Linear list r-hued colorings of graphs with bounded maximum subgraph average degrees

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Abstract

For an integer $r > 0$, and a k-list assignment L to vertices of a graph G, a linear (L, r)-coloring of a graph G is an coloring c of the vertices of G such that for every vertex v of degree $d(v)$, $c(v) \in L(v)$, the number of colors used by the neighbors of v is at least $\min\{d_G(v), r\}$, and such that for any two distinct colors i and j, every component of $G[c^{-1}(\{i, j\})]$ must be a path. The linear list r-hued chromatic number of a graph G, denoted $\chi^{\ell}_{L,r}(G)$, is the smallest integer k such that for every k-list L, G has a linear (L, r)-coloring. In this paper, the

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behavior and bounds of linear r-hued chromatic number of a graph G are investigated. Let Mad(G) denotes the maximum subgraph average degree of a graph G. We prove that if G is a graph with maximum degree Δ, then each of the following holds.
(1) If $\Delta \geq 9$ and $\mathrm{Mad}(G) < \frac{7}{3}$, then $\chi^{\ell}_{L,r}(G) \leq \max\{\lceil \Delta/2 \rceil + 1, r+1\}$.
(2) If $\Delta \geq 7$ and $\mathrm{Mad}(G) < \frac{12}{5}$, then $\chi^{\ell}_{L,r}(G) \leq \max\{\lceil \Delta/2 \rceil + 2, r+2\}$.
(3) If $\Delta \geq 7$ and $\mathrm{Mad}(G) < \frac{5}{2}$, then $\chi^{\ell}_{L,r}(G) \leq \max\{\lceil \Delta/2 \rceil + 3, r+3\}$.

Key words. Linear r-hued coloring, linear r-hued chromatic number, linear list r-hued chromatic number

1 Induction

Graphs in this paper are simple and finite. Undefined terminologies and notations are referred to [2]. Thus for a graph G, $\Delta(G)$, $\delta(G)$, $\chi(G)$ and $\chi_L(G)$ denote the maximum degree, the minimum degree, chromatic number and the list chromatic number of G, respectively. For $v \in V(G)$, let $N_G(v)$ denote the set of vertices adjacent to v in G, and $d_G(v) = |N_G(v)|$. When G is understood from the context, we often use $N(v)$ and $d(v)$ for $N_G(v)$ and $d_G(v)$, respectively. For an integer $i \geq 0$, let $D_i(G)$ denote the set of all vertices of degree i in G; vertices in $D_i(G)$ are called i-vertices of G.

Definition 1.1. *Let k, r be integers with $k > 0$ and $r > 0$, $\bar{k}$ =$\{1, 2, \ldots, k\}$ and $2^{\bar{k}}$ be the family of all subsets of $\bar{k}$. If $c : V(G) \mapsto \bar{k}$ is a mapping, and if $V' \subseteq V(G)$, then define $c(V') = \{c(v) | v \in V'\}$. A (k, r)-coloring of a graph G is a mapping $c : V(G) \mapsto \bar{k}$ satisfying both the following.*

(C1) $c(u) \neq c(v)$ *for every edge* $uv \in E(G)$;
(C2) $|c(N_G(v))| \geq min\{d_G(v), r\}$ *for any* $v \in V(G)$.

The condition (C2) is often referred to as the r-hued condition. For a fixed integer $r > 0$, the **r-hued chromatic number** of G, denoted by $\chi_r(G)$, is the smallest integer k such that G has a (k, r)-coloring. When $r = 2$, 2-hued colorings are also called dynamic coloring as in [1, 7, 10], among others.

Definition 1.2. *Let* $c : V(G) \mapsto \bar{k}$ *be a* (k, r)*-coloring of a graph* G. *If for all* $i, j \in \bar{k}$ *with* $i \neq j$,

$$\text{every component of } G[c^{-1}(i) \cup c^{-1}(j)] \text{ is a path,} \qquad (1)$$

then c *is a* **linear (k, r)-coloring** *of* G. *The* **linear r-hued chromatic number** *of* G, *denoted by* $\chi_r^\ell(G)$, *is the smallest integer* k *such that* G *has a linear* (k, r)*-coloring. We define* $\chi^\ell(G) = \chi_1^\ell(G)$, *called the* **linear chromatic number** *of* G.

For integers $k' \geq k > 0$, let G be a graph and $L : V(G) \mapsto 2^{\bar{k}'}$ be an assignment such that assigns to every $v \in V(G)$ a list $L(v)$ of colors available at v; such an L is a **k-list** of G if for any $v \in V(G)$, $|L(v)| \geq k$. An **L-coloring** is a proper coloring c such that $c(v) \in L(v)$, for every $v \in V(G)$. The graph G is **list k-colorable** if for every k-list L, G admits an L-coloring. The **list chromatic number** of G, denoted by $\chi_L(G)$, is the minimum number k such that G is list k-colorable.

For a given assignment L of color lists to every vertex of G, and a given positive integer r, an **(L, r)-coloring** c is an L-coloring such that $|c(N_G(v))| \geq \min\{|N_G(v)|, r\}$ for every vertex $v \in V(G)$. The **list r-hued chromatic number**, $\chi_{L,r}(G)$, is the least integer k such that for any $v \in V(G)$ and every list assignment L with $|L(v)| = k$, G has an (L, r)-coloring.

An (L, r)-coloring c of G is a **linear (L, r)-coloring** if c also satisfies (1). The graph G is **linear list (k, r)-colorable** if G

has a linear (L, r)-coloring for any list of colors $L : V(G) \mapsto 2^{\bar{k}}$ with $|L(v)| \geq k$, for every $v \in V(G)$. The **linear list r-hued chromatic number** of G, denoted $\chi^{\ell}_{L,r}(G)$, is the minimum number k such that G is linear list (k, r)-colorable. We define $\chi^{\ell}_{L}(G) = \chi^{\ell}_{L,1}(G)$, called the **linear list chromatic number** of G.

In the next section, we determine the linear list r-hued chromatic number for trees, complete k-partite graphs. In the last section, we also determine the linear list 2-hued chromatic number for graphs with maximum degree at most 4 and graphs whose maximum subgraph average degree and girth are in a certain ranges.

2 The linear r-hued chromatic number of certain graph families

Throughout this section, $r > 0$ denotes an integer. In this section, we determine the linear r-hued chromatic number of a certain families of graphs, including complete bipartite graphs, and cycles. Proposition 2.1 below follows immediately from the definitions.

Proposition 2.1. *Let $s \geq 1$ be an integer and G be a nontrivial connected graph with $\Delta = \Delta(G)$. Each of the following holds:*
(i) $\chi^{\ell}_{\Delta+s}(G) = \chi^{\ell}_{\Delta}(G) \geq \cdots \geq \chi^{\ell}_{r}(G) \geq \chi^{\ell}_{r-1}(G) \geq \cdots \geq \chi^{\ell}_{2}(G) \geq \chi^{\ell}(G)$ *and* $\chi^{\ell}_{L,\Delta+s}(G) = \chi^{\ell}_{L,\Delta}(G) \geq \cdots \geq \chi^{\ell}_{L,r}(G) \geq \chi^{\ell}_{L,r-1}(G) \geq \cdots \geq \chi^{\ell}_{L,2}(G) \geq \chi^{\ell}_{L}(G)$.
(ii) $|V(G)| \geq \chi^{\ell}_{L,r}(G) \geq \chi^{\ell}_{r}(G) \geq \max\{\chi_r(G), \chi^{\ell}(G)\} \geq \chi(G)$.

Lemma 2.2. *(Theorem 2.2 of [6]) If G is a tree with $|V(G)| \geq 3$, then $\chi_r(G) = \min\{r, \Delta(G)\} + 1$.*

Lemma 2.3. *[5] If G is a tree with maximum degree $\Delta(G)$, then* $\chi^{\ell}(G) = \lceil \frac{\Delta(G)}{2} \rceil + 1$.

Lemma 2.4. *Let $n \geq 2$ be an integer. If $G = K_{1,n-1}$, then* $\chi_r^{\ell}(G) = \max\{\chi_r(G), \chi^{\ell}(G)\}$.

Proof. By Proposition 2.1(ii), $\chi_r^{\ell}(G) \geq \max\{\chi_r(G), \chi^{\ell}(G)\}$. We will show that $\chi_r^{\ell}(G) \leq \max\{\chi_r(G), \chi^{\ell}(G)\}$ then. By Proposition 2.1(i), we may assume that $r \leq n-1$. By Lemmas 2.2 and 2.3, we observe that $\chi^{\ell}(G) = \lceil \frac{n-1}{2} \rceil + 1$ and $\chi_r(G) = \min\{r, n-1\} + 1$. Let $k_1 = \lceil \frac{n-1}{2} \rceil + 1$ and $k_2 = r + 1$.

If $r \leq \lceil \frac{n-1}{2} \rceil$, then we have $\max\{\chi_r(G), \chi^{\ell}(G)\} = k_1$ by Lemmas 2.2 and 2.3, and so G has a linear k_1-coloring c_1. As $r \leq \lceil \frac{n-1}{2} \rceil$, c_1 is a linear (k_1, r)-coloring of G. If $\lceil \frac{n-1}{2} \rceil + 1 \leq r \leq n-1$, then $\max\{\chi_r(G), \chi^{\ell}(G)\} = k_2$, and so G has a linear k_2-coloring c_2. As $\lceil \frac{n-1}{2} \rceil + 1 \leq r \leq n-1$, c_2 is a linear (k_2, r)-coloring of G. Thus in any case, $\chi_r^{\ell}(G) \leq \max\{\chi_r(G), \chi^{\ell}(G)\}$, which justifies the lemma. □

Theorem 2.5. *If G is a tree with $|V(G)| \geq 3$, then* $\chi_r^{\ell}(G) = \max\{\chi_r(G), \chi^{\ell}(G)\}$.

Proof. We argue by induction on $n = |V(G)|$. The theorem holds trivially if $n \leq 2$. If $n = 3$, then $G = K_{1,2}$ and so the theorem follows from Lemma 2.4. Hence we assume that G is a tree on $n \geq 4$ vertices and that the theorem holds for smaller values of n. By Lemma 2.4, we assume that $G \neq K_{1,n-1}$.

Let v be a vertex of degree 1 in G such that the degree of its neighbor is minimized, and let u be the only vertex adjacent to v in G. By induction, $\chi_r^{\ell}(G-v) = k = \max\{\chi_r(G-v), \chi^{\ell}(G-v)\}$.

Since $G \neq K_{1,n-1}$, and we choose v such that $|N_G(u)|$ is minimized, we have $\Delta(G-v) = \Delta(G)$. By induction, $G-v$ has a linear (k, r)-coloring $c' : V(G-v) \mapsto \bar{k}$, where $k = \max\{\chi_r(G-$

$v), \chi^\ell(G-v)\}$. By Lemmas 2.2 and 2.3, and since $\Delta(G-v) = \Delta(G)$, we have $k = \max\{\chi_r(G), \chi^\ell(G)\}$.

Since $|N_G(u)| \leq \Delta(G)$, by Lemmas 2.2 and 2.3, there must be a color $i_0 \in \bar{k}$ such that at most one vertex in $N_{G-v}(u)$ is colored with i_0 under c' and such that $c'(u) \neq i_0$. Further more, if $k \geq |c'(N_{G-v}(u))| + 2$, then we choose such an $i_0 \in \bar{k} - (c'(N_{G-v}(u)) \cup \{c'(u)\})$. Define $c : V(G) \mapsto \bar{k}$ by $c(z) = c'(z)$ if $z \neq v$ and $c(v) = i_0$.

Case 1. $r \leq |N_{G-v}(u)|$.

Then c also satisfies the r-hued condition (C2). Since i_0 occurs in the neighbors of u in G at most twice, and since c' satisfies (1), c also satisfies (1), and so in this case, c is a linear (k,r)-coloring of G.

Case 2. $r > |N_{G-v}(u)|$.

We claim that we always have $k \geq |c'(N_{G-v}(u))| + 2$. In fact, by induction and by Lemmas 2.2 and 2.3, if $r \geq \Delta(G)$, then $k = \max\{\chi_r(G-v), \chi^\ell(G-v)\} = \Delta(G)+1 \geq |c'(N_{G-v}(u))|+2$; if $\Delta(G) > r$, then $k = \max\{r, \lceil \frac{n-1}{2} \rceil\}+1$, and so by $r > |N_{G-v}(u)|$, we also have $k \geq |c'(N_{G-v}(u))| + 2$. This justifies the claim.

By this claim, there is always a color $i_0 \in \bar{k} - (c'(N_{G-v}(u)) \cup \{c'(u)\})$, and so c is a linear (k,r)-coloring of G in this case also. This completes the proof of the theorem. □

Next we determine the linear list r-hued chromatic number of complete bipartite graphs. To this aim, we need the help of the following two former results.

Theorem 2.6. *(Esperet, Montassier, Raspaud, Proposition 3 of [5]) If $m \geq n \geq 1$ are integers, then $\chi_L^\ell(K_{m,n}) = \chi^\ell(K_{m,n}) = \lceil m/2 \rceil + n$.*

Theorem 2.7. *(Theorem 3 of [6]) Suppose that $m \geq n \geq 2$, then $\chi_r(K_{m,n}) = \min\{2r, n+m, r+n\}$.*

Theorem 2.8. *Suppose that $m \geq n \geq 2$, then*

$$\chi^{\ell}_{L,r}(K_{m,n}) = \begin{cases} n+m, & m \leq r \\ r+n, & r < m < 2r \\ n+\lceil \frac{m}{2} \rceil, & m \geq 2r. \end{cases} \quad (2)$$

Proof. Let (X, Y) denote the vertex bipartition of $K_{m,n}$ with $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. In the rest of the proof, we assume that $k' \geq k \geq 1$ are integers, and for given values of k, let $L : V(K_{m,n}) \mapsto 2^{\bar{k}'}$ be an arbitrary list assignment to the vertices with $|L(v)| \geq k$, for every $v \in V(K_{m,n})$.

If $m \leq r$, then let $k = \chi^{\ell}_{L,r}(K_{m,n})$, and $c : V(K_{m,n}) \mapsto \bar{k}$ be a proper linear (k, r)-coloring. For any $i = 1, 2, \ldots, m$, by (C2), $|c(N_G(x_i))| = n$ and so $|c(Y)| = n$. Similarly, $|c(X)| = m$. By (C1), for any i with $1 \leq i \leq m$, and for any j with $1 \leq j \leq n$, we have $c(x_i) \neq c(y_j)$, and so $k \geq m+n$. But since $|V(K_{m,n})| = m+n$, it follows from Proposition 2.1(ii) that $k \leq m+n$. Thus if $m \leq r$, then $\chi^{\ell}_{L,r}(K_{m,n}) = m+n$. Hence we assume that $m > r$ and shall utilize the following algorithm.

Define

$$k = \begin{cases} n+\lceil \frac{m}{2} \rceil & \text{if } m \geq 2r. \\ n+r & \text{if } r < m < 2r. \end{cases} \quad (3)$$

We first to present a linear (L, r)-coloring of $K_{m,n}$. Define a coloring $c : V(K_{m,n}) \mapsto \bigcup_{v \in V(K_{m,n})} L(v)$ as follows. Firstly, for each $y_i \in Y$, choose $c(y_i) \in L(y_i)$ so that $|c(Y)| = n$. Since $k > n$, this can be done.

To color vertices in X, we randomly pick $c(x_1) \in L(x_1) - c(Y)$. If $m \geq 2r$, then let $m' = \lceil \frac{m}{2} \rceil$, and choose $c(x_{m'+1}) \in L(x_{m'+1}) - c(Y)$; for $2 \leq j \leq \cdots \lceil \frac{m}{2} \rceil$, pick $c(x_j) \in L(x_j) - (c(Y) \cup \{c(x_1), \cdots, c(x_{j-1})\})$ and $c(x_{m'+j}) \in L(x_{m'+j}) - (c(Y) \cup \{c(x_{m'+1}), \cdots, c(x_{m'+j-1})\})$. If $r < m < 2r$, then choose $c(x_{r+1}) \in L(x_{r+1}) - c(Y)$; for $2 \leq j \leq \cdots r$, pick $c(x_j) \in L(x_j) -$

$(c(Y) \cup \{c(x_1), \cdots, c(x_{j-1})\})$ and $c(x_{r+j}) \in L(x_{r+j}) - (c(Y) \cup \{c(x_{r+1}), \ldots, c(x_{r+j-1})\})$. By (3), this coloring process can be done. Thus for every vertex $x_i \in X$, the colors of neighbors of x_i are mutually distinct; and for every vertex $y_j \in Y$, any color can occur in the neighbors of y_j at most twice. It follows by definition that c is a linear (L, r)-coloring of $K_{m,n}$, and so by definition, $\chi^{\ell}_{L,r}(K_{m,n}) \leq k$.

To prove (2), we note that if $m \geq 2r$, then by Proposition 2.1(i) and Theorem 2.6, $\chi^{\ell}_{L,r}(K_{m,n}) \geq \chi^{\ell}_{r}(K_{m,n}) \geq \chi^{\ell}(K_{m,n}) = n + \lceil \frac{m}{2} \rceil$; if $n < r$, then by Proposition 2.1(ii) and Theorem 2.7, $\chi^{\ell}_{L,r}(K_{m,n}) \geq \chi^{\ell}_{r}(K_{m,n}) \geq \chi_{r}(K_{m,n}) = n + r$.

It remains to show that if $n \geq r$, then $\chi^{\ell}_{L,r}(K_{m,n}) \geq n + r$. Let $k_0 = \chi^{\ell}_{L,r}(K_{m,n})$, and $c : V(K_{m,n}) \mapsto \bar{k}_0$ be a linear (k_0, r)-coloring of $K_{m,n}$. By (1), either $|c(X)| = m$ or $|c(Y)| = n$, as otherwise, there would be a bicolored cycle of length four. Moreover, to meet the requirement in (C2), if $|c(X)| = m$, then $|c(Y)| \geq r$; and if $|c(Y)| = n$, then $|c(X)| \geq r$. It follows that $\chi^{\ell}_{L,r}(K_{m,n}) \geq \chi^{\ell}_{r}(K_{m,n}) \geq r + n$. This completes the proof of the theorem. □

Theorem 2.9. *Suppose that $n_1 \geq n_2 \geq \cdots \geq n_m > 0$ are integers. If $m \geq r + 1$, then* $\chi^{\ell}_{L,r}(K_{n_1,\ldots,n_m}) = \sum_{i=1}^{m} n_i - \lfloor \frac{n_1}{2} \rfloor$.

Proof. Let $(V_1, V_2, \ldots, V_m)$ denote the partition of $K_{n_1,\ldots,n_m}$ with $|V_i| = n_i$ for $1 \leq i \leq m$. Let $k_1 = \chi^{\ell}_{L,r}(K_{n_1,\ldots,n_m})$. For integers $k' \geq k_1 \geq 1$, let $L : V(K_{n_1,\ldots,n_m}) \mapsto 2^{\bar{k'}}$ be an arbitrary k_1-list, and let c be a linear (L, r)-coloring of $K_{n_1,\ldots,n_m}$.

Since c is a linear coloring of $K_{n_1,\ldots,n_m}$, if for some j with $1 \leq j \leq m$, there exist two vertices $v, v' \in V_j$ such that $c(v) = c(v')$, then for any $i \neq j$, we must have $|c(V_i)| = |V_i| = n_i$. This is because that if we also have $u, u' \in V_i$ with $c(u) = c(u')$, then $\{u, u', v, v'\}$ will induce a bicolored cycle of length four, violating the assumption that c is a linear coloring. As every color can occur at most twice in the vertices of V_j, it follows that

$$\chi^{\ell}_{L,r}(K_{n_1,\ldots,n_m}) = k_1 \geq \sum_{i=1}^{m} n_i - \lfloor \frac{n_j}{2} \rfloor \geq \sum_{i=1}^{m} n_i - \lfloor \frac{n_1}{2} \rfloor. \quad (4)$$

Now, let $k_2 = \sum_{i=1}^{m} n_i - \lfloor \frac{n_1}{2} \rfloor$. For integers $k' \geq k_2 \geq 1$, let $L : V(K_{n_1,\ldots,n_m}) \mapsto 2^{k'}$ be an arbitrary list assignment to the vertices with $|L(v)| \geq k_2$. We shall present a linear (L,r)-coloring of $K_{n_1,\ldots,n_m}$ as follows. First color the vertices in $\bigcup_{i=2}^{m} V_i$ such that $|c(\bigcup_{i=2}^{m} V_i)| = |\bigcup_{i=2}^{m} V_i| = \sum_{i=2}^{m} n_i$. Since $k_2 = \sum_{i=1}^{m} n_i - \lfloor \frac{n_1}{2} \rfloor$, such a coloring c on $\bigcup_{i=2}^{m} V_i$ can be found. Next, we color the vertices in V_1. Denote $V_1 = \{v_1, v_2, \cdots, v_{n_1}\}$, and let $t = \lceil \frac{n_1}{2} \rceil$. Randomly set $c(v_1) \in L(v_1) - c(\bigcup_{i=2}^{m} V_i)$ and $c(v_{t+1}) \in L(v_{t+1}) - c(\bigcup_{i=2}^{m} V_i)$. For $2 \leq j \leq t$, choose $c(v_j) \in L(v_j) - (c(\bigcup_{i=2}^{m} V_i) \cup \{c(v_1) \cdots, c(v_{j-1})\})$ and $c(v_{t+j}) \in L(v_{t+j}) - (c(\bigcup_{i=2}^{m} V_i) \cup \{c(v_{t+1}) \cdots, c(v_{t+j-1})\})$. As each color can occur in the vertices of V_1 at most twice, this is a linear coloring. Since $m \geq r+1$, the neighbors of each vertex in $V(K_{n_1,\ldots,n_m})$ will be colored with at least r different colors. Hence (C2) is satisfied, and so c is a linear (L,r)-coloring. It follows by definition that $\chi^{\ell}_{L,r}(K_{n_1,\ldots,n_m}) \leq k_2$. This, together with (4), implies the theorem. □

Next, we determine the linear list r-hued chromatic number of cycles. The following lemma will be used.

Lemma 2.10. *Suppose that $n \geq 3$ and $r \geq 2$ are integers. Then*

$$\chi_{L,r}(C_n) = \begin{cases} 3, & n \equiv 0 (mod\ 3); \\ 5, & n = 5; \\ 4, & otherwise. \end{cases}$$

In [1], Lemma 2.10 was proved for the case when $r = 2$. Since C_n is a 2-regular graph, the proof for the general case when $r \geq 2$ is similar and will be omitted.

Proposition 2.11. *If $n \geq 3$ is a natural number, then the following holds:*

$$\chi^{\ell}_{L,r}(C_n) = \begin{cases} 3, & n \equiv 0 \ (mod\ 3); \\ 5, & n = 5; \\ 4, & otherwise. \end{cases}$$

Proof. By Proposition 2.1(i), $\chi^{\ell}_{L,2}(C_n) \geq \chi_{L,2}(C_n)$, and let L be a list assigning color sets to each vertex of C_n. Since any proper subgraph of C_n is a path, it follows that when $r \geq 2$, any (L,r)-coloring of C_n must also be a linear (L,r)-coloring. Thus $\chi^{\ell}_{L,r}(C_n) = \chi_{L,r}(C_n)$, and so the proposition follows from Lemma 2.10. □

3 Linear 2-hued colorings of graphs with bounded average degree

In this section, we shall determine the linear 2-hued chromatic number of graphs with maximum degree at most 4, or with maximum subgraph average degree not too large.

Let G be a graph with $V = V(G)$, and let $V' \subseteq V$ be a vertex subset. As in [2], $G[V']$ is the subgraph of G induced by V'. A mapping $c : V' \mapsto \bigcup_{v \in V(G)} L(v)$ is a **partial linear (L,r)-coloring** of G if c is a linear (L,r)-coloring of $G[V']$. The set $C = \bigcup_{v \in V(G)} L(v)$ is referred to as the color set. The subgraph $G[V']$ is the **support** of the partial linear (L,r)-coloring c. Suppose that c is a partial linear coloring of a graph G with support G' using the color set C. For convenience, we also refer V' as the support of c. If a vertex u in G is not in the support of c, then we define $c(v) = \{\emptyset\}$. For each vertex $v \in V(G)$, we use $c^2_G(v)$ to denote the subset of colors each of which appears exactly twice on $N_G(v)$ under c. This notation will be used frequently throughout this section. We start with some lemmas and former results.

Lemma 3.1. *Let $k, r > 0$ be integers, G be a graph with minimum degree $\delta = \delta(G)$, and let L be a k-list of G. If $\delta \geq 2r - 1$, then each of the following holds.*
(i) Every linear k-coloring of G is also a linear (k, r)-hued coloring of G. Consequently, $\chi_r^\ell(G) = \chi^\ell(G)$.
(ii) Every linear L-coloring of G is also a linear (L, r)-hued coloring of G. Consequently, $\chi_{L,r}^\ell(G) = \chi_L^\ell(G)$.

Proof. By Proposition 2.1, $\chi_r^\ell(G) \geq \chi^\ell(G)$ and $\chi_{L,r}^\ell(G) \geq \chi_L^\ell(G)$. Suppose that G has a linear k-coloring c. Assume further that when L is given, c is a linear L-coloring of G. By the definition of linear coloring, $|c(N_G(v))| \geq \lceil \frac{d(v)}{2} \rceil \geq \lceil \frac{\delta}{2} \rceil \geq \lceil \frac{2r-1}{2} \rceil \geq r = \min\{d(v), r\}$. It follows that c is also an r-hued coloring of G, and so $\chi_r^\ell(G) = \chi^\ell(G)$, and $\chi_{L,r}^\ell(G) = \chi_L^\ell(G)$. □

Theorem 3.2. *(Liu and Yu, Theorem 2 of [9]) If G is a graph with $\Delta \leq 3$ which has no component isomorphic to $K_{3,3}$ or C_5, then $\chi_L^\ell(G) \leq 4$.*

Lemma 3.3. *[5] If G is a graph with maximum degree $\Delta \leq 4$, then $\chi^\ell(G) \leq 8$.*

Theorem 3.4. *If G is a graph with maximum degree $\Delta \leq 4$, then $\chi_2^\ell(G) \leq 8$.*

Proof. We prove the theorem by induction on $|V(G)|$. If $|V(G)| \leq 8$, the result holds trivially. Assume that G is a graph with $\Delta(G) \leq 4$ and $|V(G)| \geq 9$. If $3 \leq \delta(G) \leq 4$, then by Lemmas 3.1 and 3.3, $\chi_2^\ell(G) = \chi^\ell(G) \leq 8$. Hence we assume that $\delta \leq 2$.

If $D_1(G) \neq \emptyset$, then pick $v \in D_1(G)$, let u be the only neighbor of v in G and $G' = G - v$. Thus $\Delta(G') \leq \Delta(G) \leq 4$, and

$|V(G')| < |V(G)|$. By induction, G' has a linear $(8,2)$-coloring $c : V(G) \mapsto \bar{8}$. Since $\Delta(G) \leq 4$, $|c(u) \cup c^2_{G'}(u)| \leq 1 + \Delta(G)/2 = 3$, and so we pick a color $c(v) \in \bar{8} - (c(u) \cup c^2_{G'}(u))$ and extend c from $V(G')$ to $V(G)$. By the choice of $c(v)$ in this case and by definition, c is a linear $(8,2)$-coloring.

Hence we may assume that $\delta = 2$. Let $v \in D_2(G)$, and let x and y be the neighbors of v. Define H to be the graph obtained from $G - v$ by adding a new edge xy if it does not already exist. By the definition of H, $\Delta(H) = \Delta(G) \leq 4$ and $|V(H)| < |V(G)|$. By induction, there exists a linear $(8,2)$-coloring $c : V(H) \mapsto \bar{8}$ with $c(x) \neq c(y)$. Since $\Delta \leq 4$, we have $|\{c(x), c(y)\} \cup c^2_H(x) \cup c^2_H(y)| \leq 2 + 2 \cdot \frac{\Delta}{2} = 6$, and so we can extend c by setting $c(v) \in \bar{8} \setminus \{c(x), c(y)\} \cup C_2(x) \cup C_2(y)$. By definition, the extended c is a linear (8,2)-coloring of G. This completes the proof of the theorem. □

Let G be a graph. The **maximum subgraph average degree of** G, denoted by $Mad(G)$, is defined by

$$\text{Mad}(G) = \max\{\frac{2|E(H)|}{|V(H)|}, H \subseteq G\}.$$

To prove the next result, we need some more lemmas.

Lemma 3.5. *Let $k \geq 3$ and $r > 0$ be integers, and G be a graph with maximum degree Δ. Suppose that $v \in D_1(G)$ and $k \geq \max\{\lceil \Delta/2 \rceil + 1, r + 1\}$. Each of the following holds.*
(i) If $\chi^\ell_r(G - v) = k$, then $\chi^\ell_r(G) \leq k$.
(ii) If $\chi^\ell_{L,r}(G - v) = k$, then $\chi^\ell_{L,r}(G) \leq k$.

Proof. We shall prove (ii). The argument for the proof of (i) is similar and will be omitted. Let u be the only neighbor of v in G, $G' = G - v$, $N_{G'}(u)$ denote the set of vertices adjacent to u in G'. Suppose that L is a k-list of G and let c be linear (L,r)-coloring of G'. If $d_G(u) \geq r + 1$, then $|c(N_{G'}(u)| \geq r$. And

$|c^2_{G'}(u)| \leq \frac{d_G(u)-1}{2} \leq \frac{\Delta}{2} - \frac{1}{2} \leq \lceil \Delta/2 \rceil - \frac{1}{2} \leq k-2$. Extend c to a k-coloring of G by defining $c(v) \in L(v) - (c^2_{G'}(u) \cup \{c(u)\})$. This is possible as $|L(v)| \geq k > |c^2_{G'}(u) \cup \{c(u)\}|$. By the choice of $c(v)$ and since c is a linear coloring of G', we conclude that c is a linear (L,r)-coloring of G. Otherwise, $d_G(u) \leq r$. Then since c is an (L,r)-coloring of G', by (C2), we have $|c(N_{G'}(u))| = d_G(u) - 1 \leq r-1$ and $c^2_{G'}(u) = \emptyset$. Extend c to a k-coloring of G by defining $c(v) \in L(v) - (c(N_{G'}(u)) \cup \{c(u)\})$. This is possible as $|L(v)| \geq k > |c(N_{G'}(u)) \cup \{c(u)\}|$. By the choice of $c(v)$, we have $|c(N_G(u))| = d_G(u)$, and so (C2) is also satisfied. This justifies the lemma. □

For any path $P = v_0v_1v_2 \cdots v_p$, $P^o = V(P) - \{v_0, v_p\}$ denote the set of all internal vertices of P. A path P of G is a **divalent path** of G if $V(P) \subseteq G[D_2(G)]$ and $v_0 \neq v_p$; and is **internally divalent** if $P^o \subseteq D_2(G)$. We shall take the following convention in our arguments below: If L is a k-list of G, then for a subgraph G' of G, we also use L to denote the restriction of L to $V(G')$.

Lemma 3.6. *Let G be a graph with maximum degree Δ. Suppose that $v \in D_1(G)$ and $k, q, r > 0$ be integers such that $k \geq \max\{\lceil \Delta/2 \rceil + q, r+q, 5\}$. Let L be a k-list of G. Let $P = u_1u_2 \cdots u_p$ be a divalent path of G satisfying $p + q = 5$, and let $G' = G - V(P)$. If $1 \leq q \leq 3$, then each of the following holds.*
(i) If $\chi^\ell_r(G')) = k$, then $\chi^\ell_r(G) \leq k$.
(ii) If $\chi^\ell_{L,r}(G') = k$, then $\chi^\ell_{L,r}(G) \leq k$.

Proof. We only prove Part (ii), as the proof for Part (i) is similar, and will be omitted.

Let $Q = u_0u_1u_2 \cdots u_pu_{p+1}$ be a path in G such that $P = Q - \{u_0, u_{p+1}\}$ is a divalent path of G, and let $G' = G - V(P)$. Assume that c is a linear (L,r)-coloring of G'. We then will extend c to a linear (L,r)-coloring of G.

Recall that $c^2_{G'}(v)$ is the set of colors that occur twice in $c(N_{G'}(v))$. Since c is a linear coloring of G', for $v \in \{u_0, u_{p+1}\}$, we have $|c^2_{G'}(v)| \le \frac{d_{G'}(v)}{2} \le \frac{d_G(v)-1}{2} \le \lceil \Delta/2 \rceil - \frac{1}{2}$. As $\lceil \Delta/2 \rceil$ is an integer, we have $|c^2_{G'}(v)| \le \lceil \Delta/2 \rceil - 1$.

In the following, for each case of $1 \le q \le 3$, we will define an extension of c to a coloring (also denoted by c, for notational convenience) of G. After this is done, we shall show that the extended c is indeed a linear (L, r)-coloring of G.

By assumption,

$$|L(u_1)| \ge k \ge (\lceil \Delta/2 \rceil - 1) + 5 - p \ge |c^2_{G'}(v)| + 5 - p. \qquad (5)$$

Assume that $q = 1$, and so $p = 4$ and $|L(u_1)| \ge |c^2_{G'}(u_0) \cup \{c(u_0)\}|$. Define,

$$c(u_1) \in \begin{cases} L(u_1) - (c^2_{G'}(u_0) \cup \{c(u_0)\}) & \text{if } d_G(u_0) \ge r+1, \\ L(u_1) - (c(N_{G'}(u_0)) \cup \{c(u_0)\}) & \text{if } d_G(u_0) \le r. \end{cases}$$

As $d_G(u_0) \ge r + 1$ implies $|c(N_{G'}(u_0))| \ge r$, and as $d_G(u_0) \le r$ implies $|c(N_{G'}(u_0))| = d_G(u_0) - 1 \le r - 1$ by (C2). All vertices in $N_G(u_0) \cup \{u_0\}$ satisfies the r-hued condition (C2). Similarly, we define

$$c(u_4) \in \begin{cases} L(u_4) - (c^2_{G'}(u_5) \cup \{c(u_5)\}) & \text{if } d_G(u_5) \ge r+1, \\ L(u_4) - (c(N_{G'}(u_5)) \cup \{c(u_5)\}) & \text{if } d_G(u_5) \le r. \end{cases}$$

After $c(u_1), c(u_4)$ are defined, by $k \ge 5$, we choose $c(u_2) \in L(u_2) - \{c(u_0), c(u_1), c(u_4)\}$, and then $c(u_3) \in L(u_3) - \{c(u_1), c(u_2), c(u_4), c(u_5)\}$.

Assume that $q = 2$, and so $p = 3$ and $|L(u_1)| \ge |c^2_{G'}(u_0) \cup \{c(u_0)\}|$. Define

$$c(u_1) \in \begin{cases} L(u_1) - (c^2_{G'}(u_0) \cup \{c(u_0)\}) & \text{if } d_G(u_0) \ge r+1, \\ L(u_1) - (c(N_{G'}(u_0) \cup \{c(u_0)\}) & \text{if } d_G(u_0) \le r. \end{cases}$$

Similarly, we define

$$c(u_3) \in \begin{cases} L(u_3) - (c^2_{G'}(u_4) \cup \{c(u_1), c(u_4)\}) & \text{if } d_G(u_0) \geq r+1, \\ L(u_3) - (c(N_{G'}(u_4)) \cup \{c(u_1), c(u_4)\}) & \text{if } d_G(u_0) \leq r. \end{cases}$$

After $c(u_1), c(u_3)$ are defined, by $k \geq 5$, we choose $c(u_2) \in L(u_2) - \{c(u_0), c(u_1), c(u_3), c(u_4)\}$.

Assume that $q = 3$, and so $p = 2$ and $|L(u_1)| \geq |c^2_{G'}(u_0) \cup \{c(u_0), c(u_3)\}|$. Define

$$c(u_1) \in \begin{cases} L(u_1) - (c^2_{G'}(u_0) \cup \{c(u_0), c(u_3)\}) & \text{if } d_G(u_0) \geq r+1, \\ L(u_1) - (c(N_{G'}(u_0)) \cup \{c(u_0), c(u_3)\}) & \text{if } d_G(u_0) \leq r. \end{cases}$$

and

$$c(u_2) \in \begin{cases} L(u_2) - (c^2_{G'}(u_3) \cup \{c(u_0), c(u_1), c(u_3)\}) & \text{if } d_G(u_2) \geq r+1, \\ L(u_2) - (c(N_{G'}(u_3)) \cup \{c(u_0), c(u_1), c(u_3)\}) & \text{if } d_G(u_2) \leq r. \end{cases}$$

In any case, as $k \geq \max\{\lceil \Delta/2 \rceil + q, r + q, 5\}$, the extended colorings of G are possible. In addition, we have $|c(V(P))| = |V(P)| \geq 2$ and as $c(u_0), c(u_{p+1}) \notin c(V(P))$. As c is a linear coloring of G', by definition, the extended c is a linear coloring of G. Similarly, as c is an (L, r)-coloring of G', the extended c is an (L, r)-coloring of G. This proves the lemma. □

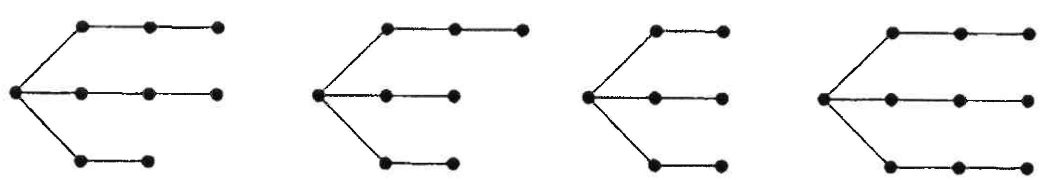

Figure A: Examples of Y_{s_1,s_2,s_3}

For integers $s_1 \geq s_2 \geq s_3 \geq 1$, let Y_{s_1,s_2,s_3} be the graph obtained from disjoint paths P_{s_1+1}, P_{s_2+1} and P_{s_3+1} by identifying an end vertex of each of these three paths. (See examples depicted in Figure A). The only vertex of degree 3 in a Y_{s_1,s_2,s_3} is called the center of it.

Let $q, r > 0$ be integers with $q = 1, 2, 3$. Define

$$k^* = \begin{cases} \max\{\lceil \Delta/2 \rceil + q, r + q, 6\}, & \text{if } q = 1, 2; \\ \max\{\lceil \Delta/2 \rceil + 3, r + 3, 7\}, & \text{otherwise.} \end{cases}$$

Lemma 3.7. *Let $k, q, r > 0$ be integers with $k \geq k^*$ and G be a graph with maximum degree Δ. Let L be a k-list of G. For $q = 1, 2, 3$, Let Y_q denote a subgraph of G with center w_0 and $D_1(Y_q) = \{w_1, w_2, w_3\}$ such that Y_1 is isomorphic to $Y_{3,3,2}$, Y_2 is isomorphic to $Y_{3,2,2}$, Y_3 is isomorphic to $Y_{2,2,2}$ and Y_4 is isomorphic to $Y_{3,3,3}$, and such that, for $1 \leq j \leq 4$, every (w_0, w_j)-path of Y_q is an internally divalent path of G. Let $G' = G - V(Y_q - D_1(Y_q))$. Then for $q = 1, 2, 3$, each of the following holds.*
(i) If $\chi^{\ell}_r(G') = k$, then $\chi^{\ell}_r(G) \leq k$.
(ii) If $\chi^{\ell}_{L,r}(G') = k$, then $\chi^{\ell}_{L,r}(G) \leq k$.

Proof. We shall prove Part (ii) only as the proof for Part (i) is similar and will be omitted. For each value $q = 1, 2, 3$, let c be a linear (L, r)-coloring of G'. We shall extend c to a linear (L, r)-coloring, also denoted by c, of G, to prove the lemma. For $1 \leq i \leq 3$, let $P_i = w_0 u^i_1 \cdots u^i_{s_i - 1} w_i$ be a path in Y_{s_1,s_2,s_3} such that $u^i_j \in D_2(G)$ and $V(P_i) \cap V(P_j) = \{w_0\}$, $1 \leq i < j \leq 3$. By assumption, $k > (\lceil \Delta/2 \rceil - 1) + 1$.

Assume that $q = 1$. Then as $|c^2_{G'}(w_1)| \leq \lceil \Delta/2 \rceil - 1$, $|L(u^1_2)| \geq k > |c^2_{G'}(w_1) \cup \{c(w_1)\}|$. Define

$$c(u^1_2) \in \begin{cases} L(u^1_2) - (c^2_{G'}(w_1) \cup \{c(w_1)\}) & \text{if } d_G(w_1) \geq r + 1, \\ L(u^1_2) - (c(N_{G'}(w_1)) \cup \{c(w_1)\}) & \text{if } d_G(w_1) \leq r. \end{cases} \tag{6}$$

Similarly, as $k \geq \max\{\lceil \Delta/2 \rceil + 1, r+1, 6\}$, define

$$c(u_2^2) \in \begin{cases} L(u_2^2) - (c_{G'}^2(w_2) \cup \{c(w_2)\}) & \text{if } d_G(w_2) \geq r+1, \\ L(u_2^2) - (c(N_{G'}(w_2)) \cup \{c(w_2)\}) & \text{if } d_G(w_2) \leq r, \end{cases}$$

$$c(u_1^3) \in \begin{cases} L(u_1^3) - (c_{G'}^2(w_3) \cup \{c(w_3)\}) & \text{if } d_G(w_3) \geq r+1, \\ L(u_1^3) - (c(N_{G'}(w_3)) \cup \{c(w_3)\}) & \text{if } d_G(w_3) \leq r, \end{cases}$$

and $c(w_0) \in L(w_0) - \{c(u_2^1), c(u_2^2), c(u_1^3), c(w_3)\}$. After $c(u_2^1)$, $c(u_2^2)$, $c(u_1^3)$ and $c(w_0)$ are defined, we choose $c(u_1^1) \in L(u_1^1) - \{c(u_2^1), c(u_1^3), c(w_1), c(w_0)\}$, and $c(u_1^2) \in L(u_1^2) - \{c(u_2^2), c(u_1^1), c(w_2), c(u_1^3), c(w_0)\}$.

Assume that $q = 2$. Since $|L(u_2^1)| \geq k > (\lceil \Delta/2 \rceil - 1) + 1 \geq |c_{G'}^2(w_1) \cup \{c(w_1)\}|$, we can define $c(u_2^1)$ the same as in (7). Similarly, define

$$c(u_1^2) \in \begin{cases} L(u_1^2) - (c_{G'}^2(w_2) \cup \{c(w_2)\}) & \text{if } d_G(w_2) \geq r+1, \\ L(u_1^2) - (c(N_{G'}(w_2)) \cup \{c(w_2)\}) & \text{if } d_G(w_2) \leq r, \end{cases}$$

$$c(u_1^3) \in \begin{cases} L(u_1^3) - (c_{G'}^2(w_3) \cup \{c(w_3), c(u_1^2)\}) & \text{if } d_G(w_3) \geq r+1, \\ L(u_1^3) - (c(N_{G'}(w_3)) \cup \{c(w_3), c(u_1^2)\}) & \text{if } d_G(w_3) \leq r, \end{cases}$$

After $c(u_2^1)$, $c(u_1^2)$ and $c(u_1^3)$ are defined, as $k \geq \max\{\lceil \Delta/2 \rceil + 2, r+2, 6\}$, we can find $c(w_0) \in L(w_0) - \{c(u_2^1), c(u_1^2), c(u_1^3), c(w_2), c(w_3)\}$ and $c(u_1^1) \in L(u_1^1) - \{c(w_0), c(w_1), c(u_2^1), c(u_1^2), c(u_1^3)\}$ to complete the extension of c.

Assume that $q = 3$. We define

$$c(u_1^1) \in \begin{cases} L(u_1^1) - (c_{G'}^2(w_1) \cup \{c(w_1)\}) & \text{if } d_G(w_1) \geq r+1, \\ L(u_1^1) - (c(N_{G'}(w_1)) \cup \{c(w_1)\}) & \text{if } d_G(w_1) \leq r, \end{cases}$$

$$c(u_1^2) \in \begin{cases} L(u_1^2) - (c_{G'}^2(w_2) \cup \{c(w_2), c(u_1^1)\}) & \text{if } d_G(w_2) \geq r+1, \\ L(u_1^2) - (c(N_{G'}(w_2)) \cup \{c(w_2), c(u_1^1)\}) & \text{if } d_G(w_2) \leq r, \end{cases}$$

$$c(u_1^3) \in \begin{cases} L(u_1^3) - (c_{G'}^2(w_3) \cup \{c(w_3), c(u_1^1), c(u_1^2)\}) & \text{if } d_G(w_3) \geq r+1, \\ L(u_1^3) - (c(N_{G'}(w_3)) \cup \{c(w_3), c(u_1^1), c(u_1^2)\}) & \text{if } d_G(w_3) \leq r, \end{cases}$$

As $k \geq 7$, we choose $c(w_0) \in L(w_0) - \{c(u_1^1), c(u_1^2), c(u_1^3), c(w_1), c(w_2), c(w_3)\}$.

Assume that $q = 4$. For $j = 1, 2, 3$, as $|c^2_{G'}(w_j)| \leq \lceil \Delta/2 \rceil - 1$, $|L(u_2^j)| \geq k > |c^2_{G'}(w_j) \cup \{c(w_j)\}|$, define

$$c(u_2^j) \in \begin{cases} L(u_2^j) - (c^2_{G'}(w_j) \cup \{c(w_j)\}) & \text{if } d_G(w_j) \geq r+1, \\ L(u_2^j) - (c(N_{G'}(w_j)) \cup \{c(w_j)\}) & \text{if } d_G(w_j) \leq r. \end{cases} \tag{7}$$

Next, we pick $c(w_0) \in L(w_0) - \{c(u_2^1), c(u_2^2), c(u_2^3)\}$. After $c(u_2^1)$, $c(u_2^2)$, $c(u_2^3)$ and $c(w_0)$ are defined, we choose $c(u_1^1) \in L(u_1^1) - \{c(u_2^1), c(w_1), c(w_0)\}$, $c(u_1^2) \in L(u_1^2) - \{c(u_2^2), c(u_1^1), c(w_2), c(w_0)\}$. and $c(u_1^3) \in L(u_1^3) - \{c(u_2^3), c(u_1^1), c(u_1^2), c(w_3), c(w_0)\}$.

Now in any case when $q = 1, 2, 3, 4$, we have obtained an extended (L, r)-coloring c of G. Since c is a linear coloring of G' and since, for each $1 \leq i \leq 4$, $|\{c(w_0), c(u_1^i), c(u_2^i)\}| = 3$, it follows by definition that the extended c is a linear coloring of G. Since c satisfies (C2) in G', by the definition of the extended c, the extended c also satisfied (C2). Hence c is a linear (L, r)-coloring of G. □

Theorem 3.8. *Let G be a graph with maximum degree Δ:*
(i) If $\Delta \geq 9$ and $Mad(G) < \frac{7}{3}$, then $\chi^{\ell}_{L,r}(G) \leq \max\{\lceil \Delta/2 \rceil + 1, r+1\}$.
(ii) If $\Delta \geq 7$ and $Mad(G) < \frac{12}{5}$, then $\chi^{\ell}_{L,r}(G) \leq \max\{\lceil \Delta/2 \rceil + 2, r+2\}$.
(iii) If $\Delta \geq 7$ and $Mad(G) < \frac{5}{2}$, then $\chi^{\ell}_{L,r}(G) \leq \max\{\lceil \Delta/2 \rceil + 3, r+3\}$.

Since every planar or projective-planar graph G with girth $g(G)$ verifies $\mathrm{Mad}(G) < 2g(G)/(g(G) - 2)$, we obtain the following corollary:

Corollary 3.9. *Let G be a planar or projective-planar graph with maximum degree Δ:*

(i) If $\Delta \geq 9$ and $g(G) \geq 14$, then $\chi^{\ell}_{L,r}(G) \leq \max\{\lceil \Delta/2 \rceil + 1, r + 1\}$.
(ii) If $\Delta \geq 7$ and $g(G) \geq 12$, then $\chi^{\ell}_{L,r}(G) \leq \max\{\lceil \Delta/2 \rceil + 2, r + 2\}$.
(iii) If $\Delta \geq 7$ and $g(G) \geq 10$, then $\chi^{\ell}_{L,r}(G) \leq \max\{\lceil \Delta/2 \rceil + 3, r + 3\}$.

Proof of Theorem 3.8. We argue by contradiction and assume that

$$G \text{ be a counterexample to the theorem with } |V(G)| \text{ minimized.} \tag{8}$$

By (8) and Lemma 3.5, we may assume that $\delta(G) \geq 2$.
(i) Since G is a counterexample, there exists a k-list with $k \geq \max\{\lceil \Delta/2 \rceil + 1, r + 1\} \geq 6$ such that G does not have a linear (L, r)-coloring.

Claim 1. *Each of the following holds.*
(C1.1) G does not have a divalent path of length 3.
(C1.2) G does not have a divalent path of length 2 with one of the endpoints being adjacent to a vertex of degree at most 4.
(C1.3) G does not have an induced subgraph H_1 consisting of three internally divalent paths P_1, P_2 and P_3, such that $|E(P_1)| = |E(P_2)| = 3$, $|E(P_3)| \in \{2, 3\}$ and such that for some $w \in D_3(G)$, and for any $1 \leq i < j \leq 3$, $V(P_i) \cap V(P_j) = \{w\}$.

Proof. (C1.1) If G contains a divalent path $P_4 = v_1v_2v_3v_4$, then let $G' = G - V(P_4)$. As G' is a subgraph of G, we have $\text{Mad}(G') < \frac{16}{7}$. By (8), G' has a linear (L, r)-coloring. By Lemma 3.6, G has a linear (L, r)-coloring of G, contrary to (8).
(C1.2) Suppose that G contains a path $P = v_0v_1v_2v_3v_4$ such that $P_3 = v_1v_2v_3$ is a divalent path with $d(v_0) \leq 4$. Let $G' = G - \{v_1, v_2\}$. As G' is a subgraph of G, we have $\text{Mad}(G') < \frac{16}{7}$. By (8), G' has a linear (L, r)-coloring. Since $k \geq 6$ and $d(v_0) \leq$

4, we can find $c(v_1) \in L(v_1) - \{c(N(v_0) \setminus \{v_1\}), c(v_0), c(v_3)\}$, $c(v_2) \in L(v_2) - \{c(v_0), c(v_1), c(v_3), c(v_4)\}$. By definition, the extended c is a linear (L, r)-coloring of G, contrary to (8).
(C1.3) Suppose that G has such an induced H_1, which is isomorphic to a $Y_{3,3,2}$ or a $Y_{3,3,3}$. It follows from (8) and Lemma 3.7 that G would have a linear (L, r)-coloring of G, contrary to (8). This proves Claim 1. □

For $v \in V(G)$, $n_2(v) = |D_2(G) \cap N_G(v)|$. For a vertex $u \in D_2(G)$ and for $i = 0, 1, 2$, u is called a **Type-$(i+1)$ vertex** if $n_2(u) = i$. We complete the proof by applying a discharging method to find a contradiction. Set the initial charge $w_0(v) = d(v)$ for each vertex $v \in V(G)$. We then apply the following discharging rules:

R1 Each 3^+-vertex gives $\frac{1}{6}$ to each adjacent Type-1 vertex and $\frac{1}{3}$ to each adjacent Type-2 vertex.

R2 Each 5^+-vertex gives $\frac{1}{6}$ to each adjacent Type-3 vertex via the adjacent 2-vertex.

Let $w(v)$ denote the new charge at a vertex v after carrying out these discharging rules. We shall show that $w(v) \geq \frac{7}{3}$ for every $v \in V(G)$ by analyzing the following cases.

(1) $d(v) = 2$. Let $N(v) = \{x, y\}$ with $d(x) \leq d(y)$. If v is a Type-1 vertex, then each of x and y gives $\frac{1}{6}$ to v by R1, and so $w(v) \geq d(v) + 2 \cdot \frac{1}{6} = \frac{7}{3}$. If v is a Type-2 vertex, then $d(y) \geq 3$ and y gives $\frac{1}{3}$ to v by R1. Hence, $w(v) \geq d(v) + \frac{1}{3} = \frac{7}{3}$. If v is a Type-3 vertex, then $d(x) = d(y) = 2$ and each of x and y is adjacent to a 5^+-vertex by (C1.2). It follows that $w(v) \geq d(v) + 2 \cdot \frac{1}{6} = \frac{7}{3}$ by R2.

(2) $d(v) = 3$. By (C1.3), v is adjacent to at most two Type-2 vertices. If v is adjacent to at most one Type-2 vertex, then $w(v) \geq d(v) - \frac{1}{3} - 2 \cdot \frac{1}{6} = \frac{7}{3}$ by R1. Otherwise, v is not adjacent to any Type-1 vertex by (C1.3). Hence, $w(v) \geq d(v) - 2 \cdot \frac{1}{3} = \frac{7}{3}$

by R1.
(3) $d(v) \geq 4$. If $d(v) = 4$, then v gives at most four times of $\frac{1}{3}$ by R1 and so $w(v) \geq d(v) - 4 \cdot \frac{1}{3} = \frac{8}{3}$. If $d(v) \geq 5$, then $w(v) \geq d(v) - d(v) \cdot (\frac{1}{3} + \frac{1}{6}) = \frac{d(v)}{2} \geq \frac{5}{2}$ by R1 and R2.

Therefore, $w(v) \geq \frac{7}{3}$ for every vertex. Since

$$\sum_{v \in V(G)} w(v) = \sum_{v \in V(G)} w_0(v) = \sum_{v \in V(G)} d(v) = 2|E(G)|,$$

we have $\text{Mad}(G) \geq \dfrac{2|E(G)|}{|V(G)|} = \dfrac{\sum_{v \in V(G)} w(v)}{|V(G)|} \geq \dfrac{7}{3}$, contrary to the assumption that $\text{Mad}(G) < \frac{7}{3}$.

(ii) Since G is a counterexample, there exists a k-list with $k \geq \max\{\lceil \Delta/2 \rceil + 2, r + 2\} \geq 6$ such that G does not have a linear (L, r)-coloring. Using arguments similar to those in the proof of Claim 1, we have the following claim.

Claim 2. *Each of the following holds.*
(C2.1) G does not have a divalent path of length 2.
(C2.2) G does not have an induced subgraph H_2 consisting of three internally divalent paths P_1, P_2 and P_3, such that $|E(P_1)| = 3$, $|E(P_2)| = |E(P_3)| = 2$ and such that for some $w \in D_3(G)$, and for any $1 \leq i < j \leq 3$, $V(P_i) \cap V(P_j) = \{w\}$.

Once again we set the initial charge $w_0(v) = d(v)$ for each vertex v and apply the following discharging rule.
R3 Each 3^+-vertex gives $\frac{1}{5}$ to each adjacent Type-1 vertex and $\frac{2}{5}$ to Type-2 vertex.

Let $w(v)$ denote the new charge after recharging by R3. We will show that $w(v) \geq \frac{12}{5}$ for all $v \in V(G)$.
(1) If $d(v) = 2$, then $n_2(v) \leq 1$ by (C2.1). If $n_2(v) = 1$, then v is a Type-2 vertex and receives $\frac{2}{5}$ from the adjacent 3^+-vertex by R3. If $n_2(v) = 0$, then v is of Type-1 and receives two times $\frac{1}{5}$

from the adjacent 3^+-vertices by R3. Thus, $w(v) \geq d(v) + \frac{2}{5} = \frac{12}{5}$.
(2) If $d(v) = 3$, then v is adjacent to at most one Type-2 vertex by (C2.2). If v is adjacent to one Type-2 vertex, then v is adjacent to at most one Type-1 vertex. It follows by R3 that $w(v) \geq d(v) - \frac{2}{5} - \frac{1}{5} = \frac{12}{5}$. Otherwise, v is adjacent to at most three Type-1 vertices, and so by R3 $w(v) \geq d(v) - 3 \times \frac{1}{5} = \frac{12}{5}$.
(3) If $d(v) \geq 4$, then v is adjacent to at most $d(v)$ 2-vertices and so v discharges at most $d(v) \cdot \frac{2}{5}$ to adjacent 2-vertices. Thus, $w(v) \geq d(v) - d(v) \cdot \frac{2}{5} = \frac{3}{5}d(v) \geq \frac{12}{5}$.

In any case, $w(v) \geq \frac{12}{5}$ for every $v \in V(G)$. Since

$$\sum_{v \in V(G)} w(v) = \sum_{v \in V(G)} w_0(v) = \sum_{v \in V(G)} d(v) = 2|E(G)|,$$

we have obtained a contradiction:

$$\frac{12}{5} > \text{Mad}(G) \geq \frac{2|E(G)|}{|V(G)|} = \frac{\sum_{v \in V(G)} w(v)}{|V(G)|} \geq \frac{12}{5}.$$

(iii) Since G is a counterexample, there exists a k-list with $k \geq \max\{\lceil \Delta/2 \rceil + 3, r + 3\} \geq 7$ such that G does not have a linear (L, r)-coloring. Again by a similar argument in the proof of Claim 1, we have the following claim.

Claim 3. *Each of the following holds.*
(C3.1) G does not have a divalent path of length 1.
(C3.2) G does not have an induced subgraph H_3 consisting of three internally divalent paths P_1, P_2 and P_3, such that $|E(P_1)| = |E(P_2)| = |E(P_3)| = 2$ and such that for some $w \in D_3(G)$, and for any $1 \leq i < j \leq 3$, $V(P_i) \cap V(P_j) = \{w\}$.

We start with our initial charge $w_0(v) = d(v)$ for each vertex $v \in V(G)$, and then apply the following discharging rule.
R4 Each 3^+-vertex gives $\frac{1}{4}$ to each adjacent 2-vertex.

Let $w(v)$ be the new charge after discharging rule R4. We will show that $w(v) \geq \frac{5}{2}$ for every $v \in V(G)$.
(1) If $d(v) = 2$, then v is adjacent to two 3^+-vertices by (C3.1) and receives two times $\frac{1}{4}$ from the adjacent 3^+-vertices by R4. It follows that $w(v) \geq 2 + 2 \cdot \frac{1}{4} = \frac{5}{2}$.
(2) If $d(v) = 3$, then v is adjacent to at most two 2-vertices by (C3.2). By R4, $w(v) \geq 3 - 2 \cdot \frac{1}{4} = \frac{5}{2}$.
(3) If $d(v) \geq 4$, then v is adjacent to at most $d(v)$ 2-vertices, so $w(v) \geq d(v) - d(v) \cdot \frac{1}{4} = \frac{3}{4}d(v) \geq 3$.

Therefore, in any case, $w(v) \geq \frac{5}{2}$ for every vertex. Since

$$\sum_{v \in V(G)} w(v) = \sum_{v \in V(G)} w_0(v) = \sum_{v \in V(G)} d(v) = 2|E(G)|,$$

we have obtained a contradiction:

$$\frac{5}{2} > \text{Mad}(G) \geq \frac{2|E(G)|}{|V(G)|} = \frac{\sum_{v \in V(G)} w(v)}{|V(G)|} \geq \frac{5}{2}.$$

This completes the proof of the theorem.

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