Line graphs containing 2-factors with bounded number of components

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Abstract

Let G be a connected simple graph of order n. We use $L(G)$ to denote the line graph of G, where $L(G)$ has the edge set of G as its vertex set and two vertices in $L(G)$ are adjacent if and only if the corresponding two edges in G share a common endvertex. A 2-factor of G is a spanning subgraph H of G such that every vertex in H has degree 2. A lot of results on the components of a 2-factor in G have appeared by studying the conditions on the

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minimum degree of G. In this paper, instead of studying the minimum degree, we use a different approach and obtain the following: if $\max\{d(x), d(y)\} \geq \frac{n-\mu}{p} - 1$ holds whenever $xy \notin E(G)$ and $|U| \geq 3$, where $U = \{v : d(v) < \frac{n-\mu}{p} - 1\}$, $p > 0$ and $\mu \geq 0$ are integers, then for n sufficiently large relative to p and μ, $L(G)$ has a 2-factor with at most $p+1$ components. Moreover, $L(G)$ has a 2-factor with at most p components if $|U| \leq 1$. Especially, it extends a result of [10] saying that if $\delta(G) \geq \frac{n}{p} - 1$, then $L(G)$ has a 2-factor with at most p components. We also show the graphs satisfying the conditions mentioned above are $(p+2)$-supereulerian, i.e., they have a spanning even subgraph with at most $p+2$ components. All results are best possible.

Keywords: 2-factor; reduced graph; line graph; dominating Eulerian subgraph; k-supereulerian graph

1 Introduction

We follow [1] for terminology and notation not defined here, and consider loopless finite graphs in which multiple edges are allowed. Let G be a graph and let $O(G)$ denote the set of all vertices in G with odd degrees. An *Eulerian* graph is a connected graph G with $O(G) = \emptyset$. The graph K_1 is an Eulerian graph. If a graph contains a spanning Eulerian subgraph, then it is called *supereulerian*. For literatures on supereulerian graphs, see the survey of Catlin [4] and its complement by Chen and Lai [5].

An Eulerian subgraph H of a graph G is *dominating* if $G - V(H)$ is edgeless, and in this case we call H a *dominating Eulerian subgraph* (DES).

We use $L(G)$ to denote the line graph of G, where $L(G)$ has $E(G)$ as its vertex set and two vertices in $L(G)$ are adjacent if and only if the corresponding two edges in G share a common endvertex. The following theorem explains the relationship between dominating Eulerian subgraphs in graph G and Hamiltonian cycles in the line graph $L(G)$.

Theorem 1. (Harary and Nash-Williams, [7]) Let G be a graph with $|E(G)| \geq 3$. Then $L(G)$ is Hamiltonian if and only if G has a DES.

A *2-factor* is a 2-regular spanning subgraph of G. A Hamiltonian cycle is then a 2-factor, and in one sense, it is the simplest 2-factor as it is composed of a single cycle. A *circuit* is an Eulerian subgraph with at least three vertices. Let F be a vertex subset of $V(G)$. An Eulerian subgraph H of G is called *F-Eulerian* if $F \subseteq V(H)$.

A *star* is the complete bipartite graph $K_{1,m}$. For a given graph G, we say that G has a *p-system that dominates* if there is a family S of edge-disjoint circuits and stars with at least three edges in G such that every edge of G is either in one of the circuits or stars, or is incident to a circuit in S, where $p = |S|$. The following result gives a characterization of graphs G such that $L(G)$ contains a 2-factor with exactly p components.

Theorem 2. (Gould and Hynds, [6]) Let G be a graph without isolated vertices. The line graph $L(G)$ contains a 2-factor with p ($p \geq 1$) components if and only if G has a p-system that dominates.

There have been efforts using minimum degree or Ore-type degree sums to study the existence of 2-factors with a bounded number of components. Niu and Xiong applied this approach to line graphs and obtained the following result.

Theorem 3. (Niu and Xiong, [10]) Let G be a connected simple graph of order n and p a positive integer such that $\delta(G) \geq \lfloor n/p \rfloor - 1$. If n is sufficiently large relative to p, then G has an even factor with at most p components, and then $L(G)$ has a 2-factor with at most p components.

Catlin [3] showed the following:

Theorem 4. (Catlin, [3]) Let G be a connected simple graph of order n, and let $p \geq 2$ be an integer. If $d(u) + d(v) > \frac{2n}{p} - 2$ whenever $uv \notin E(G)$, and if $n \geq 4p^2$, then exactly one of the following conclusions holds:

(1) G has a spanning Eulerian subgraph;

(2) G is contractible to a graph G_1 of order less than p and containing no spanning Eulerian subgraph;

(3) $p = 2$, and $G - x = K_{n-1}$ for some $x \in V(G)$ with $d(x) = 1$.

Motivated by the theorems above, in this paper, we are going to show the following main result, of which Theorem 3 is a special case.

Theorem 5. Let G be a connected simple graph of order n, and let p be a positive integer, μ a nonnegative integer, $U = \{v : d(v) < \frac{n-\mu}{p} - 1\}$, where $G[U]$ is a clique. If $n > p^3 + 6p^2 + 6p + \mu p + \mu$ and $|U| \geq 3$, and if $\max\{d(x), d(y)\} \geq \frac{n-\mu}{p} - 1$ whenever $xy \notin E(G)$, then $L(G)$ has a 2-factor with at most $p+1$ components. Moreover, if $|U| \leq 1$, then $L(G)$ has a 2-factor with at most p components. Especially, if $\delta(G) \geq \frac{n}{p} - 1$ (this implies $U = \emptyset$), then $L(G)$ has a 2-factor with at most p components.

We organize the paper as follows. In Section 2, we present Catlin's reduction method which will be used in the proof of Theorem 5 (in Section 3); Section 4 is devoted to a corollary; the sharpness of Theorem 5 is presented in the last section.

2 Introduction to Catlin's reduction method

In 1988, Catlin defined collapsible graphs in [2]. Let G be a graph. For $R \subseteq V(G)$, a subgraph Γ of G is called an *R-subgraph* if both $O(\Gamma) = R$ and $G - E(\Gamma)$ is connected. A graph is *collapsible* if G has an R-subgraph for every even set $R \subseteq V(G)$. Apparently, K_1 is a collapsible graph. Let H be a connected subgraph of G. We use G/H to denote the graph obtained from G by contracting H, that is to say, we replace H with a vertex v_H such that the number of edges in G/H joining any $v \in V(G) - V(H)$ to v_H in G/H equals the number of edges joining v in G to H. We say G is *contractible* to G' if G contains pairwise vertex-disjoint connected subgraphs $H_1, H_2, \cdots, H_k$ with $\bigcup_{i=1}^{k} V(H_i) = V(G)$ such that G' is obtained from G by successively contracting $H_1, H_2, \cdots, H_k$. Each subgraph H_i of G is called the *preimage* of the vertex v_{H_i} in G', and v_{H_i} is called the *image* of H_i. If H_i is not a single vertex in G, then we call v_{H_i} a *nontrivial* vertex in G'. For any vertex $v \in V(H_i)$, we also say that v_{H_i} is the image of the vertex v. Catlin [2] showed that every graph G has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs $H_1, H_2, \cdots, H_k$ such that $\bigcup_{i=1}^{k} V(H_i) = V(G)$. The *reduction* of G is the graph obtained from G by successively contracting $H_1, H_2, \cdots, H_k$. A nontrivial vertex in the reduction of G is a vertex which is the image of a nontrivial connected subgraph of G. If a graph is the reduction of some graph, then we say the graph is *reduced*.

Theorem 6. (Catlin, [2]) Let G be a connected graph and G' the reduction of G. Then each of the following holds.

(a) G is supereulerian if and only if G' is supereulerian;

(b) G' is triangle-free with $\delta(G') \leq 3$;

(c) If G is reduced, then G is a simple graph with $\delta(G) \leq 3$ and with either $G \in \{K_1, K_2\}$, or $|E(G)| \leq 2|V(G)| - 4$;

(d) If G is collapsible, then G is supereulerian, i.e., G has a spanning Eulerian subgraph;

(e) Let L be a collapsible subgraph of G, v_L the vertex in G/L to which L is contracted, and $M \subseteq V(G) - V(L)$. Then G has an Eulerian subgraph H such that $M \cup V(L) \subseteq V(H)$ if and only if G/L has an Eulerian subgraph H' such that $M \cup \{v_L\} \subseteq V(H')$.

3 Proof of Theorem 5

Define

$$J_p(G) = \{v \in V(G) : d(v) \geq \frac{n-\mu}{p} - 1\}.$$

Let G' be the reduction of G and $n' = |V(G')|$. Let p be a positive integer, and consider the condition that for any $xy \notin E(G)$,

$$\max\{d(x), d(y)\} \geq \frac{n-\mu}{p} - 1. \tag{3.1}$$

We shall assume that (3.1) holds and that n is sufficiently large (say $n > p^3 + 6p^2 + 6p + \mu p + \mu$). Let $U = \{v : d(v) < \frac{n-\mu}{p} - 1\}$, where $G[U]$ is clearly a complete subgraph (clique) of G. Moreover, $d(v) \geq \frac{n-\mu}{p} - 1$ for any $v \in V(G) \setminus U$.

Let $c = p + 5$ and let

$$W = \{v \in V(G') : d_{G'}(v) \leq c\} \text{ and } W' = \{v \in W : \ v \text{ is nontrivial}\}.$$

We shall prove several claims to help us establish the conclusion in our main result.

Claim 1. $|W \setminus W'| \leq 1$.

Proof: Since every vertex v of $W \setminus W'$ is trivial, $d_G(v) = d_{G'}(v) \leq c < \frac{n-\mu}{p} - 1$ when $n > p^2 + 6p + \mu$. Hence $W \setminus W' \subseteq U$. Recall that $G[U]$ is a complete subgraph of G. Since every complete graph with order at least 3 is collapsible, $G[U]$ is contracted to one vertex in G' if $|U| \geq 3$. Hence $W \setminus W'$ contains at most one vertex in G'. This proves Claim 1. □

Furthermore, Claim 1 implies the stronger statement that if $W \neq W'$, then it forces $|U| = 1$ and $W - W' = U$.

Claim 2. *For any $v \in W'$, if H_v denotes the preimage of v in G and either $|U| \leq 1$ or $G[U] \nsubseteq H_v$, then*

$$|V(H_v)| \geq \frac{n-\mu}{p} - d_{G'}(v). \tag{3.2}$$

Proof: Since v is nontrivial, $|V(H_v)| \geq 3$. Hence we can take a vertex $x \in V(H_v) - U$ because either $|U| \leq 1$ or $G[U] \nsubseteq H_v$. Then we have $\frac{n-\mu}{p} - 1 \leq d_G(x) \leq d_{H_v}(x) + d_{G'}(v)$, which implies $d_{H_v}(x) \geq \frac{n-\mu}{p} - 1 -$

$d_{G'}(v)$. So $|V(H_v)| \geq d_{H_v}(x) + 1 \geq \frac{n-\mu}{p} - d_{G'}(v)$. This proves Claim 2.
□

Claim 3. *If $|U| \leq 1$, then $|W'| \leq p$; if $|U| \geq 3$, then $|W'| \leq p+1$.*

Proof: Suppose first that $|U| \leq 1$. For any $v \in W'$, by Claim 2 we have $|V(H_v)| \geq \frac{n-\mu}{p} - d_{G'}(v)$, so $n \geq |W'| \left(\frac{n-\mu}{p} - c\right)$. This is equivalent to $|W'| \leq \frac{np}{n-\mu-pc}$. Since $|W'|$ is an integer, we have $|W'| \leq p$ when $n > p^3 + 6p^2 + 5p + \mu p + \mu$.

Suppose next that $|U| \geq 3$. Since the collapsible complete subgraph $G[U]$ is contracted to one vertex in G', there must be exactly one vertex $u \in W'$ such that $G[U] \subseteq H_u$. By Claim 2, $n-1 \geq (|W'|-1)\left(\frac{n-\mu}{p} - c\right)$. This is equivalent to $|W'| \leq \frac{(n-1)p}{n-\mu-pc} + 1$. Since $|W'|$ is an integer, we have $|W'| \leq p+1$ when $n > p^3 + 6p^2 + 4p + \mu p + \mu$. □

Claim 4. $V(G') = W$.

Proof: By contradiction, we assume that $V(G') \setminus W \neq \emptyset$. Note that every vertex in $V(G') \setminus W$ has degree at least $c+1$ in G'. Since G' is simple (G' is reduced), this means

$$n' \geq c + 2. \tag{3.3}$$

We count the adjacencies to get $c|V(G') \setminus W| \leq 2|E(G')| \leq 4n' - 8$ by Theorem 6 (c), which means $|V(G') \setminus W| \leq \frac{4n'-8}{c}$. So it follows that

$$|W| = n' - |V(G') \setminus W| \geq \left(1 - \frac{4}{c}\right) n' + \frac{8}{c}. \tag{3.4}$$

By Claims 1 and 3, $|W| \leq |W'| + 1 \leq p + 2$. Hence by (3.3) and (3.4),

$$\left(1 - \frac{4}{c}\right)(c+2) + \frac{8}{c} \leq \left(1 - \frac{4}{c}\right) n' + \frac{8}{c} \leq p + 2.$$

It follows that $p + 5 = c \leq p + 4$, a contradiction. Therefore, we must have $V(G') = W$. □

Claim 5. *Every vertex in $J_p(G)$ is contained in the preimage of some vertex in W'.*

Proof: Since $n > p^2 + 6p + \mu$, the degree of vertices in $J_p(G)$ will exceed c, and so Claim 5 follows from Claim 4. □

Note that by Claim 5 and by Theorem 6 (e), if G' has a W'-Eulerian subgraph, then G has a $J_p(G)$-Eulerian subgraph. Here we do not distinguish whether G' has a W'-Eulerian subgraph or not. Since all vertices in W' are nontrivial, we can suppose that $W' = \{v_1, v_2, \cdots, v_m\}$. For any $v_i \in W'$, the pre-image of v_i in G denoted by H_i is collapsible and hence has a spanning Eulerian subgraph by Theorem 6. Moreover, $|V(H_i)| \geq 3$ since every vertex in W' is nontrivial. By Claim 1, we divide G' into m parts $P_1, P_2, \cdots, P_m$, where each P_i is an induced subgraph of v_i and its neighbors (trivial) from $W \setminus W'$. Therefore, G is divided into m parts which are the corresponding pre-images of $P_1, P_2, \cdots, P_m$ and each part has a dominating Eulerian subgraph H_i, thus $\bigcup_{i=1}^{m} H_i$ is an m-system of G that dominates. By Theorem 2, $L(G)$ has a 2-factor with m components. Since $m = |W'| \leq p+1$ (by Claim 3), it follows that $L(G)$ has a 2-factor with at most $p+1$ components. Moreover, if $|U| \leq 1$, then $m = |W'| \leq p$ by Claim 3, so $L(G)$ has a 2-factor with at most p components. Therefore, the proof of Theorem 5 is completed. □

4 A Corollary

A graph is called *k-supereulerian*, if G has a spanning even subgraph with at most k components. The following result was proved recently.

Theorem 7. (Niu, Lai and Xiong, [9]) Let G be a connected graph and G' be the reduction of G. Then G is k-supereulerian if and only if G' is k-supereulerian.

By Theorem 7 and by the proof of Theorem 5, we obtain the following consequence.

Corollary 8. Let G be a connected simple graph of order n, and let p be a positive integer and μ a nonnegative integer. If $\max\{d(x), d(y)\} \geq \frac{n-\mu}{p} - 1$ holds for any $xy \notin E(G)$, then for $n > p^3 + 6p^2 + 6p + \mu p + \mu$, G is $(p+2)$-supereulerian if $|U| \geq 3$, where $U = \{v : d(v) < \frac{n-\mu}{p} - 1\}$, and G is $(p+1)$-supereulerian if $|U| \leq 1$. Moreover, G is p-supereulerian if $U = \emptyset$.

5 Sharpness

In this section, we shall give an example to show the sharpness of Theorem 5 and Corollary 8. Let $G_1, G_2, \ldots, G_p$ be p vertex-disjoint complete graphs of order $\frac{n-\mu}{p}$ and G_μ an additional complete graph of order μ. Obtain G by joining exactly one vertex of G_i to exactly one vertex of G_μ for $i = 1, 2, \ldots, p$. Note that $\max\{d(x), d(y)\} \geq \frac{n-\mu}{p} - 1$ whenever $xy \notin E(G)$ (where $\mu = |\{v : d(v) < \frac{n-\mu}{p} - 1\}|$) and the equation can be achieved for some pairs of nonadjacent vertices x, y. If $\mu > 2$ and n is sufficient large, then $L(G)$ does not have a 2-factor with at most p components; if $\mu = 1$, then $L(G)$ has no 2-factor with $p - 1$ components; if $\mu = 2$ and $p \geq 3$, then $L(G)$ has no 2-factor with at most p components by the fact that G has no p-system that dominates. Especially, when $\mu = 2$, if $p \leq 2$ and any one of the two vertices in G_μ is adjacent to at most one of G_i, then $L(G)$ has no 2-factors at all since G has no system that dominates. For $\mu = 0$, let G' be the graph of order n obtained from $G_1, G_2, \ldots, G_p$ such that $G/\{G_1, G_2, \ldots, G_p\}$ is a tree. Then G' satisfies the condition of Theorem 5 and $L(G')$ has a 2-factor with p components. This shows Theorem 5 is best possible.

If $\mu = 2$, G is $(p+2)$-supereulerian but not $(p+1)$-supereulerian; if $\mu \neq 2$, then G is $(p+1)$-supereulerian but not p-supereulerian; if $\mu = 0$ then G' is p-supereulerian but not $(p-1)$-supereulerian. This shows that Corollary 8 is best possible.

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