, then G is an r-regular graph.*

Proof: By Lemma 2.4, $\Delta(G) \leq r$. Let $e = uv \in E(G)$ be any edge of G. Throughout the proof of this theorem, let $k = \chi_r(G-e)$ and c be a given (k, r)-colouring of $G{-}e$. By Lemma 2.4, we have $c(u) \neq c(v)$.

(i) We may assume that $c(u) = 1, c(v) = 2$. By Lemma 2.4, there exists a vertex $u' \in N_{G-e}(u)$ with $c(u') = 2$ and a vertex $v' \in N_{G-e}(v)$ with $c(v') = 1$. As $\Delta(G) \leq r$ and by Definition 1.1(C2), $|c(N_{G-e}(u'))| = |N_{G-e}(u')|$, and so $u'v' \notin E(G)$. To show (i), it suffices to show that $N_G(u') \cap N_G(v') \neq \emptyset$. Assume to the contrary that $N_G(u') \cap N_G(v') = \emptyset$. Define

$$c_2(z) = \begin{cases} c(z), & \text{if } z \notin \{u', v'\} \\ k+1, & \text{if } z \in \{u', v'\}. \end{cases} \tag{3}$$

As c is a (k, r)-colouring of $G{-}e$, it follows by (3) that c_2 is a $(k+1, r)$-colouring of G, contradicting the fact that $\chi_r(G) = k + 2$. This proves (i).

(ii) By (i), G must be a connected graph with maximum degree 2, in which every edge lies in a 5-cycle. This implies that $G \cong C_5$.

(iii) Suppose that $r \geq 3$ and $\chi_r(G) \geq (r-1)r + 2$. By contradiction, assume that G is not r-regular. By (i), $\Delta(G) \leq r$, and so there must be an edge $e = uv \in E(G)$ with $d_G(u) \leq r - 1$. Let $k = \chi_r(G-e)$. Then $k = \chi_r(G) - 2 \geq (r-1)r$. Fix an arbitrary (k, r)-colouring c of $G{-}e$. By Lemma 2.4, we have

$$c(u) \neq c(v), \quad c(v) \in c(N_{G-e}(u)) \quad \text{and} \quad c(u) \in c(N_{G-e}(v)). \tag{4}$$

Let $N^2(u) = \{v \in V(G)$ and v is of distance at most 2 from u in $G\}$, and $C = c(N^2(u))$. Since $d_G(u) \leq r - 1$, we have that $|C| \leq r(r-1) - 1 < k$. Therefore, there must be an $\alpha \in [k] \backslash C$, which can be used to colour the vertex u to modify c to a colouring c_3. By Definition 1.1, c_3 is also a (k, r)-colouring of $G{-}e$. But as $c_3(u) \notin c_3(N_{G-e}(v))$, this is a violation of Lemma 2.4(ii). This contradiction proves (iii). ■

2.3. Graphs with $\chi_r(G - e) = \chi_r(G) + 2$ for some $e \in e(G)$

Throughout this subsection, $r \geq 2$ denotes an integer. The purpose of this subsection is to investigate graphs G with the property that $\chi_r(G - e) = \chi_r(G) + 2$ for some $e \in E(G)$. Corollary 2.6 follows immediately from Lemma 2.2.

Corollary 2.6: *Let G be a connected graph. If $\chi_r(G - e) = \chi_r(G) + 2$ holds for some edge $e = uv \in E(G)$, then each of the following must hold for any $(\chi_r(G), r)$-colouring c of G.*

(i) $\min\{d_G(u), d_G(v)\} \geq r + 1$.
(ii) *$c(u)$ occurs only once in $N_G(v)$ and $c(v)$ occurs only once in $N_G(u)$.*
(iii) *For any vertex $w \in \{u, v\}$, $|c(N_{G-e}(w))| = r - 1$ and there exist at least two vertices $w_1, w_2 \in N_G(w)$ such that $c(w_1) = c(w_2)$.*

The main result of this subsection is the following two theorems. Theorem 2.7 implies the nonexistence of graphs with $\chi_r(G - e) = \chi_r(G) + 2$ for every $e \in E(G)$, which is equivalent to Theorem 1.4(ii). Theorem 2.8 is equivalent to Theorem 1.4(iii), which extends Theorem 1.3 from $r = 2$ to all integers $r \geq 2$.

Theorem 2.7: *For any positive integer r, there exist no graph G such that $\chi_r(G - e) = \chi_r(G) + 2$ for every $e \in E(G)$.*

Proof: Assume that there exists a graph G such that $\chi_r(G - e) = \chi_r(G) + 2$ for every $e \in E(G)$. Let c be a $(\chi_r(G), r)$-colouring of G. By assumption, Corollary 2.6(i)–(iii) must hold for any edge $e \in E(G)$ under c. Choose $e_0 = uv \in E(G)$, by Corollary 2.6(iii), there exist $v_1, v_2 \in N_{G-e_0}(u)$ with $c(v_1) = c(v_2)$, and so $c(v_1)$ is present at least twice in the neighbourhood of u. As Corollary 2.6(ii) also holds for the edge $e_1 = uv_1$, we conclude that $c(v_1)$ must occur only once in $N_G(u)$. This contradiction completes the proof. ∎

Theorem 2.8: *Let r be an integer with $r \geq 2$ and G be a connected graph with $|V(G)| \geq 2$. If G does not contain a subdivision of $K_{3,3}$, then $\chi_r(G - e) \leq \chi_r(G) + 1$ for every $e \in E(G)$.*

Proof: To the contrary, we assume that there exists a connected graph G which does not contain a subdivision of $K_{3,3}$ and $\chi_r(G - e) \geq \chi_r(G) + 2$ for some edge $e = uv \in E(G)$. By Theorem 2.3, we have $\chi_r(G - e) = \chi_r(G) + 2$. As we must have $G \notin \{K_1, K_2\}$, it follows by Lemma 2.1, that $\chi_r(G) \geq 3$, and so both $\chi_r(G - e) = \chi_r(G) + 2 \geq 5$ and $|V(G)| \geq 5$. Without loss of generality, we assume that $d_G(u) \geq d_G(v)$. Let $k = \chi_r(G)$ and $c : V(G) \mapsto [k]$ be a (k, r)-colouring of G. Thus Corollary 2.6 is applicable for this edge e and this colouring c of G. We have the following observations.

Observation 2.9: Each of the following holds.

(i) As c is a (k, r)-colouring of G, c is also a k-colouring of $G-e$. Moreover, for any vertex $z \in V(G - e) - \{u, v\}$, we have $|c(N_{G-e}(z))| \geq \min\{d_{G-e}(z), r\}$. Thus we have $B_r(G - e, c) \subseteq \{u, v\}$.
(ii) By Corollary 2.6(i), we observe that $d_G(u) \geq d_G(v) \geq r + 1$.
(iii) By Corollary 2.6(ii) and (iii), we observe that $|c(N_G(u) - \{v\})| = |c(N_G(v) - \{u\})| = r - 1$. Furthermore, there exist at least two vertices $u_1, u_2 \in N_{G-e}(u)$ with $c(u_1) = c(u_2)$ and two vertices $v_1, v_2 \in N_{G-e}(v)$ with $c(v_1) = c(v_2)$.

Assured by Observation 2.9(iii), in the following, we let $u_1, u_2 \in N_{G-e}(u)$ be vertices satisfying $c(u_1) = c(u_2)$ and $v_1, v_2 \in N_{G-e}(v)$ be vertices satisfying $c(v_1) = c(v_2)$.

Claim 1: $c(u_1) = c(u_2) \neq c(v_1) = c(v_2)$.

If not, define

$$c_1(z) = \begin{cases} c(z), & \text{if } z \in V(G-e) - \{u_1, v_1\} \\ k+1, & \text{if } z \in \{u_1, v_1\}. \end{cases} \tag{5}$$

By Observation 2.9(i), $B_r(G-e,c) \subseteq \{u,v\}$. It follows from (5) and Observation 2.9(iii) that $u, v \notin B_r(G-e,c_1)$, and so c_1 is a $(k+1,r)$-colouring of $G-e$, contrary to to the assumption that $\chi_r(G-e) \geq \chi_r(G)+2$.

Claim 2: *For any possible choices of $i,j \in \{1,2\}$, one of the following holds.*

(i) $u_iv_j \in E(G)$, *or*
(ii) *there exists a vertex $w_{i,j} \in N_G(u_i) \cap N_G(v_j)$ such that $c(u_i)$ and $c(v_j)$ appear only once in the neighbourhood of $w_{i,j}$.*

If not, there must exist one pair of vertices u_i and v_j with $u_iv_j \notin E(G)$ such that either $N_G(u_i) \cap N_G(v_j) = \emptyset$ or the colours $c(u_i)$, $c(v_j)$ totally appear more than two times in the neighbourhood of every vertex $w \in N_G(u_i) \cap N_G(v_j)$. Define

$$c_2(z) = \begin{cases} c(z), & \text{if } z \in V(G-e) - \{u_i, v_j\} \\ k+1, & \text{if } z \in \{u_i, v_j\}. \end{cases} \tag{6}$$

Since $B_r(G,c) \subseteq \{u,v\}$, it follows from (6) and the assumption on u_i and v_j that $u, v \notin B_r(G,c_2) \subseteq B_r(G,c)$. Thus c_2 is a $(k+1,r)$-colouring of $G-e$, contrary to to the assumption that $\chi_r(G-e) \geq \chi_r(G)+2$. This proves the claim.

Claim 3: *If Claim* 2(ii) *holds for more than one pairs of $i,j \in \{1,2\}$, the vertices $w_{i,j}$ must be distinct.*

If there are two different pairs (i_1,j_1) and (i_2,j_2) with $(1 \leq i_1,i_2,j_1,j_2 \leq 2)$, such that $w_{i_1,j_1} = w_{i_2,j_2} = w$, then $\{u_{i_1}, v_{j_1}, u_{i_2}, v_{j_2}\} \subseteq N_G(w)$. Since (i_1,j_1) and (i_2,j_2) are different pairs, $|\{u_{i_1}, v_{j_1}, u_{i_2}, v_{j_2}\}| \geq 3$. This contradicts the definition of w_{i_1,j_1} or w_{i_2,j_2}, which completes the proof of the claim.

For any i,j with $(1 \leq i,j \leq 2)$, let

$$E_{i,j} = \begin{cases} \{u_iv_j\}, & \text{if } u_iv_j \in E(G) \\ \{u_iw_{i,j}, w_{i,j}v_j\}, & \text{otherwise}. \end{cases}$$

Let $E' = \{uv, uu_1, uu_2, vv_1, vv_2\} \cup E_{1,1} \cup E_{1,2} \cup E_{2,1} \cup E_{2,1}$. By Claims 1–3 above, $G[E']$ contains a subdivision of $K_{3,3}$, contrary to the assumption of the theorem, which completes the proof of Theorem 2.8. ∎

3. Sensitivity of vertex deletions

The purpose of this section is to investigate the sensitivity of $\chi_r(G)$ under vertex removal. A sharp lower bound of $\chi_r(G-v)$ in terms of $\chi_r(G)$ will be obtained. We start with the following result.

Theorem 3.1: *For any integer $r \geq 2$ and any graph G, if $v \in V(G)$, then $\chi_r(G) \leq \chi_r(G-v)+r$.*

Proof: Let $k = \chi_r(G-v)$ and c be a (k,r)-colouring of $G-v$. Firstly we obtain a new colouring c' of G by assigning a new colour not in $c(V(G-v))$ to v. If at least $\min\{d_G(v), r\}$ appear in the neighbourhood of v, then c' is a $(k+1,r)$-colouring of G, and so $\chi_r(G) \le \chi_r(G-v)+1$. Otherwise, by introducing at most additional $r-1$ new colours to replace the colours of some vertices in $N_G(v)$, we obtained a colouring of G, also denoted by c', such that $|c'(N_G(v))| \ge \min\{d_G(v), r\}$. As c is a (k,r)-colouring of $G-v$, this new colouring c' is a (k',r)-colouring of G with $k' \le k+r$. Hence $\chi_r(G) \le \chi_r(G-v)+r$. ■

Lemma 3.2: *Let r be an integer with $r \ge 2$. If $\chi_r(G) = \chi_r(G-v)+r$ holds for some vertex v of a graph G, then by letting $k = \chi_r(G-v)$, each of the following holds.*

(i) $d_G(v) \ge r$.
(ii) *For any integer ℓ with $1 \le \ell \le r-1$ and for any $(k+\ell-1,r)$-colouring c of $G-v$, there exist at most ℓ colours present at the neighbourhood of v in G.*

Proof: Let $d_G(v) = d$ and $N_G(v) = \{z_1, z_2, \dots, z_d\}$.

(i) Assume that $d \le r-1$. Let $c : V(G-v) \mapsto [k]$ be a (k,r)-colouring of $G-v$. Define

$$c_1(z) = \begin{cases} c(z), & \text{if } z \in V(G-e) - \{v, z_1, z_2, \dots, z_{d-1}\} \\ k+j, & \text{if } z = z_j, 1 \le j \le d-1 \\ k+d, & \text{if } z = v. \end{cases} \tag{7}$$

As c is a (k,r)-colouring of $G-v$, it follows by (7) that c_1 is a $(k+d,r)$-colouring of G, and so $\chi_r(G) \le \chi_r(G-v)+d < \chi_r(G-v)+r$, contrary to the assumption that $\chi_r(G) = \chi_r(G-v)+r$. Hence we must have $d_G(v) \ge r$.

(ii) Let us assume that, there exist an integer ℓ with $1 \le \ell \le r-1$ and a $(k+\ell-1,r)$-colouring c of $G-v$, such that $|c(N_G(v))| \ge \ell+1$. Without loss of generality, we assume that $c : V(G-v) \mapsto [k+\ell-1]$ and $c(z_{d-\ell}), c(z_{d-\ell+1}), \dots, c(z_d)$ are mutually distinct. Define

$$c_2(z) = \begin{cases} c(z), & \text{if } z \in V(G-e) - \{v, z_1, \dots, z_{r-\ell-1}\} \\ k+\ell-1+j, & \text{if } z = z_j, 1 \le j \le r-\ell-1 \\ k+r-1, & \text{if } z = v. \end{cases} \tag{8}$$

As c is a $(k+\ell-1,r)$-colouring $G-v$, $d \ge r$ and as $c(z_{d-\ell}), c(z_{d-\ell+1}), \dots, c(z_d)$ are mutually distinct, it follows by (8) that c_2 is a $(k+r-1,r)$-colouring of G, contrary to the assumption that $\chi_r(G) = \chi_r(G-v)+r = k+r$. This completes the proof. ■

The next result indicates that the bound in Theorem 3.1 is best possible in the sense that there exists graphs, namely, $K_{1,d}$ with $d \ge r$ such that the inequality in Theorem 3.1 can be reached.

Theorem 3.3: *Let r be a positive integer and G be a connected graph. If there is a vertex $v \in V(G)$ such that $\chi_r(G-v) = \chi_r(G)-r$, then each of the following holds.*

(i) $\chi_r(G) = r+1$ *if and only if $G = K_{1,d}$ for some integer $d \ge r$ and $d_G(v) = d$.*
(ii) $\chi_r(G) \ne r+2$ *for $r \ge 2$, and* $\chi_r(G) \ne r+3$ *when $r \ge 3$.*

Proof: Let $v \in V(G)$ be a vertex satisfying $\chi_r(G-v) = \chi_r(G)-r$. Let $k = \chi_r(G-v)$ and $d_G(v) = d$. By Lemma 3.2(i), $d_G(v) \ge r$. By Lemma 3.2(ii) with $\ell = 1$,

$$\text{for any } (k,r)\text{−colouring } c \text{ of } G-v, \quad |c(N_G(v))| = 1. \tag{9}$$

Thus $N_G(v)$ is an independent set in G. We claim that if $\chi_r(G-v) \ge 2$, then there must be one component of $G-v$ that contains all vertices of $N_G(v)$. In fact, if $N_G(v)$ are in different components of $G-v$, then by permutation of two colours of one of the components of $G-v$ intersecting $N_G(v)$, we can obtain a (k,r)-colouring c of $G-v$ such that $|c(N_G(v))| \geq 2$, contrary to Lemma 3.2(ii). It follows by this claim that

$$\text{if } \chi_r(G-v) \geq 2, \text{ then as } G \text{ is connected, } G-v \text{ must be connected.} \tag{10}$$

(i) By Theorem 2.3 of [10], $\chi_r(K_{1,d}) = r+1$ when $d \geq r$ and $K_{1,d}$ has the property described in Theorem 3.3. Suppose that $\chi_r(G) = r+1$. Then $\chi_r(G-v) = 1$, and so $E(G-v) = \emptyset$. As G is connected, we conclude that $G = K_{1,d}$ for some integer $d \geq r$ and $d_G(v) = d$.

(ii) Suppose that $\chi_r(G) = r+2$. Then $k = \chi_r(G-v) = 2$, and $G-v$ is connected. Hence $G-v = K_2$ by Lemma 2.1. Since $N_G(v)$ is an independent set of G, only one of the vertices of $G-v$ belongs to $N_G(v)$. It follows that G is a path on 3 vertices, and so $\chi_r(G) = 3$, contrary to the facts that $r \geq 2$ and $\chi_r(G-v) = \chi_r(G) - r$. This justifies the statement that $\chi_r \neq r+2$.

Suppose, by contradiction, that $r \geq 3$ and $\chi_r(G) = r+3$, and so $k = \chi_r(G-v) = 3$. As $r \geq 3$, if $G-v$ has a vertex w with $N_{G-v}(w) \geq 3$, then any r-hued colouring of $G-v$ requires at least 4 different colours to colour $N_{G-v}(w) \cup \{w\}$ so that (C2) of Definition 1.1 can be satisfied. Therefore, we must have $\Delta(G-v) \leq 2$. If $\Delta(G-v) \leq 1$, then $\chi_r(G-v) \leq 2$. Therefore we must have $\Delta(G-v) = 2$, and so by (10) and as $r \geq 3$, $|V(G-v)| = s > 2$ and $G-v$ is either a path or a cycle. Let $P = z_1 z_2 \cdots z_s$ denote a spanning path of $G-v$ such that if $G-v$ is a cycle, then $z_1 z_s \in E(G)$ and the subscript of the vertices are taken modulo s.

By Lemma 3.2(i), $d \geq 3$. Fix an arbitrary $(3,r)$-colouring $c : V(G-v) \mapsto \{1,2,3\}$ of $G-v$. By (9), we may assume that $N_G(v) \subseteq c^{-1}(1)$. We shall construct a $(4,r)$-colouring c_1 of $G-v$ satisfying the condition $|c_1(N_G(v))| \geq 3$, which will contradicts to Lemma 3.2(ii) with $k=3$ and $\ell = 2$. Once such a contradiction is derived, the proof is complete. We shall show below that, either $G-v$ is a cycle or a path, such a colouring c_1 always exists to complete the proof. In the following, we denote the set of all integers modulo 3 by $\{1,2,3\}$.

Since $N_G(v) \subseteq c^{-1}(1)$, we denote $N_G(v) = \{z_{i_1}, z_{i_2}, \ldots, z_{i_d}\}$ such that $1 \leq i_1 < i_2 < \cdots < i_d \leq s$. Since $c : V(G-v) \mapsto \{1,2,3\}$ is a $(3,r)$-colouring, we observe that $i_1 \equiv i_2 \equiv \cdots . \equiv i_d \pmod 3$, and we may assume that, for any $z_j \in c^{-1}(1)$, if $z_{j+1}, z_{j+2} \in V(G-v)$, then $c(z_{j+1}) = 2$ and $c(z_{j+2}) = 3$; and if $z_{j-1}, z_{j-2} \in V(G-v)$, then $c(z_{j-1}) = 3$ and $c(z_{j-2}) = 2$. Obtain a colouring $c' : V(G) \mapsto \{1,2,3,4\}$ as follows:

Step 1A. If $i_3 - i_2 = 3$, then define, assuming $z_{i_3+1}, z_{i_3+2} \in V(G-v)$,

$$(c'(z_{i_2-2}), c'(z_{i_2-1}), c'(z_{i_2}), c'(z_{i_2+1}), c'(z_{i_2+2}), c'(z_{i_3}), c'(z_{i_3+1}), c'(z_{i_3+2})) = (2,4,3,1,4,2,3,4). \tag{11}$$

Step 1B. If $i_3 - i_2 \geq 6$, then define, assuming $z_{i_3+1} \in V(G-v)$,

$$(c'(z_{i_2-2}), c'(z_{i_2-1}), c'(z_{i_2}), c'(z_{i_2+1}), c'(z_{i_2+2})) = (2,4,3,2,4)$$
$$(c'(z_{i_3-2}), c'(z_{i_3-1}), c'(z_{i_3}), c'(z_{i_3+1})) = (2,3,4,2). \tag{12}$$

Step 2. Define $c'(z) = c(z)$ if $z \in V(G-v)$ is not recoloured in Step 1A–1B.

By Steps 1 and 2, we obtained a proper colouring $c' : V(G-v) \mapsto \{1,2,3,4\}$. For any vertex z that are distance at least 3 from z_{i_2} or z_{i_3}, since c is a $(3,r)$-colouring, and as the colour 4 is a newly introduced colour, condition (C2) is satisfied at z. By definition and by (11) and (12), c' is in fact a $(4,r)$-colouring of $G-v$ satisfying $|c'(N_G(v))| = 3$, as desired. Thus Lemma 3.2(ii) with $k=3$ and $\ell = 2$ is violated. This completes the proof of the proposition. ∎

4. Sensitivity of edge additions

By Definition 1.1, adding a new edge e joining two nonadjacent vertices u and v in a graph G could change the r-hued chromatic number. The change may be caused by the fact that, with $k = \chi_r(G)$, any (k, r)-colouring c of G would violate either (C1) and/or (C2) of Definition 1.1. In this section, we investigate the optimal intervals around $\chi_r(G)$ which contains $\chi_r(G + e)$.

Theorem 4.1: *Let G be a graph with $|V(G)| \geq 2$, and u,v be any pair of nonadjacent vertices in $V(G)$. Each of the following holds.*

- (i) *If u,v are in the same component of G, then $\chi_r(G) - 2 \leq \chi_r(G + uv) \leq \chi_r(G) + 2$.*
- (ii) *If u,v are in different components of G, then $\chi_r(G) - 1 \leq \chi_r(G + uv) \leq \chi_r(G) + 1$.*
- (iii) *For any integer $r \geq 2$, there exists a connected graph G with $\Delta(G) \geq r$ such that $\chi_r(G + uv) = \chi_r(G) - 2$ for at least one pair of nonadjacent vertices $u, v \in V(G)$.*
- (iv) *For any integer $r \geq 2$, there exists a connected graph G with $\Delta(G) \geq r$ such that $\chi_r(G + uv) = \chi_r(G) + 2$ for at least one pair of nonadjacent vertices $u, v \in V(G)$.*
- (v) *For any integer $r \geq 1$, there exists a disconnected graph with two components G_1 and G_2 such that $\chi_r(G + uv) = \chi_r(G) + 1$ for some vertices $u \in V(G_1)$ and $v \in V(G_2)$.*
- (vi) *For any integer $r \geq 2$, there exists a disconnected graph with two components G_1 and G_2 such that $\chi_r(G + uv) = \chi_r(G) - 1$ for some vertices $u \in V(G_1)$ and $v \in V(G_2)$.*

Proof: (i) Assume G has ω components and let G_1 be the component of G which contains u,v. Let $G_1' = G_1 + uv$. By Theorem 2.3(i), we have $\chi_r(G_1') - 2 \leq \chi_r(G_1) \leq \chi_r(G_1') + 2$. That is, $\chi_r(G_1) - 2 \leq \chi_r(G_1') \leq \chi_r(G_1) + 2$. Since $\chi_r(G) = \max\{\chi_r(G_1), \dots, \chi_r(G_\omega)\}$ and $\chi_r(G + uv) = \max\{\chi_r(G_1'), \dots, \chi_r(G_\omega)\}$, then $\chi_r(G) - 2 \leq \chi_r(G + uv) \leq \chi_r(G) + 2$.

(ii) Assume G has ω components, G_1 is the component of G containing u and G_2 is the component of G containing v. Without loss of generality, assume that $d_G(u) \geq d_G(v)$.

Let $k_1 = \chi_r(G)$ and $c : V(G) \mapsto [k_1]$ be a (k_1, r)-colouring of G such that $c(u) \neq c(v)$. If $d_G(v) \geq r$, the colouring c is also a (k_1, r)-colouring of G+uv; If $d_G(v) \leq r - 1$ and $d_G(u) \geq r$, recolour u with $k_1 + 1$; If $d_G(v) \leq d_G(u) \leq r - 1$, then firstly permute two colours $c(u), c(v)$ in G_2; and then recolour u with $k_1 + 1$. In the two cases above, the resulting colouring is a $(k_1 + 1, r)$-colouring of G+uv. So $\chi_r(G + uv) \leq \chi_r(G) + 1$.

Let $k_2 = \chi_r(G + uv)$ and $c : V(G + uv) \mapsto [k_2]$ be a (k_2, r)-colouring of G+uv. If $d_{G+uv}(u) \leq r$, the colouring c confined to G is also a (k_2, r)-colouring of G; If there exists some vertex $w \in \{u, v\}$ such that $d_{G+uv}(w) \geq r + 1$ and $|c(N_G(w))| = r - 1$, there must exist at least two neighbours of w in G, (say) w_1, w_2 with $c(w_1) = c(w_2)$. Introduce a new colour to one of them, for example set $c(w_1) = k_2 + 1$ (if both u and v have this property, set $c(u_1) = c(v_1) = k_2 + 1$). The resulting colouring is a $(k_2 + 1, r)$-colouring of G. So $\chi_r(G) \leq \chi_r(G + uv) + 1$. That is, $\chi_r(G + uv) \geq \chi_r(G) - 1$.

(iii) and (iv) are consequences of Theorem 2.3(ii) and (iii), respectively.

(v) For any integers $r \geq 1$, let G_1 be any graph with $\chi_r(G_1) \leq r$ and $G_2 \cong K_r$. Choose any vertex $u \in V(G_1)$, $v \in V(G_2)$, let $G = G_1 + G_2$. It is routine to verify that $\chi_r(G) = r$ and $\chi_r(G + uv) = r + 1 = \chi_r(G) + 1$.

(vi) Let $k \geq r + 1$ and G_1 be any graph with $\chi_r(G_1) \leq k$. Let H be a complete k-partite graph with set partition $V_1, V_2, \dots, V_k$, such that $|V_1| = |V_2| = \dots = |V_{r-1}| = 2$. Let G_2 be the graph obtained from H by adding a new vertex v and join v to every vertex in every set V_i for $1 \leq i \leq r - 1$. Denote $G = G_1 + G_2$. Choose any vertex of G_1 as u. As $d(v) \geq r$ and as the neighbours of v are only in $r-1$ partition classes, there must be some $i \in \{1, 2, \dots, r - 1\}$ such that V_i which needs at least 2 colours. Hence $\chi_r(G) \geq k + 1$. It is routine to verify that $\chi_r(G) = k + 1$ and $\chi_r(G + uv) = k$. ∎

Proposition 4.2: *Let $r \geq 2$ be a positive integer and G be any non-complete graph. Then there is a pair of nonadjacent vertices $u, v \in V(G)$ such that $\chi_r(G + uv) \neq \chi_r(G) + 2$.*

Proof: For contradiction, we assume that there exists a non-complete graph G such that $\chi_r(G + uv) = \chi_r(G) + 2$ for every pair of nonadjacent vertices $u, v \in V(G)$. Let $k = \chi_r(G)$ and c be any (k, r)-colouring c of G. By a similar analysis as in the proof of Lemma 2.4, we have that $c(u) \neq c(v)$

holds for every pair of nonadjacent vertices $u, v \in V(G)$. Since c is also a proper colouring of G, $c(u) \neq c(v)$ holds for every pair of adjacent vertices u,v. Hence each vertex has a different colour and so $k = |V(G)| \geq \chi_r(G + uv)$, contrary to the assumption that $\chi_r(G + uv) = k + 2$. ∎

Lemma 4.3: *Let $r \geq 2$ be an integer and G be a graph. If there exists a pair of nonadjacent vertices $u, v \in V(G)$ such that $\chi_r(G + uv) = \chi_r(G) - 2$, then for any $(\chi_r(G + uv), r)$-colouring c of $G + uv$, each of the following holds.*

(i) $\min\{d_G(u), d_G(v)\} \geq r$.
(ii) *$c(u)$ occurs only once in $N_{G+uv}(v)$ and $c(v)$ occurs only once in $N_{G+uv}(u)$.*
(iii) $|c(N_G(u))| = |c(N_G(v))| = r - 1$.
(iv) *There exist two vertex disjoint induced paths u_1uu_2 and v_1vv_2 in G such that $c(u_1) = c(u_2)$ and $c(v_1) = c(v_2)$.*

Proof: Let $e = uv$. We first observe that Lemma 4.3(i)–(iii) can be obtained by replacing G with $G+e$ and replacing $G-e$ with G, respectively, in Corollary 2.6. Now by Lemma 4.3(i)–(iii), there exist two different vertices $u_1, u_2 \in N_G(u)$ and two distinct vertices $v_1, v_2 \in N_G(v)$ such that $c(u_1) = c(u_2)$ and $c(v_1) = c(v_2)$. If $\{v_1, v_2\} \cap \{v_1, v_2\} \neq \emptyset$, assume that $u_1 = v_1$. Recolour u_1 with a new colour, the resulting colouring of G is a $(\chi_r(G + uv) + 1, r)$-colouring of G, which contradicts the assumption $\chi_r(G + uv) = \chi_r(G) - 2$. Hence we may assume that $\{v_1, v_2\} \cap \{v_1, v_2\} = \emptyset$, whence both u_1uu_2 and v_1vv_2 are two vertex disjoint induced paths in G. This proves (iv). ∎

Let $P(10)$ denote the Petersen graph. It is routine to verify that $\chi_3(P(10)) = 10$ (as seen in [13]) and $\chi_r(P(10) + uv) = 8 = \chi_r(P(10)) - 2$ for every pair of nonadjacent vertices $u, v \in V(P(10))$. Therefore, there exist graphs G such that for some $r \geq 2$, and for every pair of nonadjacent vertices $u, v \in V(G)$, we have $\chi_r(G + uv) = \chi_r(G) - 2$. Characterizing such graphs seems quite nontrivial. Nevertheless, Lemma 4.3 leads to the following observation, stated as Proposition 4.4 below.

Proposition 4.4: *Let G be a connected graph such that $\chi_r(G + uv) = \chi_r(G) - 2$ for every pair of nonadjacent vertices u,v. Then the following hold.*

(i) $\delta(G) \geq \min\{r, |V(G)| - 2\}$;
(ii) *there exist two disjoint induced paths u_1uu_2 and v_1vv_2 for every pair of nonadjacent vertices u,v.*

5. Remarks

In the study of sensitivities of the r-hued chromatic numbers, we have observed that certain graphs have some interesting properties, such as the Peterson graph and 5-cycle. To the best of our knowledge, there are not many studies on these graphs. It is also of interest to determine the graphs G such that for every edge $e \in E(G)$, $\chi_r(G - e) = \chi_r(G) - 2$; and the graphs G such that $\chi_r(G + e) = \chi_r(G) - 2$ for every $e = uv \notin E(G)$, where $u, v \in V(G)$.

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