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ARTICLE

Supereulerian bipartite digraphs

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Abstract

A digraph D is supereulerian if D has a spanning closed ditrail. Bang-Jensen and Thomassé conjectured that if the arc-strong connectivity $\lambda(D)$ of a digraph D is not less than the independence number $\alpha(D)$, then D is supereulerian. A digraph is bipartite if its underlying graph is bipartite. Let $\alpha'(D)$ be the size of a maximum matching of D. We prove that if D is a bipartite digraph satisfying $\lambda(D) \geq \lfloor \frac{\alpha'(D)}{2} \rfloor + 1$, then D is supereulerian. Consequently, every bipartite digraph D satisfying $\lambda(D) \geq \lfloor \frac{\alpha(D)}{2} \rfloor + 1$ is supereulerian. The bound of our main result is best possible.

KEYWORDS

arc-strong connectivity, eulerian digraph, independence number, matching number, supereulerian bipartite digraph

1 | INTRODUCTION

We consider finite graphs and finite digraphs. Undefined terms and notation will follow [9] for graphs and [5] for digraphs. In particular, $\kappa(G), \lambda(G), \alpha(G)$, and $\alpha'(G)$ denote the connectivity, the edge connectivity, the independence number, the matching number of a graph G, respectively; and $\kappa(D)$ and $\lambda(D)$ denote the vertex-strong connectivity and the arc-strong connectivity of a digraph D. Throughout this article, we use the notation (u, v) to denote an arc oriented from u to v in a digraph; and use $[u, v]$ to denote either (u, v) or (v, u). We often use $G(D)$ to denote the underlying undirected graph of D, the graph obtained from D by erasing all orientation on the arcs of D. A digraph D is called a **bipartite digraph** if $G(D)$ is a bipartite graph. Let D be a bipartite digraph with a vertex bipartition (X, Y) and with $|X| = a$ and $|Y| = b$. If for every $x \in X$ and $y \in Y$, both $(x, y), (y, x) \in A(D)$, then D is a **complete bipartite digraph**, and is denoted by $K(a, b)$ in this article. The independence number and matching number of a digraph D are defined respectively as follows

$$\alpha(D) = \alpha(G(D)) \quad \text{and} \quad \alpha'(D) = \alpha'(G(D)).$$

Let M be a matching of D, if $V(M) = V(D)$, then M is called a **perfect matching**. Following [5], for a digraph D with $X, Y \subseteq V(D)$, define

$$(X,Y)_D = \{(x,y) \in A(D) : \ x \in X, y \in Y\}.$$

When $Y = V(D) - X$, we define

$$\partial_D^+(X) = (X, V(D) - X)_D \quad \text{and} \quad \partial_D^-(X) = (V(D) - X, X)_D.$$

When $X \subset V(D)$, we denote by $D[X]$ the subdigraph induced by X. For a vertex $v \in V(D)$, $d_D^+(v) = |\partial_D^+(\{v\})|$, and $d_D^-(v) = |\partial_D^-(\{v\})|$ are the **out-degree** and the **in-degree** of v in D, respectively. Define $d_D(v) = d_D^+(v) + d_D^-(v)$, $\delta^+(D) = \min\{d_D^+(v) : v \in V(D)\}$ and $\delta^-(D) = \min\{d_D^-(v) : v \in V(D)\}$. The **minimum semidegree** of D, is $\delta(D) := \min\{\delta^+(D), \delta^-(D)\}$. $\Delta^+(D) = \max\{d_D^+(v) : v \in V(D)\}$ and $\Delta^-(D) = \max\{d_D^-(v) : v \in V(D)\}$. Let $N_D^+(v) = \{u \in V(D) - v : (v,u) \in A(D)\}$ and $N_D^-(v) = \{u \in V(D) - v : (u,v) \in A(D)\}$ denote the **out-neighborhood** and **in-neighborhood** of v in D, respectively. We call the vertices in $N_D^+(v)$ and $N_D^-(v)$ are the **out-neighbors** and **in-neighbors** of v, respectively. Let $N_D(v) = N_D^+(v) \cup N_D^-(v)$. When the digraph D is understood from the context, we often omit the subscript D.

In 1977, Boesch et al. [8] proposed the supereulerian problem, which seeks to characterize graphs that have spanning eulerian subgraphs. They indicated that this problem would be very difficult. Pulleyblank [21] later in 1979 proved that determining whether a graph is supereulerian, even within planar graphs, is NP-complete. Since then, supereulerian graphs have been studied extensively, as seen in the survey by Catlin [10], and the updates [11,20].

It is natural to study supereulerian digraphs. A digraph D is **eulerian** if D is weakly connected and for every $v \in V(D)$, $d_D^+(v) = d_D^-(v)$; and is **supereulerian** if D contains a spanning eulerian subdigraph. Earlier studies were done by Gutin [14,15]. Some of the recent developments can be found in [1–3,6,16, 17], among others. In particular, the following have been proved. In [16], an infinite family $\mathcal{F}_0(k_1, k_2, 2)$ of nonsupereulerian digraphs have been constructed to show the sharpness of Theorem 1.1 below.

Theorem 1.1 (Hong et al., Theorem 3.4 of [16]). *Let D be a strong digraph of order n and minimum out-degree $\delta^+(D) \geq 4$ and minimum in-degree $\delta^-(D) \geq 4$. If $\delta^+(D) + \delta^-(D) \geq n - 4$, then the following are equivalent.*

- **(i)** *D has a spanning eulerian subdigraph.*
- **(ii)** *Either $\delta^+(D) + \delta^-(D) > n - 4$, or for some integer $k_1, k_2, \delta^+(D) = k_1, \delta^-(D) = k_2$ but $D \notin \mathcal{F}_0(k_1, k_2, 2)$.*

Theorem 1.2 (J. Bang-Jensen and A. Maddaloni, Theorem 2.4 of [6]). *Let D be a digraph. If $\lambda(D) \geq \alpha(D)$, then D has a spanning subdigraph H such that for any $v \in V(H)$, $d_H^+(v) = d_H^-(v) > 0$.*

Theorem 1.3 (J. Bang-Jensen and A. Maddaloni, Theorem 3.6 of [6]). *Let D be a strong digraph on n vertices. If $d_D(x) + d_D(y) \geq 2n - 3$ for any pair of nonadjacent vertices x and y, then D is supereulerian.*

A well-known theorem of Chvátal and Erdős states that if $|V(G)| \geq 3$ and if $\kappa(G) \geq \alpha(G)$, then G is hamiltonian [13]. Thomassen [22] gave an infinite family of nonhamiltonian (but supereulerian) digraphs such that $\kappa(D) = \alpha(D) = 2$, showing that the Chvátal–Erdős Theorem does not hold for digraphs. Bang-Jensen and Thomassé (2011, see [6]) proposed the following conjecture.

Conjecture 1.1. *Let D be a digraph. If $\lambda(D) \geq \alpha(D)$, then D is supereulerian.*

Theorem 1.2 is an effort toward this conjecture. In [6], Conjecture 1.1 has been verified for semicomplete multipartite digraphs and quasitransitive digraphs. In [6], Bang-Jensen and Maddaloni proved that if $\lambda(G) \geq \alpha(G)$ for a graph G, then G is supereulerian. In [1], Algefari and Lai proved that if $\lambda(D) \geq \alpha'(D)$ for a strong digraph D, then D is supereulerian.

It has been observed that when studying the supereulerian problem, the lower bound of a degree condition to warrant a supereulerian graph may be reduced by about half for bipartite graphs, as seen in Theorem 5 [19]. This motivates the current research. The main goal of this article, stated below, is to prove that Conjecture 1.1 holds for bipartite digraphs with the lower bound being about half of the conjectured bound.

Theorem 1.4. *Let D be a strong bipartite digraph with a vertex bipartition (X, Y) satisfying $|X| \leq |Y|$. Each of the following holds.*

(i) *If $\delta(D) \geq \lfloor \frac{\alpha'(D)}{2} \rfloor + 1$, then D is supereulerian.*

(ii) *Suppose that $\alpha'(D)$ is even and $\alpha'(D) < |X|$. If $\delta(D) \geq \frac{\alpha'(D)}{2}$, then D is supereulerian.*

Theorem 1.4 has a few applications. In particular, Theorem 1.4 implies that Conjecture 1.1 holds for bipartite digraphs.

Theorem 1.5. *Let D be a strong bipartite digraph. If $\lambda(D) \geq \lfloor \frac{\alpha(D)}{2} \rfloor + 1$, then D is supereulerian.*

Let D be a bipartite digraph with a vertex bipartition (X, Y) satisfying $|X| \leq |Y|$. As $\alpha(D) \geq |Y| \geq |X| \geq \alpha'(D)$, we conclude that Theorem 1.5 follows from Theorem 1.4 (i). Similarly, as $\delta(D) \geq \lambda(D) \geq \kappa(D)$, thus, $\delta(D)$ can be replaced by either $\lambda(D)$ or $\kappa(D)$ in Theorem 1.4.

In Section 2, we shall display examples of nonsupereulerian strong bipartite digraphs to show that Theorem 1.4 is sharp in some sense. Tools and needed mechanisms in our arguments will be presented in Section 3. The last section is devoted to the proof of Theorem 1.4.

2 | EXAMPLES

In this section, we will present, for any integer $k > 0$, an infinite family $\mathcal{D}(k)$ of nonsupereulerian bipartite digraphs D with $\alpha'(D) = k$ satisfying $\delta(D) = \lfloor \frac{k}{2} \rfloor$, thereby showing that Theorem 1.4 is best possible. Following [5], a **ditrail** in D is an alternating sequence $T = v_1 a_1 v_2 a_2 v_3 \ldots v_{k-1} a_{k-1} v_k$ of vertices v_i and arcs a_i from D such that $a_i = (v_i, v_{i+1})$ for each i with $1 \leq i \leq k-1$, and such that all the arcs are mutually distinct. Let D be a digraph and $W \subset V(D)$. We call a collection of ditrails $T_1, T_2, \ldots, T_t$ of the induced subdigraph $D[W]$ a **cover** of W if $\bigcup_{i=1}^{t} V(T_i) = W$ and $A(T_i) \cap A(T_j) = \emptyset$, whenever $i \neq j$. The minimum value of such t is denoted by $t(W)$. For any subset $A \subseteq V(D) - W$, define $B = V(D) - W - A$. Let

$$h(W, A) = \min\left\{\left|\partial_D^+(A)\right|, \left|\partial_D^-(A)\right|\right\} + \min\{|(W, B)_D|, |(B, W)_D|\} - t(W).$$

Then we have the following proposition.

Proposition 2.1 (Hong, Lai, and Liu, Proposition 2.1 of [16]). *If D has a spanning eulerian subdigraph, then for any $W \subset V(D)$, we have $h(W, A) \geq 0$.*

Example 2.1. Let $k > 0$ and $\ell \geq \lfloor \frac{k}{2} \rfloor + 1$ be integers, a, b be even integers with $a \leq b$ and $a + b = 2k$, and let A and B be two disjoint sets of vertices with $|A| = a$ and $|B| = b$. Let W be a set of vertices

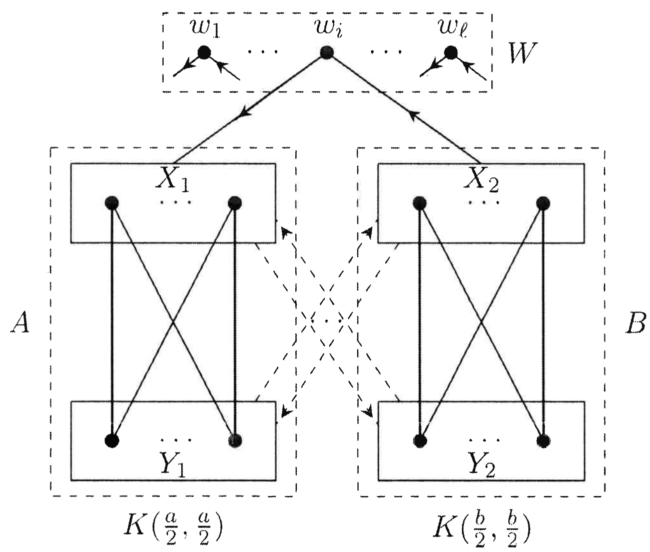

FIGURE 1 The digraph $D(a, b, k, \ell)$

disjoint from $A \cup B$ with $|W| = \ell$. We construct a digraph $D = D(a, b, k, \ell)$ such that $V(D) = A \cup B \cup W$ and $A(D)$ consists of exactly the arcs satisfying (D1),(D2), and (D3) below. (See Figure 1.)

(D1) $D[A]$ is a complete bipartite digraph $K(\frac{a}{2}, \frac{a}{2})$ with vertex bipartition (X_1, Y_1) such that $|X_1| = |Y_1| = \frac{a}{2}$; and $D[B]$ is a complete bipartite digraph $K(\frac{b}{2}, \frac{b}{2})$ with vertex bipartition (X_2, Y_2) such that $|X_2| = |Y_2| = \frac{b}{2}$.

(D2) $|(X_1, Y_2)_D \cup (Y_1, X_2)_D| = \lfloor \frac{k}{2} \rfloor$ and $|(X_2, Y_1)_D \cup (Y_2, X_1)_D| = \lfloor \frac{k}{2} \rfloor$.

(D3) For every vertex $w \in W$, and for every $x' \in X_1$ and $x'' \in X_2$, we have both $(w, x'), (x'', w) \in A(D)$. Moreover, $A(D[W]) = \emptyset$. Hence for any $w \in W$, we have $N_D^+(w) = X_1$ and $N_D^-(w) = X_2$.

Proposition 2.2. *Let $D = D(a, b, k, \ell)$ for some given parameters k and ℓ as defined in Example 2.1 such that if k is even, then $a = b = k$; and if k is odd, then $a = k - 1$, $b = k + 1$. Each of the following holds.*

(i) *$D = D(a, b, k, \ell)$ is a strong bipartite digraph with $\alpha'(D) = k$.*

(ii) *If k is even, then $\delta(D) = \frac{k}{2}$; if k is odd, then $\delta(D) = \frac{k-1}{2}$.*

(iii) *D is not supereulerian.*

Proof. By definition of $D = D(a, b, k, \ell)$, D is a bipartite digraph with vertex bipartition (X, Y), where $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2 \cup W$. Hence $|X| = |X_1 \cup X_2| = \frac{a}{2} + \frac{b}{2} = k$, $|Y| = |Y_1 \cup Y_2 \cup W| = \frac{a}{2} + \frac{b}{2} + \ell = k + \ell$, and so by (D1), we have $\alpha'(D) = k = |X| \le |Y|$. It is routine to show that for any pair of vertices $u, v \in V(D)$, D has a (u, v)-dipath, and D is strong. This proves (i).

By definition of $D = D(a, b, k, \ell)$, if k is even, then $\delta(D) = \frac{a}{2} = \frac{k}{2}$; if k is odd, then $\delta(D) = \frac{a}{2} = \frac{k-1}{2}$. This proves (ii).

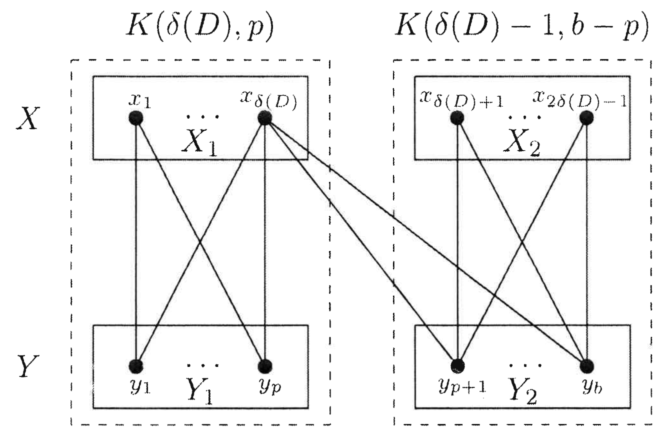

FIGURE 2 The bipartite digraph $D_1(a, b)$

We apply Proposition 2.1 to show that D is not supereulerian. By (D2), $|(X_1, Y_2)_D \cup (Y_1, X_2)_D| = \lfloor \frac{k}{2} \rfloor$, and so $|\partial_D^+(A)| = \lfloor \frac{k}{2} \rfloor$. By (D3), $|(W, B)_D| = 0$ and so $t(W) = |W| \geq \lfloor \frac{k}{2} \rfloor + 1$. It follows that

$$h(W, A) = \left|\partial_D^+(A)\right| + |(W, B)_D| - t(W) = \left\lfloor \frac{k}{2} \right\rfloor - |W| < 0,$$

and so by Proposition 2.1, D is not supereulerian. This proves (iii). ■

3 | MECHANISMS

In this section, we present some tools needed in our arguments. Throughout this section, if D is a digraph in the context, then $G = G(D)$ is the underlying graph of D. If M is edge subset of a graph G, we use $V(M)$ to denote the set of vertices in G that are incident with an edge in M. Given a matching M of G, a path P is an M**-augmenting path** if the edges of P are alternately in M and in $E(G) - M$, and if both end vertices of P are not in $V(M)$. The following theorem is fundamental.

Theorem 3.1 (**C. Berge, Theorem 1 of [7]**). *A matching M in G is a maximum matching if and only if G does not have M-augmenting paths.*

Two graphs, defined below, are of particular interests in our discussion.

Definition 3.1 (**Amar, [4]**). Let D be a bipartite digraph with a vertex bipartition (X, Y) satisfying $|X| = a \leq b = |Y|$. Let $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$. Define a family of digraphs $\mathcal{D}_1(a, b)$ and a digraph $D_2(3, b)$ as follows.

(i) Let $b > a = 2\delta(D) - 1$. Let p be an integer satisfying $\delta(D) \leq p \leq b - \delta(D)$, and $K(\delta(D), p)$ and $K(\delta(D) - 1, b - p)$ be complete bipartite digraphs with vertex bipartitions (X_1, Y_1) and (X_2, Y_2), respectively. We obtain a digraph $D_1(a, b)$ from the disjoint union of $K(\delta(D), p)$ and $K(\delta(D) - 1, b - p)$, together with all the arcs in both directions between exactly one vertex of X_1 (without loss of generality, we choose the vertex $x_{\delta(D)}$) and all the vertices of Y_2. (See Figure 2, where unoriented edges represents a directed 2-cycle.) Let $\mathcal{D}_1(a, b)$ be the family of all digraphs isomorphic to one of such $D_1(a, b)$'s.

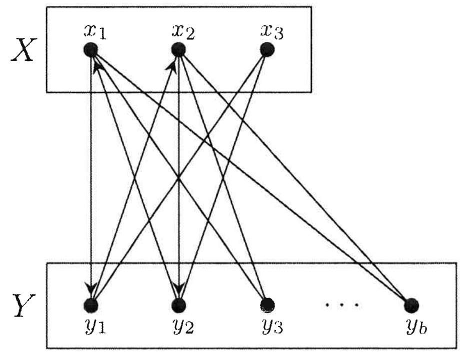

FIGURE 3 The bipartite digraph $D_2(3, b)$

(ii) Let $b \geq 3$ be an integer. Define $D_2(3, b)$ to be the bipartite digraph with $V(D_2(3, b)) = X \cup Y$, where $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, \ldots, y_b\}$, and

$$A(D_2(3, b)) = \{(x_1, y_1), (y_1, x_2), (x_2, y_2), (y_2, x_1)\} \cup \{(x_3, y_h), (y_h, x_3)\} \cup \{(x_i, y_j), (y_j, x_i)\},$$

where $h, i \in \{1, 2\}$ and $j \in \{3, 4, \ldots, b\}$. (See Figure 3, where an unoriented edge represents a directed 2-cycle.)

Theorem 3.2 (Amar, Theorem 3 of [4]). *Let $D = (X, Y)$ be bipartite digraph with $|X| = a \leq b = |Y|$ and let $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$. If $a \leq 2\delta(D) - 1$, then D has a dicycle of length $2a$ unless:*

(i) *$b > a = 2\delta(D) - 1$ and D is isomorphic to a member in $\mathcal{D}_1(a, b)$ or*

(ii) *$\delta(D) = 2$, $b \geq a = 3$, and D is isomorphic to $D_2(3, b)$.*

Thus, we have the following proposition.

Proposition 3.1. *Any member in $\mathcal{D}_1(a, b)$ and the digraph $D_2(3, b)$ are bipartite and are Eulerian digraphs.*

Lemma 3.1. *Let $k > 0$ be an integer, D be a bipartite digraph with a vertex bipartition (X, Y) satisfying $|X| = a \leq b = |Y|$, and M be a matching of D with $|M| = k$. Suppose that $V(D) - V(M)$ contains a subset Z satisfying*

$$|Z \cap X| \geq 1, |Z \cap Y| \geq 1 \quad \text{and for each } z \in Z, d(z) \geq k. \tag{1}$$

If there exists a vertex $z' \in Z$ with $d(z') \geq k + 1$, then M is not a maximum matching of D.

Proof. Let $z, z' \in Z$ be distinct vertices such that $d(z') \geq k + 1$, by contradiction, we assume that M is a maximum matching of D. By (1), we have $|Z \cap X| \geq 1$ and $|Z \cap Y| \geq 1$. Without loss of generality, we assume that $z \in Z \cap X$ and $z' \in Z \cap Y$. Since M is maximum, by Theorem 3.1, D has no M-augmenting path. In particular, z and z' are not adjacent in D. Hence all vertices adjacent to z must be in $Y \cap V(M)$ and all vertices adjacent to z' must be in $X \cap V(M)$.

Denote $M = \{e_1, \ldots, e_k\}$. For $v \in \{z, z'\}$, define $M_v \subseteq M$ be the set of arcs in M each of which is incident with at least one vertex in $N_D(v)$. Since $d(z) \geq k$ and $d(z') \geq k + 1$, we conclude that $|M_z| \geq \lfloor \frac{k+1}{2} \rfloor$ and $|M_{z'}| \geq \lfloor \frac{k+2}{2} \rfloor$. It follows that there exists at least one arc $e = [x, y] \in M$, with $x \in X$ and $y \in Y$, such that $[z', x], [y, z] \in A(D)$, and so $\{[z', x], [x, y], [y, z]\}$ induces an M-augmenting path in D, contrary to the assumption that M is a maximum matching. This proves the lemma. ■

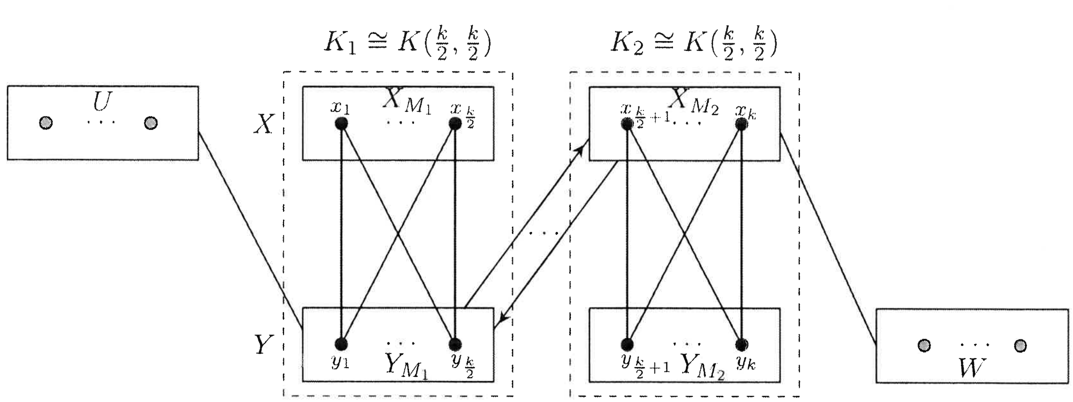

FIGURE 4 The bipartite digraph $D_3(a, b, k)$

Corollary 3.1. *If D is a bipartite digraph, then $\lambda(D) \le \alpha'(D)$.*

Proof. Let (X, Y) be a vertex bipartition of D and M be the maximum matching of D with $k = |M| = \alpha'(D)$. By contradiction, we assume that $\lambda(D) \ge k + 1$. If M is a perfect matching of D, then $|V(D)| = 2k$ and $|X| = |Y| = k$. It follows that $\min\{\triangle^+(D), \triangle^-(D)\} \le k$, and so $k + 1 \le \lambda(D) \le \min\{\triangle^+(D), \triangle^-(D)\} \le k$, a contradiction.

Hence, we assume that D has no perfect matching. Since $\lambda(D) \ge k + 1$, we have $\min\{|X|, |Y|\} \ge k + 1$, and so both $|(V(D) - V(M)) \cap X| \ge 1$ and $|(V(D) - V(M)) \cap Y| \ge 1$. It follows by Lemma 3.1 that M is not a maximum matching of D, which is a contradiction. This proves the corollary. ■

Definition 3.2. For an even integer $k > 0$ and given integers $b \ge a \ge k$, we define a bipartite digraph $D = D_3(a, b, k)$ with a vertex bipartition (X, Y) with $|X| = a$ and $|Y| = b$ in the following steps.

(i) Let X_{M_1} and X_{M_2} be disjoint subsets of X with $|X_{M_1}| = |X_{M_2}| = \frac{k}{2}$ and Y_{M_1} and Y_{M_2} be disjoint subsets of Y with $|Y_{M_1}| = |Y_{M_2}| = \frac{k}{2}$; and let $U = X - (X_{M_1} \cup X_{M_2})$ and $W = Y - (Y_{M_1} \cup Y_{M_2})$.

(ii) For $i \in \{1, 2\}$, define $K_i \cong K(\frac{k}{2}, \frac{k}{2})$ to be the complete bipartite digraph with a vertex bipartition (X_{M_i}, Y_{M_i}).

(iii) Define $D_3(a, b, k)$ to be a bipartite digraph with vertex bipartition (X, Y) such that

(iii-a) for $i \in \{1, 2\}$, the induced subdigraphs $D[X_{M_i} \cup Y_{M_i}] = K_i \cong K(\frac{k}{2}, \frac{k}{2})$;

(iii-b) $D[Y_{M_1} \cup U] \cong K(\frac{k}{2}, |U|)$, $D[X_{M_2} \cup W] \cong K(\frac{k}{2}, |W|)$, $(X_{M_2}, Y_{M_1})_D \ne \emptyset$ and $(Y_{M_1}, X_{M_2})_D \ne \emptyset$, and

(iii-c) $A(D) = (\bigcup_{i=1}^{2} A(D[X_{M_i} \cup Y_{M_i}])) \cup A(D[Y_{M_1} \cup U]) \cup A(D[X_{M_2} \cup W]) \cup (X_{M_2}, Y_{M_1})_D \cup (Y_{M_1}, X_{M_2})_D$.
(See Figure 4, where an unoriented edge represents all possible arcs in both directions.)

As stated in Definition 3.2 (iii-b), for given values a, b, and k, there would be more than one way to define $D_3(a, b, k)$. We then use $\mathcal{D}_3(a, b, k)$ to denote the family of digraphs being constructed by the steps in Definition 3.2. The observation below, stated in (2), follows from Definition 3.2 immediately.

$$\text{Every } D \in \mathcal{D}_3(a, b, k) \text{ satisfies that } \alpha'(D) = k. \tag{2}$$

Lemma 3.2. *For given integers $b \geq a > 0$, and an even integer $k > 0$, let D be a strong bipartite digraph with a vertex bipartition (X, Y) satisfying $\alpha'(D) = k$, $|X| = a$ and $|Y| = b$. Let M be a maximum matching of D, $U = (V(D) - V(M)) \cap X$ and $W = (V(D) - V(M)) \cap Y$. If $k < a$ and $\delta(D) \geq \frac{k}{2}$, then each of the following holds.*

- **(i)** *For each vertex $v \in U \cup W$, we have $d_D^+(v) = d_D^-(v) = \frac{k}{2}$.*
- **(ii)** *The digraph D is isomorphic to a member in $\mathcal{D}_3(a, b, k)$.*

Proof. Since $k < a$, we have $|U| \geq 1$ and $|W| \geq 1$. Since $\delta(D) \geq \frac{k}{2}$, we have

$$\min\left\{d_D^+(v), d_D^-(v)\right\} \geq \frac{k}{2}, \text{ for every vertex } v \in V(D). \tag{3}$$

If there exists one vertex $v \in U \cup W$ such that either $d_D^+(v) \geq \frac{k}{2} + 1$ or $d^-)D(v) \geq \frac{k}{2} + 1$, then as $\delta(D) \geq \frac{k}{2}$, we have $d_D(v) \geq k + 1$. It follows from Lemma 3.1 that M is not a maximum matching, contrary to the assumption. Hence (i) must hold.

We are to prove (ii). Firstly, we show that

$$(U, W)_D \cup (W, U)_D = \emptyset. \tag{4}$$

Since D is a bipartite digraph, we have $N_D^+(U) \cup N_D^-(U) \subseteq Y$ and $N_D^+(W) \cup N_D^-(W) \subseteq X$. If there is an arc $e \in (U, W)_D \cup (W, U)_D$, then by the definition of U and W, $M \cup \{e\}$ is also a matching of D, contrary to the assumption that M is a maximum matching of D. This proves (4).

Next, we prove that $N_D^+(U) = N_D^-(U)$ and $N_D^+(W) = N_D^-(W)$. By (i), we have $|N_D^+(U)| = |N_D^-(U)| \geq \frac{k}{2}$ and $|N_D^+(W)| = |N_D^-(W)| \geq \frac{k}{2}$. Thus if $N_D^+(U) \neq N_D^-(U)$, then $|N_D^+(U) \cup N_D^-(U)| \geq \frac{k}{2} + 1$. Let $Y^* = N_D^+(U) \cup N_D^-(U)$ and $X^* = N_D^+(W) \cup N_D^-(W)$. Then by (4), $Y^* \subseteq Y - W$ and $X^* \subseteq X - U$. Define $X(Y^*) = \{x \in X : \exists y \in Y^* \text{ such that } [x, y] \in M\}$. As M is a matching, we have $|X(Y^*)| = |Y^*| \geq \frac{k}{2} + 1$ and $X(Y^*) \subseteq X - U$. By the definition of U, we have $|X - U| = k$. Since $X(Y^*) \cup X^* \subseteq X - U$, we have

$$|X(Y^*) \cap X^*| \geq |X(Y^*)| + |X^*| - |X - U| \geq \frac{k}{2} + 1 + \frac{k}{2} - k = 1.$$

It follows that there must exist an arc $e = [x, y] \in M$ such that $x \in N_D^+(W) \cup N_D^-(W)$ and $y \in N_D^+(U) \cup N_D^-(U)$. We assume that $[x, w], [y, u] \in A(D)$ for some $w \in W$ and $u \in U$, then $(M - \{e\}) \cup \{[x, w], [y, u]\}$ is also a matching of D with size $|M| + 1$, contrary to the assumption that M is a maximum matching of D. Thus we must have $N_D^+(U) = N_D^-(U) \subseteq Y$. Similarly, we also have $N_D^+(W) = N_D^-(W) \subseteq X$.

Let

$$Y_{M_1} = N_D^+(U) = N_D^-(U) \quad \text{and} \quad X_{M_2} = N_D^+(W) = N_D^-(W). \tag{5}$$

Then direct computation yields that $|Y_{M_1}| = |X_{M_2}| = \frac{k}{2}$. With the analysis above, we conclude that $((X_{M_2}, Y_{M_1})_D \cup (Y_{M_1}, X_{M_2})_D) \cap M = \emptyset$.

Let

$$X_{M_1} = X - (X_{M_2} \cup U) \quad \text{and} \quad Y_{M_2} = Y - (Y_{M_1} \cup W), \tag{6}$$

then direct computation yields that $|X_{M_1}| = |Y_{M_2}| = \frac{k}{2}$. By (5), we have $(U, Y_{M_2})_D = \emptyset$ and $(W, X_{M_1})_D = \emptyset$. We shall justify the following:

$$(X_{M_1}, Y_{M_2})_D \cup (Y_{M_2}, X_{M_1})_D = \emptyset. \tag{7}$$

Since $((X_{M_2}, Y_{M_1})_D \cup (Y_{M_1}, X_{M_2})_D) \cap M = \emptyset$. By (5) and (6), there must exist $\frac{k}{2}$ arcs in $A(D[X_{M_2} \cup Y_{M_2}]) \cap M$. Similarly, there must exist $\frac{k}{2}$ arcs in $A(D[X_{M_1} \cup Y_{M_1}]) \cap M$. By contradiction, we assume that $(X_{M_1}, Y_{M_2})_D \cup (Y_{M_2}, X_{M_1})_D \neq \emptyset$, without loss of generality, we assume that $[x', y''] \in A(D)$ for $x' \in X_{M_1}$, $y'' \in Y_{M_2}$ and $[y', x'], [y'', x''] \in M$ for $y' \in Y_{M_1}$, $x'' \in X_{M_2}$. By (5), for some $u \in U$ and $w \in W$, we have $[u, y'], [x'', w] \in A(D)$. It follows that $(M - \{[y', x'], [y'', x'']\}) \cup \{[u, y'], [x', y''], [x'', w]\}$ is also a matching of D with size $|M| + 1$, contrary to the assumption that M is a maximum matching of D. Hence we must have $(X_{M_1}, Y_{M_2})_D \cup (Y_{M_2}, X_{M_1})_D = \emptyset$. This proves (7).

Since $N_D^+(X_{M_1}) \subseteq Y - W = Y_{M_1} \cup Y_{M_2}$. By (7), $N_D^+(X_{M_1}) \subseteq Y_{M_1}$. By (3), $|N_D^+(X_{M_1})| \geq \frac{k}{2} = |Y_{M_1}|$, and so $N_D^+(X_{M_1}) = Y_{M_1}$. Similarly, we have $N_D^-(X_{M_1}) = Y_{M_1}$, $N_D^+(Y_{M_2}) = X_{M_2}$ and $N_D^-(Y_{M_2}) = X_{M_2}$. It follows that

$$N_D^+(X_{M_1}) = N_D^-(X_{M_1}) = Y_{M_1} \quad \text{and} \quad N_D^+(Y_{M_2}) = N_D^-(Y_{M_2}) = X_{M_2}. \tag{8}$$

By (3), (6), and (8), we have $D[X_{M_1} \cup Y_{M_1}] \cong K(\frac{k}{2}, \frac{k}{2})$ and $D[X_{M_2} \cup Y_{M_2}] \cong K(\frac{k}{2}, \frac{k}{2})$. For $i \in \{1, 2, \ldots, k\}$, let $e_i = [x_i, y_i]$ and $M = \{e_1, \ldots, e_k\}$, then $|M| = k$. Since D is strong, we have $|(X_{M_2}, Y_{M_1})_D| \geq 1$ and $|(Y_{M_1}, X_{M_2})_D| \geq 1$. (See Figure 4.) This completes the proof for *(ii)*. ■

4 | PROOF OF THE MAIN RESULT

The main goal of this section is to prove Theorem 1.4. Throughout this section, let $a, b, k > 0$ be integers and D be a bipartite digraph with a vertex bipartition (X, Y) satisfying $|X| = a \leq b = |Y|$, and let M be a maximum matching of D with $|M| = k$. Since D is a bipartite digraph, we have that $\alpha'(D) = k \leq a$. We shall prove the following theorem, which is a restatement of Theorem 1.4.

Theorem 4.1. *Let D be a strong bipartite digraph with a vertex bipartition (X, Y) satisfying $|X| \leq |Y|$. Each of the following holds.*

(i) *If $k = a$, and if $\delta(D) \geq \lfloor \frac{k}{2} \rfloor + 1$, then D is supereulerian.*

(ii) *If $k < a$, and if*

$$\delta(D) \geq \left\lfloor \frac{k}{2} \right\rfloor + \frac{1 - (-1)^k}{2}, \tag{9}$$

then D is supereulerian.

Proof.

(i). Suppose that $k = a$ and $\delta(D) \geq \lfloor \frac{k}{2} \rfloor + 1$. As $\delta(D) \geq \lfloor \frac{k}{2} \rfloor + 1$, we conclude that if a is even, then $a = k \leq 2\delta(D) - 2$; if a is odd, then $a = k \leq 2\delta(D) - 1$. It follows by Theorem 3.2 that either D is isomorphic to $D_1(a, b)$ or $D_2(3, b)$, or D has a dicycle of length $2a$. If D is isomorphic to $D_1(a, b)$ or $D_2(3, b)$, then by Proposition 3.1, D is supereulerian. Hence, we assume that D has a dicycle C of length $2a$. As $|X| = a$, we have $X \subseteq V(C)$. If $b = a$, then $Y \subseteq V(C)$, and dicycle C is a hamiltonian dicycle, implying that D is supereulerian. Hence we may assume that $b > a$. As C is a dicycle of length $2a$, we have $Y \setminus V(C) = \{z_1, z_2, \dots, z_{b-a}\} \neq \emptyset$. Since $\delta(D) \geq \lfloor \frac{k}{2} \rfloor + 1 = \lfloor \frac{|X|}{2} \rfloor + 1$, for each i with $1 \leq i \leq b-a$, there exists a vertex $x_i \in X$ such that $(x_i, z_i), (z_i, x_i) \in A(D)$. Consequently, $D[A(C) \cup \{(x_1, z_1), (z_1, x_1), \dots, (x_{b-a}, z_{b-a}), (z_{b-a}, x_{b-a})\}]$ is a spanning eulerian subdigraph of D, and so D is supereulerian. This proves (i).

(ii). Since $k < a$, we have

$$(V(D) - V(M)) \cap X \neq \emptyset \quad \text{and} \quad (V(D) - V(M)) \cap Y \neq \emptyset. \tag{10}$$

If k is odd, then $\delta(D) \geq \lfloor \frac{k}{2} \rfloor + 1 = \frac{k+1}{2}$, and so for any vertex $v \in V(D)$ both $d_D^+(v) \geq \frac{k+1}{2}$ and $d_D^-(v) \geq \frac{k+1}{2}$. Since M is a maximum matching, for any $u \in (V(D) - V(M)) \cap X$, we must have $N_D(u) \subseteq V(M) \cap Y$. Similarly, for any $w \in (V(D) - V(M)) \cap Y$, we must have $N_D(w) \subseteq V(M) \cap X$. It follows from $\delta(D) \geq \frac{k+1}{2}$ that there exists an $e = [x, y] \in M$, with $x \in X$ and $y \in Y$, such that $[w, x], [y, u] \in A(D)$. Hence $\{[w, x], [x, y], [y, u]\}$ induces an M-augmenting path in D, contrary to the assumption that M is a maximum matching.

From now on, we assume that k is even, and so $\delta(D) \geq \frac{k}{2}$. Since $k < a$, by Lemma 3.2, we have $D \in \mathcal{D}_3(a, b, k)$. The proof of the theorem will be completed once we justify the claim below. ■

Claim 1. Every digraph in $\mathcal{D}_3(a, b, k)$ is supereulerian.

In the proof of Claim 1, we use the notation of Definition 3.2, as depicted in Figure 4. By Definition 3.2 (iii-a) and (iii-b), $H_1 = D[U \cup X_{M_1} \cup Y_{M_1}]$ and $H_2 = D[W \cup X_{M_2} \cup Y_{M_2}]$ are complete bipartite digraphs, and so H_1 and H_2 are eulerian digraphs. Pick a vertex $x_1 \in X_{M_1}$ and a vertex $y_k \in Y_{M_2}$.

By Definition 3.2 (iii-b), there exist arcs $(y_1', x_2'), (x_2'', y_1'') \in A(D)$ with $y_1', y_1'' \in Y_{M_1}$ and $x_2', x_2'' \in X_{M_2}$. If $y_1' = y_1''$ and $x_2' = x_2''$, then $D[A(H_1) \cup A(H_2) \cup \{(y_1', x_2'), (x_2'', y_1'')\}]$ is a supereulerian subdigraph of D. If $x_2' = x_2''$ and $y_1' \neq y_1''$, then $D[(A(H_1) - \{(x_1, y_1''), (y_1', x_1)\}) \cup \{(y_1', x_2'), (x_2'', y_1'')\} \cup A(H_2)]$ is a spanning eulerian subdigraph of D. If $x_2' \neq x_2''$ and $y_1' = y_1''$, then $D[A(H_1) \cup \{(y_1', x_2'), (x_2'', y_1'')\} \cup (A(H_2) - \{(x_2'', y_k), (y_k, x_2')\})]$ is a spanning eulerian subdigraph of D. If $x_2' \neq x_2''$ and $y_1' \neq y_1''$, then $D[(A(H_1) - \{(x_1, y_1''), (y_1', x_1)\}) \cup \{(y_1', x_2'), (x_2'', y_1'')\} \cup (A(H_2) - \{(x_2'', y_k), (y_k, x_2')\})]$ is a spanning eulerian subdigraph of D. This justifies Claim 1, and completes the proof of the theorem.

5 | REMARKS

This research is motivated by Conjecture 1.1 and by the observation that when studying the supereulerian problem, the lower bound of a sufficient degree condition may be reduced by about half for bipartite graphs. While we verify Conjecture 1.1 for bipartite digraphs, the conjecture remains unsettled. For a digraph D, Jackson [18] defined $\alpha_2(D)$ to be the largest cardinality of a vertex subset $S \subseteq V(D)$ such that $D[S]$ contains no dicycle of length 2. By definition, for any digraph D, we have $\alpha(D) \leq \alpha_2(D)$. The following relaxation of Conjecture 1.1 is also open.

Conjecture 5.1. *Let D be a digraph. If $\lambda(D) \geq \alpha_2(D)$, then D is supereulerian.*

From the arguments presented in this article, it is also possible that any sufficient degree conditions for supereulerian digraphs, such as those stated in Theorems 1.1 and 1.3, may be reduced by about half when restricted to bipartite digraphs. This remains to be investigated.

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