Strongly Spanning Trailable Graphs with Short Longest Paths

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Abstract

For a graph G and edges $e = u_1v_1, e' = u_2v_2 \in E(G)$, the graph $G(e, e')$ is obtained from G by replacing $e = u_1v_1$ by a path $u_1v_ev_1$ and by replacing $e' = u_2v_2$ by a path $u_2v_{e'}v_2$, where $v_e, v_{e'}$ are two new vertices not in $V(G)$. A graph G is strongly spanning trailable if for any $e = u_1v_1, e' = u_2v_2 \in E(G)$, $G(e, e')$ has a spanning $(v_e, v_{e'})$-trail. Luo et al. [Discrete Mathematics 306 (2006) 87-98] proved that every 4-edge-connected graph is spanning trailable. In this paper, we show that, for a 3-edge-connected graph G which is not the Wagner graph, if every pair of edges is joined by a longest path of length at most 8, then G is strongly spanning trailable.

Keywords: supereulerian graphs, eulerian-connected graphs, spanning trailable graphs.

1 Introduction

We consider finite loopless graphs. Undefined terms and notation will follow [2]. In particular, for a graph G, $\kappa'(G)$ denotes the edge-connectivity

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of G. A cycle on n vertices is often called an **n-cycle**. Given $u, v \in V(G)$, a (u, v)-trail is a trail from u to v. Let $O(G)$ denote the set of odd degree vertices in a graph G. If G is connected with $O(G) = \emptyset$, then G is **eulerian**. If G has a spanning eulerian subgraph, then G is **supereulerian**. Boesch et al. [1] first posed the problem of characterizing supereulerian graphs. Pulleyblank [15] proved that determining if a 3-edge-connected planar graph is supereulerian is NP-complete. Catlin [5] gave a survey on supereulerian graphs, which was supplemented and updated in [10, 12].

Let $e, e' \in E(G)$. An **(e, e')-trail** is a trail having the end-edges e and e'. An (e, e')-trail is **dominating** if each edge of G is incident with at least one internal vertex of the trail; it is **spanning** if it is a dominating trail which contains all the vertices of G. A graph G is **spanning trailable** if for each pair of edges e_1 and e_2, G has a spanning (e_1, e_2)-trail.

Strongly spanning trailable graphs are a special class of supereulerian graphs. Suppose that $e = u_1v_1, e' = u_2v_2 \in E(G)$ are two edges of G. If $e \neq e'$, then the graph $G(e, e')$ is obtained from G by replacing $e = u_1v_1$ by a path $u_1v_ev_1$ and by replacing $e' = u_2v_2$ by a path $u_2v_{e'}v_2$, where $v_e, v_{e'}$ are two new vertices not in $V(G)$. If $e = e'$, then $G(e, e')$ is also denoted by $G(e)$ and is obtained from G by replacing $e = u_1v_1$ by a path $u_1v_ev_1$. A graph G is **strongly spanning trailable** if for any $e, e' \in E(G)$, $G(e, e')$ has a spanning $(v_e, v_{e'})$-trail. By definition,

every strongly spanning trailable graph is also spanning trailable. (1.1)

As shown explicitly in [14] (see Theorem 1.2 below) and implicitly in Theorem 4 of [6], every 4-edge-connected graph is strongly spanning trailable. However, it is routine to see that the Wagner graph W_8 depicted in Figure 1 below is spanning trailable but not strongly spanning trailable. Thus strongly spanning trailable and spanning trailable are not equivalent concepts in graphs with edge-connectivity at most 3. As $e = e'$ is possible, strongly spanning trailable graphs are supereulerian. The following Catlin-Jaeger Theorem indicates that it suffices to study supereulerian graphs with edge-connectivity at most 3.

Theorem 1.1 *(Catlin [4] and Jaeger [11]) Every 4-edge-connected graph is supereulerian.*

The four cycle is an example that a supereulerian graph may not be spanning trailable. Luo, Chen and Chen [14] first explicitly studied spanning trailable graphs (called eulerian-connected graphs in [14]). The following theorem improves Theorem 1.1.

Theorem 1.2 *(Luo, Chen and Chen [14]) Every 4-edge-connected graph is spanning trailable.*

An edge-cut X of a graph G is **essential** if both sides of $G - X$ have edges. A graph G is essentially k-edge-connected if G does not have essential edge-cuts of size less than k. The following was implicitly implied by Theorem 4 of [6].

Theorem 1.3 *(Catlin and Lai [6]) If G has two edge-disjoint spanning trees, then G is strongly spanning trailable if and only if G is essentially 3-edge-connected.*

Spanning trailable graphs have several useful applications. Shao [16] indicated that spanning trailable graphs have applications in the investigation of hamiltonian-connected line graphs. For fixed distinct edges $e, e' \in E(G)$, $G^*(e, e')$ is obtained from $G(e, e')$ by adding a new vertex z and new edges $zv_e, zv_{e'}$. Then $G(e, e')$ has a spanning $(v_e, v_{e'})$-trail if and only if $G^*(e, e')$ is supereulerian. As Pulleyblank [15] indicated that determining if a 3-edge-connected graph is supereulerian is NP-complete, determining if a 3-edge-connected graph is strongly spanning trailable is at least as hard as the supereulerian graph problem.

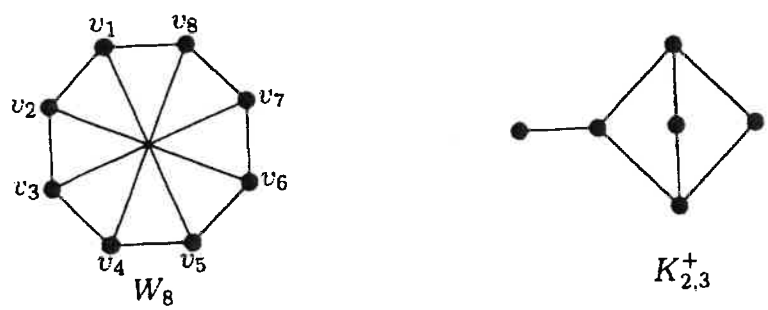


Figure 1. The graph W_8 and $K^+_{2,3}$

Let W_8 denote the **Wagner graph** as depicted in Figure 1. For $e = v_1v_5$, $e' = v_3v_7 \in E(W_8)$, it is routine to check that $W_8(e, e')$ does not have a spanning $(v_e, v_{e'})$-trail, and so any longest $(v_e, v_{e'})$-path in $W_8(e, e')$ has length 8. Using case analysis, Wang ([17]) has verified that W_8 is the 3-edge-connected non-spanning trailable graph with fewest number of vertices. Our main results of this paper are the following. In the statement of Theorem 1.4, notations for W_8 in Figure 1 will be used.

Theorem 1.4 *Let G be a 3-edge-connected graph. Let $e, e' \in E(G)$ be two edges. If the length of a longest $(v_e, v_{e'})$-path in $G(e, e')$ is at most 8, then* *either G has a spanning $(v_e, v_{e'})$-trail or $G = W_8$ with. up to isomorphism, $e = v_1v_5$ and $e' = v_3v_7$.*

Corollary 1.5 *Let G be a 3-edge-connected graph. If for any edges $e, e' \in E(G)$, the length of a longest $(v_e, v_{e'})$-path in $G(e, e')$ is at most 8, then either G is strongly spanning trailable.*

Corollary 1.5 follows from Theorem 1.4 immediately as for edges $e_1 = v_1v_2$ and $e_2 = v_7v_8$, a longest (v_{e_1}, v_{e_2})-path $v_{e_1}v_2v_3v_4v_8v_1v_5v_6v_7v_{e_2}$ has length 9. Hence W_8 does not satisfy the hypothesis of Corollary 1.5. The paper is organized as follows. We present Catlin's reduction method in the next section. The proof of Theorem 1.4 will be given in the last two sections.

2 Collapsible graphs and Catlin's reduction method

Let G be a graph. The **girth** of G, denoted $girth(G)$, is the length of a shortest cycle in G. Otherwise, we define $girth(G) = \infty$. For a vertex $v \in V(G)$ and a subgraph H of G, define $N_G(v) = \{u \in V(G) \mid vu \in E(G)\}$, $N_H(v) = N_G(v) \cap V(H)$, $d_H(v) = |N_H(v)|$, and $d_G(v) = |N_G(v)|$. For an integer $i \geq 0$, define $d_i(G) = |\{v \in V(G) | d_G(v) = i\}|$.

For a $P = u_1u_2 \cdots u_n$ and for $1 \leq i < j \leq n$, we use $P[u_i, u_j]$ to denote the subpath $u_iu_{i+1}...u_{j-1}u_j$ in P from u_i to u_j, and $P^-[u_j, u_i]$ to denote $u_ju_{j-1}...u_{i+1}u_i$, the subpath in the reversing order.

Let G be a graph and $X \subseteq E(G)$ be an edge subset. The **contraction** G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. We define $G/\emptyset = G$. If H is a subgraph of G, we write G/H for $G/E(H)$. If H is a connected subgraph of G, and if v_H is the vertex in G/H onto which H is contracted, then H is the **preimage** of v_H, and is denoted by $PI_G(v_H)$.

Collapsible graphs were introduced by Catlin [4]. A graph G is **collapsible** if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph G_R such that $O(G_R) = R$. Catlin showed in [4] that for any graph G, every vertex of G lies in a unique maximal collapsible subgraph of G. The **reduction** of G, denoted by G', is obtained from G by contracting all maximal collapsible subgraphs of G. A graph is **reduced** if it is the reduction of some graph. Let $F(G)$ be the minimum number of additional edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees.

Theorem 2.1 *Let G be a connected graph, H be a collapsible subgraph of G, v_H the vertex in G/H with $PI_G(v_H) = H$, and G' the reduction graph of G. Then each of the following holds.*
(i) (Theorems 3 and 8 of [4]) G is collapsible if and only if G/H is collapsible; and G is supereulerian if and only if G' is supereulerian. In particular, G is collapsible if and only if the reduction $G' = K_1$.
(ii) (Theorem 5 of [4]) G is reduced if and only if G has no nontrivial collapsible subgraphs.
(iii) (Theorem 8 of [4]) G' is simple, girth$(G') \geq 4$ and $\delta(G') \leq 3$.
(iv) (Lemma 1 of [3]) Every subdivision of K_4 with at most 6 vertices is collapsible. In particular, $K_{3,3}^-$ is collapsible, where $K_{3,3}^-$ is the graph obtained from $K_{3,3}$ by deleting an edge.
(v) (Theorem 1.3 of [7]) If G is connected and if $F(G) \leq 2$, then $G' \in \{K_1, K_2\} \cup \{K_{2,t} | t \geq 1\}$.
(vi) If G is reduced, then $F(G) = 2|V(G)| - |E(G)| - 2$.
(vii) (Theorem 11 of [3]) For an integer $t > 0$, let $G(t)$ be the graph with $2t+3$ vertices and $3t+3$ edges with $V(G(t)) = \{x_0, x_1, \dots, x_t, y_0, y_1, \dots, y_t, v\}$, where $E(G(t))$ consists of the edges $\{x_0y_0, x_0v\} \cup \left(\bigcup_{i=1}^{t}\{x_iy_i, x_{i-1}x_i, y_{i-1}y_i\}\right)$ and exactly one edge in $\{vx_t, vy_t\}$. Then $G(t)$ is collapsible if and only if $G(t)$ is not bipartite.

Lemma 2.2 *Let G be a connected simple graph with $n \geq 3$ vertices.*
(i) (Li et al., Lemma 2.1 of [13]) If $n \leq 8$ and if $d_1(G) = 0$ and $d_2(G) \leq 2$, then the reduction of G is in $\{K_1, K_2, K_{2,3}\}$.
(ii) (Chen [8], also Theorem 2.5 of [9]) If G is reduced with $|V(G)| \leq 11$ and $\kappa'(G) \geq 3$, then $G = K_1$ or G is the Petersen graph.
(iii) (Theorem 2.4 of [9]) If G is a connected reduced graph with $6 \leq |V(G)| \leq 11$ and $F(G) \leq 3$, then either G is the Petersen graph, or G must have at least 3 vertices of degree at most 2.
(iv) (Wang [17]) If $n \leq 12$, $d_1(G) = 0$ and $d_2(G) \leq 1$, then either the reduction of G is in $\{K_1, K_2, K_{1,2}, K_{2,3}, K_{2,3}^+, P(10), P(10)(e)\}$, or G is a supereulerian graph on 12 vertices, where $K_{2,3}^+$ is defined in Figure 1, $P(10)$ is the Petersen graph, and $P(10)(e)$ is from $P(10)$ by subdividing an edge.

Definition 2.3 *Let $C = w_1w_2w_3w_4$ be a 4-cycle in G with a partition $\pi(C) = (\{w_1, w_3\}, \{w_2, w_4\})$. Following [3], we define $G/\pi(C)$ to be the graph obtained from $G - E(C)$ by identifying w_1 and w_3 to form a vertex w', by identifying w_2 and w_4 to form a vertex w'', and by adding an edge $e_{\pi(C)} = w'w''$.*

Theorem 2.4 *(Caltin [3]) Let G be a graph containing a 4-cycle C and let $G/\pi(C)$ be defined as in Definition 2.3. If $G/\pi(C)$ is collapsible, then*

G is collapsible.

Lemma 2.5 *Let $u, w \in V(G)$, H be a collapsible subgraph of G, and let v_H denote the vertex in G/H onto which H is contracted, and*

$$u' = \begin{cases} u, & \text{if } u \notin V(H) \\ v_H, & \text{if } u \in V(H) \end{cases} \quad \text{and } w' = \begin{cases} w, & \text{if } w \notin V(H) \\ v_H, & \text{if } w \in V(H) \end{cases}.$$

If G/H has a (u', w')-trail L' containing v_H, then G has a (u, w)-trail L such that $(V(L') - \{v_H\}) \cup V(H) \subseteq V(L)$.

Proof. By the definition of contraction, $E(L') \subseteq E(G)$. Let L'' be the subgraph of G with edge set $E(L')$ and define

$$\begin{aligned} R &= [\{v \in V(H) \mid d_{L''}(v) \text{ is odd }\} \cup \{u, w\}] \\ &\quad -[\{v \in V(H) \mid d_{L''}(v) \text{ is odd }\} \cap \{u, w\}]. \end{aligned}$$

Since L' is a (u', w')-trail of G/H, $d_{L'}(v_H)$ is odd if and only if $|V(H) \cap \{u, w\}| = 1$. It follows that $|R| \equiv 0 \pmod 2$. Since H is collapsible, H has a spanning connected subgraph H_R such that $O(H_R) = R$. Let $L = G[E(L') \cup E(H_R)]$ be an edge induced subgraph of G. Since L' contains v_H, L is connected. By the definition of R and the choice of H_R, $O(L) = \{u, w\}$, and so L satisfies $(V(L') - \{v_H\}) \cup V(H) \subseteq V(L)$. ∎

Lemma 2.6 *Let G be a graph with $\kappa'(G) \geq 3$, and let $G_1, G_2, \cdots, G_k$ be the blocks of G. Then the following are equivalent.*
(i) G is strongly spanning trailable.
(ii) For every $i = 1, 2, \cdots, k$, G_i is strongly spanning trailable.

Proof. Since each block of G is also 3-edge-connected, (i) implies (ii). To prove (ii) implies (i), we argue by induction on k, the number of blocks of G. As (ii) trivially implies (i) when $k = 1$, we assume that $k > 1$ and for any graph with fewer than k blocks, (ii) implies (i).

Since $k \geq 2$, G has two connected subgraphs H and L and a vertex z_0 such that $G = H \cup L$ and $V(H) \cap V(L) = \{z_0\}$. Let $e, e' \in E(G)$. If $\{e, e'\} \cap E(L) = \emptyset$, then by induction, $H(e, e')$ has a spanning $(v_e, v_{e'})$-trail Q_1. By induction, for any edge $e'' \in E(L)$, $L(e'')$ has a spanning $(v_{e''}, v_{e''})$-trial, and so L has a spanning closed trail Q_2. It follows that $Q = G[E(Q_1) \cup E(Q_2)]$ is a spanning $(v_e, v_{e'})$-trail of G. The proof for the case when $\{e, e'\} \subseteq E(L)$ is similar, and will be omitted. Hence we may assume that $e \in E(H)$ and $e' \in E(L)$.

Since $\kappa'(H) \geq \kappa'(G) \geq 3$, and so H has an edge $e'' \in E_H(z_0) - \{e\}$. By induction, H has a spanning closed $(v_e, v_{e''})$-trail T_1'. Assume that $e'' = z_0 w$. Define

$$T_1 = \begin{cases} T_1' - z_0 v(e''), & \text{if } z_0 v(e'') \text{ is the last edge in } T_1' \\ H[E(T_1' - v(e'')) \cup \{e''\}], & \text{if } wv(e'') \text{ is the last edge in } T_1' \end{cases}.$$

Thus T_1 is a spanning $(v(e), z_0)$-trail of H. Similarly, L has a spanning $(z_0, v(e'))$-trail T_2. It follows $T = T_1 \cup T_2$ is a spanning $(v_e, v_{e'})$-trail. ∎

In the rest of this section, we will discuss properties of a graph H as specified in the next definition.

Definition 2.7 *Let $k > 0$ be an integer. A connected graph H with two distinct vertices $w_1, w_2 \in V(H)$ is said to have* **Property R(k)** *if each of the following holds.*
(i) For any $v \in V(H) - \{w_1, w_2\}$, $d_H(v) \geq 3$.
(ii) For any longest (w_1, w_2)-path P and for any $u \in V(H) - V(P)$, H has three edge-disjoint paths Q_1, Q_2, Q_3 from u to $V(P)$ with $|V(Q_i) \cap V(P)| = 1$, $(1 \leq i \leq 3)$ and with $|\cup_{i=1}^{3} V(Q_i) \cap V(P)| \geq 2$.
(iii) every (w_1, w_2)-path in H has length at most k.

Throughout the rest of this section in Lemmas 2.8-2.11, H always denotes a 2-connected graph with Property R(6) and with the same notations in Definition 2.7, and

$$P = z_0 z_1 ... z_h \text{ is a longest } (w_1, w_2)\text{-path in } H. \tag{2.1}$$

Lemma 2.8 *If for a pair of distinct vertices z_i, z_j $(0 \leq i < j \leq h \leq 6)$, $H - E(P)$ has a longest (z_i, z_j)-path $Q = x_0 x_1 ... x_k$ with $x_0 = z_i$ and $x_k = z_j$ such that $k = |E(Q)| \geq 4$ and $V(P) \cap V(Q) = \{z_i, z_j\}$, then either*
(i) H is not reduced, or
(ii) $H[V(P) \cap V(Q)]$ is spanned by the graph L depicted in Figure 2 below.

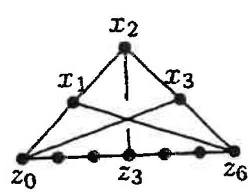


Figure 2. The graph L

Proof. We assume that H is reduced to prove Lemma 2.8(ii). By Theorem 2.1 (iii), $girth(H) \geq 4$. Since H is connected, for each i $(1 \leq$

$i < k$), H has a shortest path T_i from x_i to P with $V(T_i) \cap V(P) = \{z_{t_i}\}$. Then we have the following observations: either (A0), or one of (A1) or (A2) must hold, and either (B0) or (B1) must hold. Moreover, if $k = 6$, (C0) holds.

(A0) $V(T_1) \cap V(Q) - (V(P) \cup \{x_1\} - \{z_i, z_j\}) \neq \emptyset$.

Note that if (A0) fails, then $t_1 \leq j$. Otherwise, assume that $t_1 > j$. By (2.1), $j \geq i + |E(Q)| \geq i + 4$, and so $6 \geq t_1 \geq j + 4 \geq i + 8$, a contradiction.

(A1) If (A0) fails and $i < t_1 < j$, then by $girth(H) \geq 4$, $t_1 \geq i + 2$. Hence $6 \geq j \geq t_1 + |E(Q)| \geq t_1 + 4 \geq i + 6$. It follows that we must have $i = 0$, $t_1 = 2$, $j = 6$, $x_1 z_2 \in E(H)$ and $|E(Q)| = 4$.

(A2) If (A0) fails and $t_1 < i$, then by $girth(H) \geq 4$, $i \geq t_1 + 2$. Hence $6 \geq j \geq i + |E(Q)| \geq i + 4 \geq t_1 + 6$. It follows that we must have $t_1 = 0$, $i = 2$, $j = 6$, $x_1 z_0 \in E(H)$ and $|E(Q)| = 4$.

(B0) $V(T_2) \cap V(Q) - (V(P) \cup \{x_2\} - \{z_i, z_j\}) \neq \emptyset$.

Note that if (B0) fails, then $i \leq t_2 \leq j$. Otherwise, if $t_2 < i$, then by (2.1), $i \geq t_2 + 3$, whence $6 \geq j \geq i + |E(Q)| \geq i + 4 \geq t_1 + 7$, a contradiction. Thus $i \leq t_2$. By symmetry, $t_2 \leq j$.

(B1) If (B0) fails and $i < t_2 < j$, then by (2.1), $t_2 \geq i + 3$ and so $6 \geq j \geq t_2 + |E(Q[x_2, x_k])| \geq t_2 + 3 \geq i + 6$. It follows that we must have $i = 0$, $t_2 = 3$, $j = 6$, $x_2 z_3 \in E(F)$ and $|E(Q)| = 4$.

(C0) If $k = 6$, then $V(T_3) \cap V(Q) - (V(P) \cup \{x_3\} - \{z_i, z_j\}) \neq \emptyset$. Otherwise, symmetrically, we may assume that $j \geq t_3 \geq 3$ and so $P[z_0, z_{t_3}]x_3 Q[x_3, z_j]$ $P[z_j, z_h]$ is longer than P, contrary to (2.1).

By symmetry, these observations can also be applied to x_{k-1} (as symmetric to x_1) and to x_{k-2} (as symmetric to x_2).

Claim 1. (A0) holds for x_1.

If not, then assume first that (A1) holds for x_1. Thus (B1) does not hold as otherwise $t_1 = 2$ and $t_2 = 3$, and so $P[z_0, z_2]T_1^-[x_1, z_2]T_2[x_2, z_3]$ $P[z_4, z_h]$ is longer than P. Hence (B0) must hold. By $|E(Q)| = 4$, $t_2 \in \{0, 6\}$. If $t_2 = 0$, then by $girth(H) \geq 4$, $|E(T_2)| \geq 2$, and so $T_2^-[x_2, z_0]x_1 T_1[x_1, z_2]P[z_2, z_6]$ is longer than P, contrary to (2.1). Hence $t_2 = 6$ and $T_2 = x_2 x_3' z_6$. Applying the observations of (A0)-(A2) to x_3 (by the symmetry between x_1 and x_3), we conclude that $t_3 = 0$, whence $z_0 x_3 x_2 x_1 P[z_2, z_6]$ is longer than P, contrary to (2.1). This proves that (A1) does not hold. The proof for (A2) does not hold is similar. This proves Claim 1.

Since $girth(H) \geq 4$, and since Q is longest, $T = T_1[x_1, x_s]$ satisfies $V(T) \cap V(Q) = \{x_1, x_s\}$ such that if $s = 3$, then $|E(T)| \geq 2$. As T_1 is shortest, $s \geq 3$.

Claim 2. If $x_s \neq z_j$, then (B0) holds for x_2.

If (B0) does not hold for x_2, then (B1) holds for x_2, implying $s = 3$. As Q is longest, $T = x_1 x_2' x_3$, and so (B0) or (B1) must also hold for x_2'. If (B1) holds for x_2', then $x_2' z_3 \in E(H)$, and so $P[z_0, z_3] x_x' x_1 x_2 x_3 z_6$ is longer than P, contrary to (2.1). Hence (B0) holds for x_2'. By symmetry, and since $girth(H) \geq 4$ and since Q is longest, we may assume that H has a vertex $x_1' \notin V(P) \cup V(Q)$ such that $z_0 x_1', x_1' x_2' \in E(H)$. (See Figure 3). Thus $z_0 x_1' x_2' x_1 x_2 z_3 P[z_3, z_6]$ is longer than P, contrary to (2.1). This proves Claim 2.

(A1) and (A2)

Figure 3. (A0),(B1) and (B0) for $x_{2'}$

If $t_2 = i$, then by $girth(H) \geq 4$ and as Q is longest, $T_2 = x_2 x_1' z_i$ for some $x_1' \notin V(P) \cup V(Q)$. By the symmetry between x_1 and x_1', we must also have (A0) holds for x_1'. Let T' be an $(x_1', x_{s'})$-path with $V(T') \cap V(Q) = \{x_1', x_{s'}\}$ and $V(T') \cap V(P) = \emptyset$.

Claim 3. If $x_s \neq z_j$ and $t_2 = i$, then $s = s'$. Moreover, $s > 3$.

If $s' \neq s$, then as $girth(H) \geq 4$, H would have an (z_i, z_j)-path of length at least 7 (See Figure 4). Hence we assume that $s = s' \geq 3$. If $s = s' = 3$, then by $girth(H) \geq 4$ and by (2.1), $|E(T')| = 2$ and $E(T_1[x_1, x_3]) = 2$. It follows that $H[\{x_0, x_2\} \cup V(T') \cup V(T_1[x_1, x_3])] \cong G(2)$ defined in Theorem 2.1 (vii), and so by Theorem 2.1 (vii), H is not reduced, contrary to the assumption that H is reduced. This proves Claim 3.

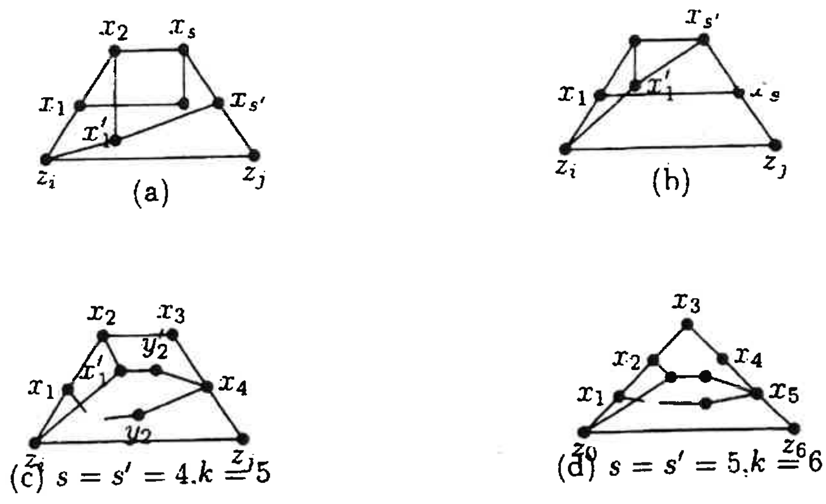


Figure 4.

Claim 4. If $x_s \neq z_j$ and $t_2 = i$, then $s = s' = 5$ (and so $k = 6$).

If $s = s' = 4$, and if $min\{|E(T')|, |E(T_1[x_1, x_4])|\} = 1$, then by Theorem 2.1 (vii), H is not reduced. Hence when $s = s' = 4$, by (2.1), we must have $|E(T')| = |E(T_1[x_1, x_4])| = 2$, and so we may denote $T' = x_1' y_2' x_4$ and $T_1[x_1, x_4] = x_1 y_2 x_4$. If $k = 5$, then by symmetry with x_2, either (B0) or (B1) holds (with x_3 replacing x_2). If $i < t_3 < j$, then $z_i x_1 x_2 T'[x_1', x_4] x_3 T_3[x_3, z_{t_3}] P[z_{t_3}, z_j]$ is longer than $P[z_i, z_j]$, contrary to (2.1). Therefore (B0) must hold for x_3, and so H has an (x_3, z)-path T_3' such that $V(T_3') \cap (V(Q) \cup V(T') \cup V(T_2)) = \{x_3, z\}$. Table 1 below shows that for any values of x, either H has a nontrivial collapsible subgraph, or (2.1) is violated, and so Claim 4 is proved.

z	longer (z_0, z_h)-path or H is not reduced	Explanations
x_1	$P[z_0, z_i](T_3')^-[x_1, x_3]x_2 x_1' x_4 P[z_j, z_h]$,	$girth(H) \geq 4$, $\|E(T_3')\| \geq 2$.
x_2	$P[z_0, z_i]x_1(T_3')^-[x_2, x_3]x_4 P[z_j, z_h]$,	$girth(H) \geq 4$, $\|E(T_3')\| \geq 3$.
x_4	$P[z_0, z_i]Q[x_1, x_3]T'[x_3, x_4]P[z_j, z_h]$,	$girth(H) \geq 4$, $\|E(T_3')\| \geq 3$.
x_1'	$P[z_0, z_i]x_1 x_2 x_3 (T_3')^-[x_1', x_3]y_2' x_4 P[z_j, z_h]$,	$girth(H) \geq 4$, $\|E(T_3')\| \geq 2$.
y_2	$P[z_0, z_i]x_1 y_2 x_3 x_2 x_1' y_2' x_4 P[z_j, z_h]$.	
z_i	$H[V(Q[x_0, x_4]) \cup x_1', y_2, y_2'\}]$ is not reduced,	by Lemma 2.2(i), if $\|E(T_3')\| = 1$.
z_i	$P[z_0, z_i](T_3')^-[z_i, x_3]x_2 x_1 T_1[x_1, x_4]P[z_j, z_h]$,	if $\|E(T_3')\| \geq 2$.
z_j	$P[z_0, z_i]T_1[x_1, x_4](T')^-[x_4, x_1']x_2 T_3'[x_3, z_j]P[z_j, z_h]$.	

Table 1: When $s = s' = 4$, a contradiction is found

Claim 5. If $x_s \neq z_j$, then $t_2 \neq i$, and so $t_2 = j$.

If not, then by Claim 4, we have $s = s' = 5$ and $k = 6$. Thus $i = 0$ and $j = 6$. By Definition 2.7 (ii), H has an (y_2, y)-path T_4' such that $x_1'y_2, y_2x_5 \notin E(T_4')$ and $V(T_4') \cap (V(P) \cap V(Q) \cup \{x_1', y_2, y_2'\}) = \{y_2, y\}$. If $y = z_t$ for some $0 < t < 6$, then either $P[z_0, z_t](T_4')^-[y_2, z_t]x_1'Q[x_2, z_6]$ or $z_0x_1x_2x_1'T_4'[y_2, z_t]P[z_t, z_6]$ is longer than P, contrary to (2.1). Hence $y \in V(Q) \cup \{x_1', y_2, y_2'\}$. Table 2 below indicates that for any possibilities of y, a violation to (2.1) always occurs, and so Claim 5 follows.

y	longer (z_0, z_h)-path in H	Explanations
x_1	$z_0x_1'T_4'[y_2, x_1]Q[x_1, z_6]$.	
x_2	$z_0x_1'T_4'[y_2, x_2]Q[x_2, z_6]$.	
x_3	$z_0x_1x_2x_1'T_4'[y_2, x_3]Q[x_4, z_6]$.	
x_4	$z_0x_1x_2x_1'T_4'[y_2, x_4]x_5z_6$.	
x_5	$z_0x_1x_2x_1'T_4'[y_2, x_5]z_6$,	by $girth(H) \geq 4$, $\|E(T_4')\| \geq 3$.
x_1'	$z_0x_1x_2x_1'(T_4')[y_2, x_1']x_5z_6$,	by $girth(H) \geq 4$, $\|E(T_4')\| \geq 3$.
y_2'	$z_0x_1x_2x_1'T_4'[y_2, y_2']x_5z_6$,	by $girth(H) \geq 4$, $\|E(T_4')\| \geq 2$.
z_0	$(T_4')^-[z_0, y_2]x_1'x_2Q[x_2, z_6]$.	
z_6	$Q[z_0, z_5]y_2T_4'[y_2, z_6]$.	

Table 2: When $s = s' = 5$, a contradiction to (2.1) is found.

Claim 6. $x_s = z_j$, and so $t_1 = j$.

If not, then by Claim 5, $t_2 = j$. If $k = 4$, then $s = 3$. By $girth(H) \geq 4$, both $|E(T[x_1, x_3])| \geq 2$ and $|E(T_2)| \geq 2$. It follows that $z_0T_1[x_1, x_3]x_2T_2[x_2, z_j]$ is longer than Q, contrary to the choice of Q. Suppose that $k = 5$ and $s = 4$. By symmetry, $t_3 = i$ and T_3 is internally disjoint from Q. If $|E(T[x_1, x_4])| = |E(T_2)| = |E(T_3)| = 1$, then $H[V(Q)]$ contains a $K_{3,3}^-$, contrary to the assumption that H is reduced. Hence $\max\{|E(T[x_1, x_4])|, |E(T_2)|, |E(T_3)|\} \geq 2$. Table 3 shows that a contradiction to the choice of Q always exists.

Cases	(z_i, z_j)-path longer than Q in $H - E(P)$
$\|E(T[x_1, x_4])\| \geq 2$	$z_iT_1[x_1, x_4]x_3'T_2[x_2, z_j]$.
$\|E(T_2)\| \geq 2$	$z_iT_1[x_1, x_4]x_3'T_2[x_2, z_j]$.
$\|E(T_3)\| \geq 2$	$T_3^-[z_i, x_3]x_2T_1[x_1, z_j]$.

Table 3: When $k = 5$ and $s = 4$, a, (z_i, z_j)-path longer than Q is found.

Therefore we must have $k = 5$ and $s = 3$. By $girth(H) \geq 4$, $|E(T_1)| \geq 2$. As Q is longest, $T_1 = x_1x_2'x_3$ and $T_2 = x_2z_j$. By (2.1), $t_4 = i$ and $T_4 = x_4z_i$. By Theorem 2.1 (vii), $H[V(Q)]$ is not reduced, contrary to the assumption that H is reduced. This proves Claim 6.

Claim 7. (B0) holds for x_2.

If not, then (B1) holds for x_2, whence $i = 0$, $t_2 = 3$, $j = 6$, $x_2z_3 \in E(F)$ and $|E(Q)| = 4$. If $|E(T_1)| \geq 2$, then $P[z_0, z_3]x_2T_1[x_1, z_6]$ is longer than P. Hence we have $T_1 = x_1z_6$. By symmetry to x_1, we also have $T_3 = x_3z_0$. Thus $H[V(P) \cap V(Q)]$ is spanned by the graph depicted in Figure 2, and so Claim 1 must hold. Hence Claim 7 follows.

By Claims 1 and 6, and by the symmetry between x_1 and x_{k-1}, we may assume that $V(T_1) \cap (V(P) \cup V(Q)) = \{z_j, x_1\}$ and $V(T_{k-1}) \cap (V(P) \cup V(Q)) = \{z_i, x_{k-1}\}$. By the maximality of Q, we have

$$T_1 = x_1z_j \text{ and } T_{k-1} = x_{k-1}z_i. \tag{2.2}$$

By Claim 7, T_2 contains a subpath $T_2' = T_2[x_2, y']$ such that $V(T_2') \cap V(Q) = \{x_2, y'\}$ and such that $x_1x_2, x_2x_3 \notin E(T_2')$. By the choice of Q, $y' \notin \{x_1, x_3\}$. By Claim 6 and by symmetry, $y' \neq x_{k-1}$.

If $k = 4$, then as Q is longest and by $girth(H) \geq 4$, we have

$$T_2 = x_2xz, \text{ where } z \in \{z_i, z_j\} \text{ and } x \notin V(P) \cup V(Q). \tag{2.3}$$

By the symmetry between x and x_1, we must have $\{x, z\} = \{z_i, z_j\}$. Thus $H[V(Q) \cup \{x\}] \cong K_{3,3}^-$, contrary to the assumption that H is reduced. Hence $k \geq 5$.

Assume that $k = 5$. As $y' \neq x_{k-1}$, $y' \in \{z_i, z_j\}$. If $y' = z_i$, then by maximality of Q, (2.3) holds. Hence by Lemma 2.2(i), $H[V(Q) \cup V(T_2)]$ is not reduced, contrary to the assumption that H is reduced. Thus we have $y' = z_j$. By the maximality of Q, $|E(T_2)| = 2$. By the symmetry between x_2 and x_3, we also have $t_3 = i$ and $|E(T_3)| = 2$. If follows by Lemma 2.2 (i) that $H[V(Q) \cup V(T_2) \cup V(T_3)]$ is not reduced, contrary to the assumption that H is reduced.

Hence $k = 6$. Then $y' \in \{z_i, x_4, z_j\}$. If $y' = z_i$, by the maximality of Q, (2.3) holds with $z = z_i$. By the symmetry between x_1 and x, and by (2.2), we have $xz_j \in E(H)$. It follows that $H[\{z_i, z_j, x_1, x, x_2, x_5\}] \cong K_{3,3}^-$, contrary to the assumption that H is reduced. Thus $y' \neq z_i$ ($t_2 \neq i$) and by symmetry, $t_4 \neq j$.

If $y' = x_4$, then $|E(T_2')| = 2$. By (C0), T_3 has a subpath $T_3' = T_3[x_3, y'']$ such that $V(T_3') \cap V(Q) = \{x_3, y''\}$ and such that $x_2x_3, x_3x_4 \notin E(T_3')$. By Claim 6 and by symmetry, $y'' \notin \{x_1, x_5\}$. Thus $y'' \in \{z_i, z_j\}$.

Assume first $y'' = z_i$. If $|E(T_3)| \geq 2$, then $z_i T_2^-[z_i, x_3] x_4 T_2^-[x_4, x_2] x_1 z_j$ has length at least 7, contrary to (2.1); if $T_3 = x_3 z_i$, then by Lemma 2.2 (i), $H[V(Q) \cup V(T_2)]$ is not reduced, contrary to the assumption that H is reduced. Thus $y'' \neq z_i$. By symmetry, we also have $y'' \neq z_j$.

Hence we must have $y' = z_j$. If $|E(T_2)| \geq 3$, then $z_i Q^-[x_5, x_2] T_2[x_2, z_j]$ has length at least 7, contrary to (2.1). Thus we must have $T_2 x_2 x' z_j$. By symmetry, we must also have $T_4 = x_4 x'' z_i$. Now applying (2.2) to the path $Q' = x_0 x'' x_4 x_3 x_2 x' x_6$, we also have $x'' z_j, x' z_i \in E(H)$. Thus by Lemma 2.2 (iii), $H[V(Q) \cup \{x', x''\}]$ is not reduced, contrary to the assumption that H is reduced. This completes the proof of the lemma. ∎

Denote $V(H) - V(P) = \{u_1, u_2, \dots\}$. By Definition 2.7 (ii), for each $1 \leq i \leq 3$ and $u_j \in V(H) - V(P)$, let

$$Q_i^j \text{ denote a } (u_j, z_{t_i^j})\text{-path with } V(P) \cap V(Q_i^j) = \{z_{t_i^j}\} \tag{2.4}$$

such that Q_1^j, Q_2^j and Q_3^j are mutually edge-disjoint.

Lemma 2.9 *Let H be a reduced graph satisfying Property R(6). If for some j, i_1 and i_2 with $i_1 \neq i_2$, we have $V(Q_j^{i_1}) \cap V(Q_{i_2}^j) - (V(P) \cup \{u_1\}) \neq \emptyset$, then $\{t_{i_1}^j, t_{i_2}^j\} = \{0, 6\}$ and for a path Q in $Q_j^{i_1} \cup Q_j^{i_2}$, $P \cup Q$ is spanned by the graph L depicted in Figure 2.*

Proof. Let $Q_i^1 = v_0^i v_1^i \dots v_{n_i}^i$ with $u_1 = v_0^i$ and $z_{t_i^1} = v_{n_i}^i$. We assume that $p < n_1$ is the largest such that $v_p^1 \in V(Q_2^1)$ and $q < n_2$ is the largest such that $v_q^2 \in V(Q_1^1)$.

Figure 5. $V(Q_1^1) \cap V(Q_2^1) - (V(P) \cup \{u_1\}) \neq \emptyset$.

Suppose first that $v_p^1 \neq v_q^2$. By symmetry, we may assume that $v_q^2 \in V(Q_1^1[u_1, v_p^1])$. Hence $t_2^1 > t_1^1$. As $girth(H) \geq 4$, H has a $(z_{t_1^1}, z_{t_2^1})$-path Q_{12} with

$$V(P) \cap V(Q_{12}) = \{z_{t_1^1}, z_{t_2^1}\} \text{ and with } |E(Q_{12}| \geq 4. \tag{2.5}$$

Next we assume that $v_p^1 = v_q^2$. By (2.6), $H - v_p^1$ has a path Q' from v_{q-1}^2 to a vertex in $V(P)$. By $girth(H) \geq 4$, we also conclude that (2.5) must hold (see Figure 5). By Lemma 2.8, we must have $\{t_1^1, t_2^1\} = \{0, 6\}$ and $H[V(P) \cup V(Q_{12})]$ is spanned by L. ∎

Lemma 2.10 *Let H be a reduced graph satisfying Property R(6) with $h = 6$, $z_0 = w_1$, $z_6 = w_2$ and P in (2.1) being a longest (w_1, w_2)-path. Each of the following must hold.*
(A) H does not have L (depicted in Figure 2) as a subgraph.
(B) For any $u_j \in V(H) - V(P)$, if $i_1 \neq i_2$, then $V(Q_{i_1}^j) \cap V(Q_{i_2}^j) \subseteq \{u_j\} \cup V(P)$.
(C) There is no path Q satisfying the condition in Claim 1 in the proof of Lemma 2.8. (Thus any path Q satisfying the condition of Claim 1 of Lemma 2.8 will be referred to a **forbidden path***).*
(D) $E(H - V(P)) = \emptyset$.

Proof. We shall use the same notations in Lemmas 2.8 and 2.9.

(A). Assume that H has L as a subgraph. Let Q be a path stated in Lemma 2.9 such that $H[V(P) \cup V(Q)]$ is spanned by L. If the neighbors of vertices in z_1, z_2, z_4 and z_5 are all on $V(P)$, then by Definition 2.7(i), we have $\kappa'(H[V(P) \cup V(Q)]) \geq 3$. and so by $|V(L)| = 10$ and by Lemma 2.2(ii), $H[V(P) \cup V(Q)]$ is not reduced, contrary to the assumption that H is reduced. Hence by symmetry we assume that z_1 or z_2 is adjacent to a vertex z' not on P. Let $z \in \{z_1, z_2\}$. By Claim 1 in the proof of Lemma 2.8, z is not adjacent to any vertex in $Q - V(P)$. Let $z' \in N_H(z) - V(P)$. By Definition 2.7 (ii), H has a (z', z_i)-path Q_z, for some $0 \leq i \leq 6$ such that $V(Q_z) \cap V(P) = \{z_i\}$. By Lemma 2.8, $V(Q_z) \cap V(Q) \subseteq \{z_0, z_6\}$. By (2.1), $z_i \notin N_H(z) \cap V(P)$. If $i \in \{4, 5\}$. then $P[z_0, z]Q_z[z, z_i]P^-[z_3, z_i]x_2x_1z_6$ is longer than P. Therefore, if $z = z_1$, then $i \neq 0$ and so $i = 6$. It follows that $z_0x_1x_2z_3z_2z_1zQ_z[z', z_6]$ is longer than P. Therefore, we must have $z = z_2$ and $N_H(z') = \{z_0, z_2, z_6\}$. It follows by Lemma 2.2 (i) that $H[V(Q) \cup \{z', z_2, z_3\}]$ is not reduced, contrary to the assumption that H is reduced. This proves (A).

(B). If not, we assume that for $j = 1$ and so $V(Q_1^1) \cap V(Q_2^1) - \{u_1\} \cup V(P) \neq \emptyset$. By Lemma 2.9, the graph L depicted in Figure 2 is a subgraph of H containing P. contrary to (A).

(C). If it does, then by Lemma 2.8, H contains L as a subgraph, contrary to (A).

(D). By contradiction, we assume that $u_1u_2 \in E(H - V(P))$. By (B), the paths in (2.4) are internally vertex disjoint. Without lose of generality, for $j \in \{1,2\}$, we assume $t_1^j \le t_2^j \le t_3^j$. By Definition 2.7 (ii), we may assume $t_1^1 \le t_2^1 < t_3^1$, and $t_2^1 < t_3^2$. By symmetry, we further assume that $t_1^1 \le t_2^2, t_2^1 \le t_3^2$. If $t_1^1 < t_2^1 < t_2^2 < t_3^2$, then since $girth(H) \ge 4$ and by (2.1), we must have $t_2^1 \ge t_1^1 + 2$, $t_2^2 \ge t_2^1 + 3$ and $t_3^2 \ge t_2^2 + 2$. Thus $6 \ge h \ge t_3^2 \ge t_1^1 + 7$, a contradiction. Hence we cannot have $t_1^1 < t_2^1 < t_2^2 < t_3^2$. Similarly, we cannot have $t_1^1 < t_2^2 < t_2^1 < t_3^2$. Hence we must have $|\{t_1^1, t_2^2, t_2^1, t_3^2\}| \le \{2,3\}$,

Case D1. $|\{t_1^1, t_2^2, t_2^1, t_3^2\}| = 3$

Thus either $t_1^1 < t_2^1 = t_2^2 < t_3^2$, or $t_1^1 = t_2^2 < t_2^1 < t_3^2$ or $t_1^1 < t_2^2 < t_2^1 = t_3^2$ must hold (see Figure 6 (a) and (b)). As $girth(H) \ge 4$, if $t_1^1 < t_2^1 = t_2^2 < t_3^2$, then $|E(Q_2^1)| + |E(Q_2^2)| \ge 3$, and so either $Q_1^1 \cup Q_2^2 \cup \{u_1u_2\}$ or $Q_2^1 \cup Q_3^2 \cup \{u_1u_2\}$ is a forbidden path; if $t_1^1 = t_2^2 < t_2^1 < t_3^2$, then $|E(Q_1^1)| + |E(Q_2^2)| \ge 3$, and so either $Q_1^1 \cup Q_3^2 \cup \{u_1u_2\}$ or $Q_2^1 \cup Q_2^2 \cup \{u_1u_2\}$ is a forbidden path. The case when $t_1^1 < t_2^2 < t_2^1 = t_3^2$ is similar to that for $t_1^1 = t_2^2 < t_2^1 < t_3^2$. Thus a contradiction to (C) always exists.

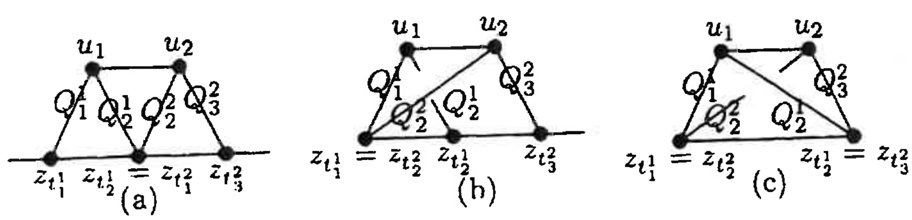


Figure 6. forbidden paths in Lemma 2.10(D).

Case D2. $|\{t_1^1, t_2^2, t_2^1, t_3^2\}| = 2$.

Thus $\{t_1^1, t_2^1\} = \{t_2^2, t_3^2\}$, and so $t_1^1 = t_2^2$ and $t_2^1 = t_3^2$. By $girth(H) \ge 4$, both $|E(Q_1^1)| + |E(Q_2^2)| \ge 3$ and $|E(Q_2^1)| + |E(Q_3^2)| \ge 3$ (see Figure 6(c)). It follows that a forbidden path must exist, contrary to (C).

This completes the proof for Case D2, and so (D) is justified. ▮

Lemma 2.11 *If H is a graph satisfying Property R(6), then H is not reduced.*

Proof. By contradiction, we assume that H is a smallest counterexample, and so H is reduced. Then by Theorem 2.1, $girth(H) \ge 4$. We shall assume and use the notations of (2.1).

If $|V(H)| \leq 8$. then by Lemma 2.2, H is not reduced. Hence we must have $|V(H) - V(P)| \geq 2$. If H has a cut vertex w, then H has two connected nontrivial subgraphs H_1 and H_2 such that $H = H_1 \cup H_2$ and $V(H_1) \cap V(H_2) = \{w\}$. If for some $i \in \{1,2\}$, $w_1, w_2 \in V(H_i)$, then by the minimality of H, H_i is not reduced. If for $i \in \{1,2\}$, $w_i \in V(H_i)$, then applying the minimality of H to w_1, w in H_1, we conclude that H_1 is not reduced. These contradictions show that

$$\kappa(H) \geq 2. \tag{2.6}$$

Claim 1. $|V(H) - V(P)| \geq 4$.

If $|V(H) - V(P)| < 4$, then as $|V(P)| \leq 7$, $|V(H)| \leq 10$. Construct a new graph G from H by adding a new vertex z, which is adjacent to both w_1 and w_2. Then z is the only vertex of degree 2 in G and $|V(G)| \leq 11$. By (2.6) and Lemma 2.2 (iv), the reduction of G is either $K_{2,3}$, whence H is not reduced; or is in $\{P(10), P(10)(e)\}$, whence a longest path connecting w_1 and w_2 in H is at least 9. These contradictions establish Claim 1.

By Lemma 2.10(D), for any $u_j \in V(H) - V(P)$, $z_{t_1^j}, z_{t_2^j}, z_{t_3^j} \in N_H(u_j) \subseteq V(P)$. By Claim 1, $|E(G)| = |E(P)| + |E(G) - E(P)| \geq |V(P)| - 1 + 3|V(H) - V(P)| = 3|V(H)| - 1 - 2|V(P)| \geq 2|V(H)| - 1 + |V(H) - V(P)| - |V(P)| \geq 2|V(H)| + 3 - 7 = 2|V(H)| - 4$. By Theorem 2.1(vi) $F(H) = 2|V(H)| - |E(H)| - 2 \leq 2$. By Theorem 2.1(vi), either H is not reduced, contrary to the assumption; or H is a $K_{2,t}$ for some integer $t \geq 11 - 2 = 9$, contrary to the fact that H has a path of length at least 6. This proves the lemma. ∎

3 Proof of Theorem 1.4

In order to prove Theorem 1.4, we need an auxiliary theorem as stated below. The proof of Theorem 3.1 will be given in Section 4.

Theorem 3.1 *Suppose G is a graph such that $G \neq W_8$, and satisfies $\kappa(G) \geq 2$ and $\kappa'(G) \geq 3$. Let $e = v_0v_1$ and $e' = v_{c-1}v_c$ be edges in G, and $P = v_ev_1 \cdots v_{c-1}v_{e'}$ be a longest $(v_e, v_{e'})$-path in $G(e,e')$ with $c = |E(P)| \leq 8$. If $G(e,e')$ is reduced and contains no spanning $(v_e, v_{e'})$-trail, then $V(G) = \{v_i : 0 \leq i \leq c\}$.*

Proof of Theorem 1.4. Let G be a counterexample to Theorem 1.4 with

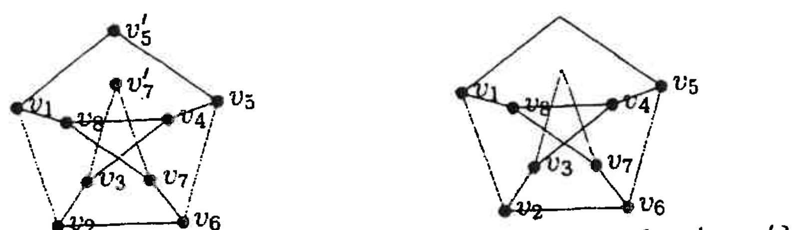
$$|V(G)| \text{ is minimized.} \tag{3.1}$$

Let $e, e' \in E(G)$ be edges such that the length of a longest $(v_e, v_{e'})$-path in $G(e, e')$ is at most 8 and such that $G(e, e')$ does not have a spanning $(v_e, v_{e'})$-trail. If $G = W_8$, then it is routine to verify that, up to isomorphism, $e = v_1v_5$ and $e' = v_3v_7$ (using notations in Figure 1 or Figure 7). Hence we assume that $G \neq W_8$. Denote $e = z_0z_1$ and $e' = z_{c-1}z_c$ and let $P = v_ez_1 \cdots z_{c-1}z_{e'}$ be a longest $(v_e, v_{e'})$-path in $G(e, e')$, By the assumption of Theorem 1.4, $c \leq 8$. By (3.1), by Lemmas 2.5 and 2.6, we may assume that

$$\kappa(G) \geq 2 \text{ and } G(e, e') \text{ is reduced.} \tag{3.2}$$

By Theorem 3.1, $V(G) = \{z_i \ : \ 0 \leq i \leq c\}$. Obtain a new graph L_w from $G(e, e')$ by adding a new vertex w and new edges wv_e and $wv_{e'}$. Then $G(e, e')$ has a spanning $(v_e, v_{e'})$-trail if and only if L_w is supereulerian. Since $|V(G)| = c + 1 \leq 9$, we have $|V(L_w)| \leq |V(G)| + 3 = 12$. As L_w has exactly one edge cut of size 2, it follows by Lemma 2.2(iv) that either L is supereulerian, or the reduction of L_z is $P(10)$ or $P(10)(f)$, for any edge $f \in E(P(10))$.

If the reduction of L_w is $P(10)$, then one vertex v_L (say) of $P(10)$ must be the image of a nontrivial collapsible subgraph of L_w containing the new vertex w. As $P(10)$ has a cycle of length 9 containing v_L, this implies that $G(e, e')$ has a $(v_e, v_{e'})$-path of length at least 9, contrary to the assumption of Theorem 1.4. If L_w is supereulerian, then $G(e, e')$ has a spanning $(v_e, v_{e'})$-trail, contrary to (3.1). Hence the reduction of L_w must be isomorphic to $P(10)(f)$. (In Figure 7, we choose $f = v_5'v_7'$ as an illustration.) Thus w must be the only vertex of degree 2 in $P(10)(f)$. If any vertex w' of $P(10)(f) - w$ is the contraction image of a nontrivial collapsible subgraph of $G(e, e')$, then $P(10)(f)$ has a cycle containing both w and w' with length 10. By definition of contraction, this cycle can be lifted to yield a (v_e, v_e')-path in $G(e, e')$ of length at least 9, contrary to the assumption of Theorem 1.4. Thus we must have $L_w = P(10)(f)$, implying that $G = W_8$ with, up to isomorphism, $e = v_1v_5$ and $e' = v_3v_7$. (See Figure 7. The labels indicates that the W_8 in Figure 7 is a different drawing of the W_8 in Figure 1.) This contradicts to the assumption that $G \neq W_8$, and completes the proof of the theorem. ∎



$P(10) - v_5'v_7'$ $\qquad$ $W_8 = (P(10) - v_5'v_7')/\{v_5v_5', v_7v_7'\}$

Figure 7. The graphs in the proof of Theorem 1.4.

4 Proof of Theorem 3.1

We denote $e = v_0v_1$ and $e' = v_{c-1}v_c$. For any longest (v_e, v_e')-path $P = v_ev_1...v_{c-1}v_{e'}$ in $G(e, e')$, define $P' = v_0v_1...v_{c-1}v_c$ and $J = J(P) = G(e, e') - (V(P) \cup \{v_0, v_c\}) = G - V(P')$. The strategy of the proof is to work on a counterexample G. First show that $E(J) = \emptyset$ for any longest (v_e, v_e')-path P in $G(e, e')$. Then we use case analysis to show that if $|V(J)| \geq 2$, then a longer (v_e, v_e')-path or a nontrivial collapsible subgraph can always be found. Finally, we prove that assuming $|V(J)| = 1$ will also yield a similar contradiction, which forces that $V(J) = \emptyset$, and completes the proof of Theorem 3.1.

If H is a subgraph of a graph Γ, we define the set of vertices of attachment of H in Γ as

$$A_\Gamma(H) = \{v \in V(H) \ : \ v \text{ is adjacent to a vertex in } V(\Gamma) - V(H)\}.$$

To prove Theorem 3.1, we assume $G \neq W_8$,

$$P = v_ev_1 \cdots v_{c-1}v_{e'} \text{ is a longest } (v_e, v_{e'})\text{-path in } G(e, e'), \quad (4.1)$$

that $G(e, e')$ has no spanning $(v_e, v_{e'})$-trail and that

$$c \leq 8 \text{ and } G(e, e') \text{ is reduced (and so } girth(G(e, e')) \geq 4). \quad (4.2)$$

If $e = e'$, then let $e = w_1w_2$. Thus in $G - e$, every longest (w, w')-path has length at most 6, and so $G - e$ has Property $R(6)$. By Lemma 2.11, $G - e$ is not reduced, contrary to (4.2). Throughout the rest of this section, we assume that $e \neq e'$. For each vertex $u \in V(J)$, as $\kappa(G) \geq 2$ and $\kappa'(G) \geq 3$, we note that G has edge-disjoint (u, v_{i_j})-path Q_i with

$$V(Q_i) \cap V(P') = \{v_{i_j}\} (1 \leq j \leq 3) \text{ and } |\{v_{i_1}, v_{i_2}, v_{i_3}\}| \geq 2. \quad (4.3)$$

The following notation will be used in the proof:

$$Q_j = uz_1^jz_2^j...z_{n_j}^jv_{i_j}, \text{ for } j = 1, 2, 3. \quad (4.4)$$

Claim 1. In each component of J, choose a vertex u and the related paths Q_i $(1 \leq i \leq 3)$ such that

$$|\{v_{i_1}, v_{i_2}, v_{i_3}\}| \text{ is maximized.} \quad (4.5)$$

Then $|\{i_1, i_2, i_3\}| = 3$.

Proof of Claim 1. By contradiction and without lose of generality, we assume that J has a component J_1 with a vertex $u \in V(J_1)$ satisfying (4.5) with $\{i_1, i_2, i_3\} = \{i_1, i_2\}$. Since u satisfies (4.5), we have $A_{G(e,e')}(J_1) = \{v_{i_1}, v_{i_2}\}$. It follows by $c \le 8$ that J_1 has Property $R(6)$ with $\{w_1, w_2\} = \{v_{i_1}, v_{i_2}\}$. By Lemma 2.11, J_1 is not reduced, contrary to (4.2). This proves Claim 1.

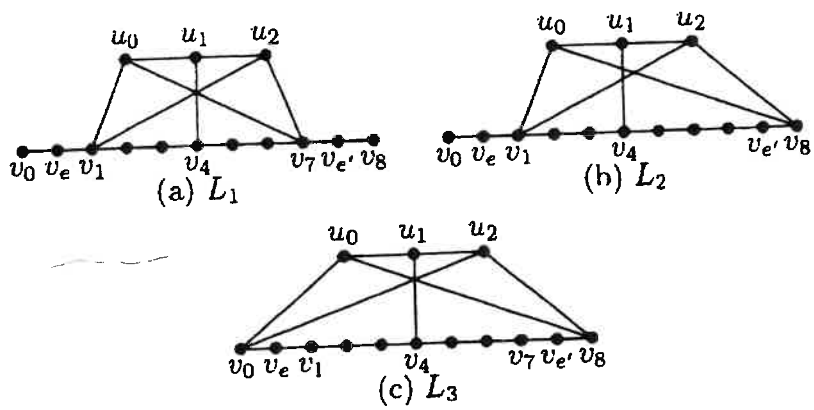


Figure 8. The subgraphs in Claim 2(B).

Claim 2. Let $u \in V(J)$ and define the Q_j's as in (4.3). Each of the following holds.
(A) J does not have an edge $u_1u_2 \in E(J)$ such that $N_G(u_1) \cup N_G(u_2) - \{u_1, u_2\} \subseteq V(P) \cup \{v_0, v_c\}$.
(B) If $v_0 \neq v_c$, and if a nontrivial component of J contains a vertex $u \in N_G(v_i)$ for some i with $1 \le i \le c-1$, then either $G(e,e')$ is not reduced or $G(e,e')$ has a $(v_e, v_{e'})$-path longer than P, or $G(e,e')$ has a subgraph isomorphic to one of the graphs depicted in Figure 8.
(C) $E(J) = \emptyset$.

Proof of Claim 2. **(A).** Assume that such u_1, u_2 exist. Let $v_{i_1}, v_{i_2} \in N_G(u_1) - \{u_2\}$ and $v_{j_1}, v_{j_2} \in N_G(u_2) - \{u_1\}$. By (4.2), i_1, i_2, j_1, j_2 are mutually distinct. By symmetry, we assume that $j_1 < j_2$ and $i_1 < \min\{i_2, j_1\} < j_2$.

Case A1. $i_1 < i_2 < j_1 < j_2$.

By (4.1), $i_2 \ge i_1 + 2$, $j_1 \ge i_2 + 3$ and $j_2 \ge j_1 + 2$. Hence $j_2 \ge i_1 + 7$. It follows that we must have $i_1 = 0$ or $j_2 = c = 8$. If $i_1 = 0$, then by (4.1), $i_2 \ge 3$, and so $j_2 \ge i_2 + 5 = 8$, forcing $j_1 = 6$ and $j_2 = 8$. But then,

$P[v_e, v_6]u_2v_8v_{e'}$ is longer than P, contrary to (4.1).

Case A2. $i_1 < j_1 < i_2 < j_2$.

By (4.1), $j_2 \geq i_1 + 3$, $i_2 \geq j_1 + 3$ and $j_2 \geq i_2 + 3$. It follows that $8 = c \geq j_2 \geq i_1 + 9$, a contradiction.

Case A3. $i_1 < j_1 < j_2 < i_2$.

By (4.1), $j_1 \geq i_1 + 3$, $j_2 \geq j_1 + 2$ and $i_2 \geq j_2 + 3$. Hence $i_2 \geq i_1 + 7$. It follows that we must have $i_1 = 0$ or $i_2 = c = 8$. If $i_1 = 0$, then by (4.1), $j_1 \geq 4$, and so $i_2 \geq j_1 + 5 = 9$, forcing $j_2 = 6$ and $i_2 = 8$. But then, $P[v_e, v_6]u_2u_1v_8v_{e'}$ is longer than P, contrary to (4.1).

(B). Let H be a nontrivial component of J with a vertex $u \in V(H)$ satisfying the hypothesis of Claim 2(B). Since $E(H) \neq \emptyset$, H contains a longest (u, u')-path Q for some vertex $u' \in A_G(H) - \{u\}$ such that u' is adjacent to a vertex v_j with $j \neq i$. Since $\kappa(G) \geq 2$, such u' exists. By symmetry, we may assume that $i < j$. Denote $Q = u_0u_1u_2...u_q$ with $u = u_0$, $u' = u_q$. By (4.1), we have $4 \geq q \geq 2$. If every path from a vertex in Q to a vertex in $P \cup \{v_0, v_c\}$ must use v_i or v_j, then by Lemma 2.11 (with w_i, w_i replaced by v_i, v_j), $G[V(H) \cup \{v_i, v_j\}]$ is not reduced. Hence we may assume that for some h with $0 \leq h \leq q$, and a (u_h, v_k)-path Q' with $k \notin \{i, j\}$.

Case B1. $i < k < j$. By (4.1), we have

$$\begin{cases} k \geq i+2, j \geq k + |E(Q)| + 2 \geq i + 6 & \text{if } h = 0 \\ k \geq i+3, j \geq k + |E(Q[u_h, u_q])| + 2 \geq i + 6 & \text{if } 0 < h < q \\ k \geq i + |E(Q)| + 2, j \geq k + 2 \geq i + 6 & \text{if } h = q \end{cases}$$

Since $i \geq 1$, we must have $i = 1$, $j \in \{7, 8\}$, $k \in \{3, 4, 5\}$, $|E(Q')| = 1$ and $|E(Q)| = 2$. Let

$$z = \begin{cases} u_1 & \text{if } h = 0 \text{ (whence } k = 3) \text{ or if } h = 2 \text{ (whence } k = 5) \\ u_0 & \text{if } h = 1 \text{ (whence } k = 4) \end{cases}$$

As $\kappa'(G) \geq 3$, $G(e, e')$ has a (z, v_l)-path T_1 such that $V(Q) \cap V(T_1) = \{z\}$. Tables 4A and 4B below indicate that either a $(v_e, v_{e'})$-path longer than P always exists, or when $k = 3$ and $h = 0$, $l = j \in \{7, 8\}$. Thus we may assume that $k = 3$ and $h = 0$, either $u_0v_7, u_2v_7 \in E(G)$ or $u_0v_8, u_2v_8 \in E(G)$. By symmetry, either $u_0v_0, u_2v_0 \in E(G)$ or $u_0v_1, u_2v_1 \in E(G)$. Thus $G(e, e')$ contains a graph in Figure 8 as a subgraph.

k	h	j	z	l	$(v_e, v_{e'})$-path longer than P (or collapsible subgraph in $G(e, e')$)
3	0	7,8	u_1	0	$v_e v_0 u_1 u_0 P[v_1, v_{e'}]$
3	0	7,8	u_1	1	$v_e v_0 u_1 P[v_1, v_{e'}]$
3	0	7,8	u_1	2, 3	$v_e v_1 u_0 u_1 P[v_l, v_{e'}]$
3	0	7,8	u_1	4, 5	$v_e v_1 v_2 v_3 u_0 u_1 P[v_l, v_{e'}]$
3	0	7,8	u_1	6	$v_e P[v_1, v_6] u_1 u_2 v_7 v_{e'}]$
3	0 0	7(or 8)	u_1	7(or 8)	$G[\{u_1, u_2, v_7\}] \cong K_3$ or $G[\{u_1, u_2, v_8\}] \cong K_3$
3	0	7	u_1	8	$v_e P[v_1, v_7] u_2 u_1 v_8 v_{e'}]$
3	0	8	u_1	7	$v_e P[v_1, v_7] u_1 u_2 v_8 v_{e'}]$

Table 4A: Case B1 with $k = 3$ and $h = 0$

(The case $k = 5$ and $h = 2$ is symmetric to this).

k	h	j	z	l	$(v_e, v_{e'})$-path longer than P (or collapsible subgraph in $G(e, e')$)
4	1	7,8	u_0	0	$v_e v_0 u_0 P[v_1, v_{e'}]$
4	1	7.8	u_0	2(or 4)	$G[\{u_1, v_1, v_2\}] \cong K_3$ or $G[\{u_1, u_2, v_4\}] \cong K_3$
4	1	7,8	u_0	3	$P[v_e, v_3] u_0 u_1 P[v_4, v_{e'}]$
4	1	7,8	u_0	5, 6	$P[v_e, v_4] u_1 u_0 P[v_l, v_{e'}]$
4	1	7	u_0	8	$P[v_e, v_7] u_2 u_1 u_0 v_8 v_{e'}]$
4	1	8	u_0	7	$P[v_e, v_7] u_0 u_1 u_2 v_8 v_{e'}]$

Table 4B: Case B1 with $k = 3$ and $h = 0$.

Case B2. $k < i < j$. If $h > 0$, then by (4.1), $i \geq k + |E(Q[u_1, u_h])| + 2$ and $8 \geq j \geq i + |E(Q)| + 2 \geq k + |E(Q[u_1, u_h])| + 5$. It follows that either $k = 1$, $i = 4$, $|E(Q)| = 2$, $h = 1$, $u_1 v_1 \in E(G)$ and $j = 8$, or $k = 0$, $i = 4$, $|E(Q)| = 2$, $h = 1$, $u_1 v_0 \in E(G)$ and $j = 8$. In any case, $P[v_e, v_4] u_0 u_1 u_2 v_8 v_{e'}$ is longer than P. Hence we must have $h = 0$. By (4.1), $i \geq k + 2$ and $j \geq i + |E(Q)| + 2 \geq k + |E(Q[u_1, u_h])| + 4$. Thus $k \in \{0, 1\}$, $i = 3$, $u_0 v_k \in E(G)$, $j \in \{7, 8\}$ and $|E(Q)| = 2$. As $\kappa'(G) \geq 3$, $G(e, e')$ has a (u_1, v_l)-path T_2 such that $V(Q) \cap V(T_2) = \{u_1\}$. Table 5 completes the proof of Case B2.

k	h	j	l	$(v_e, v_{e'})$-path longer than P (or collapsible subgraph in $G(e, e')$)
0	0	7, 8	0 (or 3, or j)	$G[\{u_0, u_1, v_0\}] \cong K_3$ or $G[\{u_1, u_2, v_3\}] \cong K_3$ or $G[\{u_1, u_2, v_j\}] \cong K_3$
0	0	7,8	1, 2	$v_e v_0 u_0 u_1 P[v_l, v_{e'}]$
0	0	7,8	4, 5	$P[v_e, v_3] u_0 u_1 P[v_l, v_{e'}]$
0	0	7,8	6	$P[v_e, v_l] u_1 u_2 v_j v_{e'}$
0	0	7	8	$P[v_e, v_7] u_2 u_1 v_8 v_{e'}$
0	0	8	7	$P[v_e, v_7] u_1 u_2 v_8 v_{e'}$

Table 5: Case B2 .

Case B3. $i < j < k$. If $h < q$, then by (4.1), $j \geq i + |E(Q)| + 2$ and $8 \geq k \geq j + |E(Q[u_h, u_q])| + 2 \geq i + 7$. It follows that $i = 1$, $j = 4$, $|E(Q)| = 2$, $h = 1$, $u_1 v_8 \in E(G)$ and $k = 8$, whence $v_e v_1 u_0 u_1 u_2 P[v_4, v_{e'}]$ is longer than P. Therefore, we must have $h = q$. By (4.1), $j \geq i + |E(Q)| + 2$ and $k \geq j + 2 \geq i + |E(Q)| + 4$. Thus $i = 1$, $j = 5$, $k \in \{7, 8\}$, $|E(Q)| = 2$ and $u_2 v_k \in E(G)$. As $\kappa'(G) \geq 3$, $G(e, e')$ has a (u_1, v_l)-path T_3 such that $V(Q) \cap V(T_3) = \{u_1\}$. Table 6 completes the proof of Case B3.

k	h	j	l	$(v_e, v_{e'})$-path longer than P (or collapsible subgraph in $G(e, e')$)
7,8	2	5	1 (or 5, or k)	$G[\{u_0, u_1, v_1\}] \cong K_3$ or $G[\{u_1, u_2, v_5\}] \cong K_3$ or $G[\{u_1, u_2, v_k\}] \cong K_3$
7,8	2	5	1	$v_e v_0 u_1 u_0 P[v_l, v_{e'}]$
7,8	2	5	2,3	$v_e v_1 u_0 u_1 P[v_l, v_{e'}]$
7,8	2	5	4	$P[v_e, v_4] u_1 u_2 P[v_5, v_{e'}]$
7,8	2	5	6	$P[v_e, v_6] u_1 u_2 v_k v_{e'}$
7	2	5	8	$P[v_e, v_7] u_2 u_1 v_8 v_{e'}$
8	2	5	7	$P[v_e, v_7] u_1 u_2 v_8 v_{e'}$

Table 6: Case B3 .

(C). By Claim 2(B), for any $i \in \{2, 3, 5, 6\}$, either $N_G(v_i) \subseteq V(P) \cup \{v_0, v_c\}$ or for some $u \in V(J)$,

$$v_i \in N_G(u) \subseteq V(P) \cup \{v_0, v_c\}. \tag{4.6}$$

Assume first that $G(e, e')$ does not have any graph depicted in Figure 8 as a subgraph. By Claim 2(B), for any nontrivial component L of J, we must have $A_G(L) = \{v_0, v_c\}$. But by Claim 1, we should have $|A_G(L)| \geq 3$.

a contradiction. Thus (C) must hold as J does not have any nontrivial components. Hence we assume that $G(e, e')$ has a graph in Figure 8 as a subgraph.

For $2 \leq i \leq 3$, by $\kappa(G) \geq 3$, $N_G(v_i) - N_P(v_i)$ contains a vertex x_i'. Define,

$$x_i = \begin{cases} x_i' & \text{if } x_i' \notin V(J) \\ \text{a vertex in } N_G(x_i') \cap V(P) - \{v_i\} & \text{if } x_i' \in V(J) \end{cases} \tag{4.7}$$

Case C1. $G(e, e')$ has L_1 as a subgraph.

Since $G(e, e')$ is reduced, if $x' \notin V(J)$, then $x \notin \{v_1, v_3, v_4\}$ and if $x' \in V(J)$, then as $|N_G(x') \cap (V(P) \cup \{v_0, v_c\})| \geq 3$, we can choose $x_2 \notin \{v_1, v_3, v_4\}$. Similarly, we may assume that $x_3 \notin \{v_1, v_2, v_4\}$. Table 7 shows that we must have $x_2 = x_2' = x_3 = x_3' = v_7$, and so $G(e, e')[\{v_2, v_3, v_7\}] \cong K_3$, contrary to (4.2).

x_2	x_3	$(v_e, v_{e'})$-path longer than P
v_0		$v_e v_0 v_2 v_1 u_0 u_1 P[v_4, v_{e'}]$ or $v_e v_0 x_2' v_2 v_1 u_0 u_1 P[v_4, v_{e'}]$
v_5, v_6 :		$v_e v_1 u_2 u_1 v_4 v_3 v_2 P[x_2, v_{e'}]$ or $v_e v_1 u_2 u_1 v_4 v_3 v_2 x_2' P[x_2, v_{e'}]$
v_8		$v_e v_1 u_2 u_1 u_0 v_7 v_6 v_5 v_4 v_3 v_2 v_8 v_{e'}$ or $v_e v_1 u_2 u_1 u_0 v_7 v_6 v_5 v_4 v_3 v_2 x_2' v_8 v_{e'}$
	v_0 v_0	$v_e v_0 v_3 v_2 v_1 u_1 u_2 P[v_4, v_{e'}]$ or $v_e v_0 x_3' v_3 v_2 v_1 u_1 u_2 P[v_4, v_{e'}]$
	v_5	$v_e v_1 u_0 u_1 v_4 v_3 P[v_5, v_{e'}]$ or $v_e v_1 u_0 u_1 v_4 v_3 x_3' P[v_5, v_{e'}]$
	v_6	$P[v_e, v_3] v_6 v_5 v_4 u_1 u_0 v_7 v_{e'}$ or $P[v_e, v_3] x_3' v_6 v_5 v_4 u_1 u_0 v_7 v_{e'}$
	v_8	$v_e v_1 u_0 u_1 u_2 v_7 v_6 v_5 v_4 v_3 v_8 v_{e'}$ or $v_e v_1 u_0 u_1 u_2 v_7 v_6 v_5 v_4 v_3 x_3' v_8 v_{e'}$

Table 7: Case C1 in Claim 2 .

Case C2. $G(e, e')$ has L_2 as a subgraph.

As in Case C1, since $G(e, e')$ is reduced, we may assume that $x_2 \notin \{v_1, v_3, v_4\}$. As shown in Table 8, we always obtain a contradiction to (4.1).

x_2	$(v_e, v_{e'})$-path longer than P
v_0	$v_e v_0 v_2 v_1 u_0 u_1 P[v_4, v_{e'}]$ or $v_e v_0 x_2' v_2 v_1 u_0 u_1 P[v_4, v_{e'}]$
v_5, v_6	$v_e v_1 u_0 u_1 v_4 v_3 v_2 P[x_2, v_{e'}]$ or $v_e v_1 u_0 u_1 v_4 v_3 v_2 x_2' P[x_2, v_{e'}]$
v_7	$P[v_e, v_2] v_7 v_6 v_5 v_4 u_1 u_2 v_8 v_{e'}$ or $P[v_e, v_2] x_2' v_7 v_6 v_5 v_4 u_1 u_2 v_8 v_{e'}$
v_8	$v_e v_1 u_2 u_1 u_0 v_8 P[v_2, v_{e'}]$ or $v_e v_1 u_2 u_1 u_0 v_8 x_2' P[v_2, v_{e'}]$

Table 8: Case C2 in Claim 2 .

Case C3. $G(e, e')$ has L_3 as a subgraph.

As in Case C1, since $G(e, e')$ is reduced, we may assume that $x_2 \notin \{v_1, v_3, v_4\}$. As shown in Table 9, we always obtain a contradiction to (4.1). This proves Case (C3) and completes the proof of Claim 2.

x_2	$(v_e, v_{e'})$-path longer than P
v_0	$v_e v_1 v_2 v_0 u_0 u_1 P[v_4, v_{e'}]$ or $v_e v_1 v_2 x_2' v_0 u_0 u_1 P[v_4, v_{e'}]$
v_5	$v_e v_0 u_0 u_1 v_4 v_3 v_2 P[x_5, v_{e'}]$ or $v_e v_0 u_0 u_1 v_4 v_3 v_2 x_2' P[x_5, v_{e'}]$
v_6, v_7	$P[v_e, v_2] P^-[x_2, v_4] u_1 u_2 v_8 v_{e'}$ or $P[v_e, v_2] x_2' P^-[x_2, v_4] u_1 u_2 v_8 v_{e'}$
v_8	$v_e v_0 u_0 u_1 u_2 v_8 P[v_2, v_{e'}]$ or $v_e v_0 u_0 u_1 u_2 v_8 x_2' P[v_2, v_{e'}]$

Table 9: Case C3 in Claim 2 .

Let $V(J) = \{u_1, u_2, \cdots\}$. By Claim 2(C), if $j \geq 1$, then $N_G(u) \subseteq V(P) \cup \{v_0, v_c\}$. Thus we may denote that $N_G(u_j) = \{v_{i_1^j}, v_{i_2^j}, v_{i_3^j}\}$ with $i_1^j < i_2^j < i_3^j$. Claim 3 below follows from the fact that P is longest.

Claim 3. Let $u_j \in V((G)(e, e')) - V(P) \cup \{v_0, v_c\}$. Then

$$i_2^j \geq \begin{cases} i_1^j + 2, & \text{if } i_1 \neq 0 \\ 3, & \text{if } i_1 = 0 \end{cases}, \text{ and } i_3^j \geq \begin{cases} i_2^j + 2, & \text{if } c > i_3^j \\ i_2^j + 3, & \text{if } c = i_3^j \end{cases}.$$

Therefore, $c \geq 6$.

Claim 4. Let $u_j \in V(J)$. If $i_2^j = i_1^j + 2$, then $N_G(v_{i_1^j+1}) \subseteq V(P) \cup \{v_0, v_c\}$. If $i_3^j = i_2^j + 2$, then $N_G(v_{i_2^j+1}) \subseteq V(P) \cup \{v_0, v_c\}$.

Proof of Claim 4. By symmetry, we only prove the case when $i_2^j = i_1^j + 2$. Let $P' = P[v_e, v_{i_1^j}] u_j P^-[v_{i_2^j}, v_{e'}]$. Then $|P| = |V(P')|$. Applying Claim 2(C) on P', we conclude that $E(G(e, e') - (V(P') \cup \{v_0, v_c\})) = \emptyset$, and so $N_G(v_{i_1^j+1}) \subseteq V(P') \cup \{v_0, v_c\}$. Since $girth(G(e, e')) \geq 4$, $u_j v_{i_1^j+1} \notin E(G)$. Thus $N_G(v_{i_1^j+1}) \subseteq V(P) \cup \{v_0, v_c\}$. This justifies Claim 4.

In Claims 5-8 below, we assume that $u_1, u_2 \in V(J)$ and define $s = \min\{i_1^1, i_2^1, i_3^1, i_1^2, i_2^2, i_3^2\}$ and $\ell = \max\{i_1^1, i_2^1, i_3^1, i_1^2, i_2^2, i_3^2\}$. For any v_i with

$1 \le i \le 7$, by $\kappa'(G) \ge 3$, there exists $x'_i \in N_G(v_i) - N_P(v_i)$. By Claim 2(C), either $x'_i \in V(P) \cup \{v_0, v_c\}$, or $x'_i \in V(J)$ with $N_G(x'_i) \subseteq V(P) \cup \{v_0, v_c\}$. Define x_i as in (4.7). By (4.2) and (4.1), when $i-2 \ge 1$ and $i+2 \le c-1$, we can always choose x_i so that

$$x_i \notin \{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\}, \text{ unless } x'_i \in V(J) \text{ and } N_G(x'_i) = \{v_{i-2}, v_i, v_{i+2}\}. \tag{4.8}$$

Claim 5. Each of the following holds.
(i) $\ell - s \le 8$. Furthermore, if $\ell - s = 8$, then $c = 8, \ell = 8$ and $s = 0$.
(ii) $i_1^2 \le i_2^1$. (By symmetry, $i_1^1 \le i_2^2$, $i_3^1 \ge i_2^2$ and $i_3^2 \ge i_2^1$.)

(iii) A $(v_e, v_{e'})$-path longer than P exists if for some $s \in \{1, 2\}$,

$$1 \le i_1^1 < i_s^2 < i_2^1 < i_{s+1}^2 < i_3^1 < c \text{ or } 1 \le i_1^2 < i_s^1 < i_2^2 < i_{s+1}^1 <, i_3^2 < c.$$

(iv) We cannot have $i_1^1 = i_1^2 < i_2^1 < i_2^2 = i_3^1 < i_3^2$. (By symmetry, we cannot have $i_1^1 < i_1^2 = i_2^1 < i_2^2 < i_3^1 = i_3^2$.)
(v) We cannot have $i_1^1 = i_1^2 < i_2^1 < i_2^2 < i_3^1 = i_3^2$.
(vi) $i_2^2 = i_2^1$.

Proof of Claim 5. (i). Claim 5(i) follows immediately from $c \le 8$.

(ii). We argue by contradiction to prove $i_1^2 \le i_2^1$. The proof for $i_3^1 \ge i_2^2$ is omitted by symmetry. Assume that $i_1^2 > i_2^1$.

Case 5(ii).1. $i_3^2 > i_3^1$.

If $i_2^2 > i_3^1$, then $i_3^2 \ge i_2^2 + 2$. As $P[v_e, v_{i_2^1}]u_1P^-[v_{i_3^1}, v_{i_1^2}]u_2P[v_{i_2^2}, v_{e'}]$ is not longer than P, we have $(i_1^2 - i_2^1) + (i_2^2 - i_3^1) \ge 4$ and so $i_3^2 - i_1^1 \ge 9$, contrary to Claim 5(i). If $i_2^2 = i_3^1$, then as $P[v_e, v_{i_2^1}]u_1P^-[v_{i_3^1}, v_{i_1^2}]u_2P[v_{i_3^2}, v_{e'}]$ is not longer than P, we have $(i_1^2 - i_2^1) + (i_3^2 - i_2^2) \ge 4$, and so $i_3^2 - i_1^1 \ge 8$. By Claim 5(i), $c = 8, i_1^1 = 0$ and $i_3^2 = 8$. Thus $i_2^1 = 3, i_3^1 = 6$, and so the path $P[v_e, v_6]u_2v_8v_{e'}$ is longer than P, contrary to (4.1). If $i_2^2 < i_3^1$, then as $P[v_e, v_{i_2^1}]u_1P^-[v_{i_3^1}, v_{i_1^2}]u_2P[v_{i_3^2}, v_{e'}]$ is not longer than P, and so $(i_3^2 - i_3^1) + (i_1^2 - i_2^1) \ge 4$. Since $i_2^1 - i_1^1 \ge 2$ and $i_2^2 - i_1^2 \ge 2$, we have $i_3^2 - i_1^1 \ge 9$, contrary to Claim 5(i).

Case 5(ii).2. $i_3^2 = i_3^1$.

By (4.1), $i_3^1 \ge i_1^1 + 7$. As $i_1^1 = 0$ implies $i_2^1 \ge 3$, we conclude that we always have $c = v_3^1 = 8$, $i_2^2 = 6$, $i_1^2 = 3$, and $i_1^1 \in \{0, 1\}$. Hence $P[v_e, v_6]u_2v_8v_{e'}]$ is longer than P, contrary to (4.1).

Case 5(ii).3. $i_3^2 < i_3^1$.

By (4.1), $i_3^1 \ge i_1^1 + 8$, and so we must have $i_1^1 = 0, i_3^1 = 8$. That $i_1^1 = 0$ forces $i_2^1 \ge 3$, and so $i_3^1 \ge 9$, contrary to Claim 5(i).

(iii). If $1 \le i^1_1 < i^2_s < i^1_2 < i^2_{s+1} < i^1_3 < c$, then either $P[v_e, v_{i^1_1}]u_1P^-[v_{i^1_2}, v_{i^2_s}]u_2P[v_{i^2_{s-1}}, v_{e'}]$ or $P[v_e, v_{i^2_s}]u_2\,P^-[v_{i^2_{s+1}}, v_{i^1_2}]u_1P[v_{i^1_3}, v_{e'}]$ is longer than P. The proof for the other case is similar. This proves Claim 5(iii).

(iv). Assume that $i^1_1 = i^2_1 < i^1_2 < i^2_2 = i^1_3 < i^2_3$.

By (4.2), $i^2_3 - i^1_1 \ge 6$. If $i^1_1 = 0$, then by (4.1), we must have $i^1_2 \ge 3$, $i^1_3 \ge 5$ and $i^2_3 \in \{7, 8\}$. Then $P[v_e, v_{i^1_3}]u_1v_0u_2v_{i^2_3}v_{e'}$ is longer than P, and so we assume that $2 \ge i^1_1 \ge 1$. If $i^1_1 = 2$, then we must have $i^1_3 = 6$, $c = 8$ and $v_8u_2 \in E(G)$, and so $P[v_e, v_6]u_2v_8v_{e'}$ is longer than P. Hence we have $(i^1_1, i^1_2, i^1_3) = (1, 3, 5)$ with $i^2_3 \in \{7, 8\}$. If $i^2_3 = 8$, then Table 10A shows that a contradiction can always be found. By (4.8), $x_4 \notin \{v_3, v_4, v_5\}$, and $x_4 \in \{v_2, v_6\}$ only if $x'_4 \ne x_4$.

x_4	$(v_e, v_{e'})$-path longer than P	Explanation
v_0	$v_ev_0v_4v_3v_2v_1u_1P[v_5, v_{e'}]$	
v_1		$G[\{v_1, v_2, v_3, v_4, v_5, u_1\}]$ is not reduced, by Lemma 2.2(i).
v_2	$v_ev_1v_2x'_4v_4v_3u_1P[v_5, v_{e'}]$	
v_6, v_7	$P[v_e, v_4]v_6v_5u_2v_8v_{e'}$ or $P[v_e, v_4]v_7v_6v_5u_2v_8v_{e'}$	
v_8	$P[v_e, v_4]v_8u_2v_5v_6v_7v_{e'}$	

Table 10A: Claim 5(iv) with $i^2_3 = 8$.

If $i^2_3 = 7$, then Table 10B shows that a contradiction can always be found. By (4.8), $x_2 \notin \{v_1, v_2, v_3\}$ and $x_2 = v_4$ only if $x'_2 \ne x_2$.

x_2	$(v_e, v_{e'})$-path longer than P	Explanation
v_0	$v_ev_0v_2v_1u_1P[v_3, v_{e'}]$	
v_4	$v_ev_1v_2x'_2v_4v_3u_1P[v_5, v_{e'}]$	
v_5		$G[\{v_1, v_2, v_3, v_4, v_5, u_1\}]$ is not reduced, by Lemma 2.2 (i).
v_6	$v_ev_1u_1v_5v_4v_3v_2v_6v_7v_{e'}$	
v_7	$(x'_2 \ne v_7)$, $v_ev_1u_1v_5v_4v_3v_2x'_2v_7v_{e'}$	$(x'_2 = v_7)$ and $G[\{v_1, v_2, v_3, v_4, v_5, v_7, u_1, u_2\}]$ is not reduced, by Lemma 2.2(i).
v_8	$v_ev_1u_2P^-[v_7, v_2]v_8v_{e'}$	

Table 10B: Claim 5(iv) with $i^2_3 = 7$.

(v). Assume that $i^1_1 = i^2_1 < i^1_2 < i^2_2 < i^1_3 = i^2_3$.

By (4.2), $i_3^2 - i_1^1 \geq 5$. If $i_3^2 - i_1^1 = 5$, then $G[V(P[v_{i_1^1}, v_{i_3^2}]) \cup \{u_1, u_2\}]$ is not reduced, by Lemma 2.2(i). Hence we assume that $i_3^2 - i_1^1 \geq 6$. If $i_1^1 = 0$, then by (4.1), $i_2^1 \geq 3$ and $i_2^2 \geq 4$, and so $P[v_e, v_{i_2^2}]u_2 v_0 u_1 P[v_{i_3^1}, v_{e'}]$ is longer than P. Thus $i_1^1 > 0$. By symmetry, $i_3^2 < c$. As $c \leq 7$ implies $i_3^2 - i_1^1 = 5$, we must have $c = 8$ and $(i_1^1, i_2^1, i_2^2, i_3^2) \in \{(1,3,4,6), (1,3,5,7), (1,3,4,7)\}$. By (4.8), $x_4 \notin \{v_3, v_4, v_5\}$ and $x_4 \in \{v_2, v_6\}$ only if $x_4' \neq x_4$; $x_2 \notin \{v_1, v_2, v_3\}$ and $x_2 = v_4$ only if $x_2' \neq x_2$.

x_4	x_2	$(v_e, v_{e'})$-path longer than P	Explanation or Conclusion
v_0		$v_e v_0 v_4 v_3 v_2 v_1 u_2 v_5 v_6 v_7 v_{e'}$	
v_1			$G[\{v_1, v_2, v_3, v_4, v_5, u_1\}]$ is not reduced, by Lemma 2.2 (i).
v_2		$v_e v_1 u_1 v_3 v_2 x_4' P[v_4, v_{e'}]$	(same for $x_2 = v_4$).
v_6		$P[v_e, v_4] x_4' v_6 v_5 u_2 v_7 v_{e'}$	
v_7			either $v_4 v_7 \in E(G)$ and $G[\{v_1, v_2, v_3, v_4, v_5, v_7, u_1, u_2\}]$ is not reduced, by Lemma 2.2(i); or $v_4, v_7 \in N_G(x_4')$, for $\pi = (\{u_2, v_6\}, \{v_5, v_7\})$, $G[(V(P) - \{v_e, v_{e'}\}) \cup \{u_1, u_2, x_4'\}]/\pi$ is collapsible by Theorem 2.4, G is not reduced.
v_5		$P[v_e, v_3] u_1 v_5 x_4' v_8 v_{e'}$	Hence $v_4 v_8 \in E(G)$.
	v_0	$v_e v_0 P[v_2, v_5] u_1 v_1 u_2 v_7 v_{e'}$	
	v_5		$G[\{v_1, v_2, v_3, v_4, v_5, u_1\}]$ is not reduced, by Lemma 2.2 (i).
	v_6	$v_e v_1 u_1 P^-[v_5, v_2] v_6 v_7 v_{e'}$	
v_8	v_7	$v_e v_1 u_1 v_3 v_2 v_7 v_6 v_5 v_4 v_8 v_{e'}$	$v_4 v_8 \in E(G)$.
v_8	v_8	$v_e v_1 u_1 v_3 v_2 v_8 P[v_4, v_{e'}]$	$v_4 v_8 \in E(G)$.

Table 11A: Claim 5(v) with $(i_1^1, i_2^1, i_2^2, i_3^2) = (1,3,5,7)$.

If $(i_1^1, i_2^1, i_2^2, i_3^2) = (1,3,4,6)$, by Theorem 2.2(i), $G(e,e')[V(P[v_1, v_6]) \cup \{u_1, u_2\}]$ is collapsible, contrary to (4.2). If $(i_1^1, i_2^1, i_2^2, i_3^2) = (1,3,5,7)$, then Table 11A first indicates that $v_4 v_8 \in E(G)$ and then shows that a contradiction can always be found.

Thus we assume $(i_1^1, i_2^1, i_2^2, i_3^2) = (1,3,4,7)$. Table 11B shows that a contradiction can always be found. By (4.7), if $v_2 v_7 \notin E(G)$, then $x_2 \neq v_7$.

x_2	$(v_e, v_{e'})$-path longer than P	Explanation or Conclusion
v_0	$v_e v_0 v_2 v_3 u_1 v_1 u_2 P[v_4, v_{e'}]$	
v_4	$v_e v_1 u_1 v_3 v_2 x'_2 P[v_4, v_{e'}]$	
v_5, v_6	$v_e v_1 v_2 x'_2 P^-[x_2, v_3] u_1 v_7 v_{e'}$	$G[V(P[v_1, v_7]) \cup \{u_1, u_2\}]$ is not reduced, by Lemma 2.2 (iv).
v_7		$G[\{v_1, v_2, v_3, v_4, v_7, u_1, u_2\}]$ is not reduced, by Lemma 2.2(i).
v_8	$v_e v_1 u_2 P^-[v_7, v_2] v_8 v_{e'}$	

Table 11B: Claim 5(v) with $(i_1^1, i_2^1, i_2^2, i_3^2) = (1, 3, 4, 7)$.

(vi). We shall prove that assuming $i_2^2 > i_2^1$ will lead to contradictions, and so $i_2^2 \leq i_2^1$. By symmetric arguments, we also have $i_2^1 \leq i_2^2$, which proves (vi). In the rest of the proof of Claim 5(vi), we assume that $i_2^2 > i_2^1$. By Claim 5(ii), (iv) and (v), $i_1^2 \leq i_2^1$, $i_3^1 \geq i_2^2$ and $i_1^1 \neq i_1^2$ when $i_1^3 \in \{i_2^2, i_3^2\}$.

Case 5(vi).1. $i_3^1 \notin \{i_2^2, i_3^2\}$.

Case 5(vi).1A. $i_3^1 > i_3^2$.

If $i_1^2 < i_1^1$, then by (4.1), either $(i_1^2, i_1^1, i_3^2, i_3^1) = (0, 1, 6, 7)$, whence $v_e v_0 u_2 P^-[v_6, v_1] u_1 v_7 v_{e'}$ is longer than P; or $(i_1^2, i_1^1, i_3^2, i_3^1) = (1, 2, 7, 8)$, whence $v_e v_1 u_2 P^-[v_7, v_2] u_1 v_8 v_{e'}$ is longer than P. If $i_1^1 = i_1^2 \in \{0, 1\}$, then $i_2^1 \geq 3$, $i_3^2 - i_2^1 \geq 3$ and $i_3^1 \in \{7, 8\}$. Thus $v_e v_{i_1^1} u_2 P^-[v_{i_3^2}, v_{i_2^1}] u_1 v_{i_3^1} v_{e'}$ is longer than P. Hence we assume $i_1^2 > i_1^1$.

If $i_1^2 = i_2^1$, then by $girth(G(e, e')) \geq 4$, $i_3^1 - i_1^1 \geq 8$, and so $i_1^1 \in \{0, 1\}$ and $(i_1^2, i_2^2, i_3^2, i_3^1) = (3, 5, 7, 8)$. By (4.8), $x_4 \notin \{v_3, v_4, v_5\}$, and $x_4 \in \{v_2, v_6\}$ only if $x'_4 \neq x_4$. Table 11C shows that a contradiction always exists.

x_4	i_1^1	$(v_e, v_{e'})$-path longer than P	Explanation or Conclusion
v_0	0, 1	$v_e v_0 v_4 v_5 v_6 v_7 u_2 v_3 u_1 v_8 v_{e'}$	
v_1	0	$v_e v_0 u_1 v_3 v_2 v_1 P[v_4, v_{e'}]$	
v_1	1	$v_e v_1 P[v_4, v_7] u_2 v_3 u_1 v_8 v_{e'}$	
v_2	0, 1	$v_e v_1 v_2 x'_4 P[v_4, v_7] u_2 v_3 u_1 v_8 v_{e'}$	
v_6	0,1	$P[v_e, v_4] x'_6 v_6 v_5 u_2 v_7 v_{e'}$	
v_7	0,1	$(x'_4 \neq v_7)$ $P[v_e, v_3] u_2 v_5 v_4 x'_4 v_7 v_{e'}$	$(x'_4 = v_7)$ $G[\{v_3, v_4, v_5, v_6, v_7, u_2\}]$ is not reduced, by Lemma 2.2(i).
v_8	0,1	$P[v_e, v_3] u_2 P^-[v_7, v_4] v_8 v_{e'}$	

Table 11C: Claim 5(vi).1A with $i_1^1 \in \{0, 1\}$ and $(i_1^2, i_2^2, i_3^2, i_3^1) = (3, 5, 7, 8)$.

Thus we have $i_1^1 < i_1^2 < i_2^1 < i_2^2 < i_3^2 < i_3^1$. By Claim 5(iii), a $(v_e, v_{e'})$-path longer than P exists. This excludes this subcase.

Case 5(vi).1B. $i_2^2 < i_3^1 < i_3^2$.

Now we have $i_1^1 < i_2^1 < i_2^2 < i_3^1 < i_3^2$, As $P[v_e, v_{i_2^1}]u_1P^-[v_{i_3^1}, v_{i_2^2}]u_2 P[v_{i_3^2}, v_{e'}]$ is not longer than P, $(i_2^2 - i_2^1) + (i_3^2 - i_3^1) \geq 4$. By Claim 5(iii), we cannot have $i_1^1 < i_1^2 < i_2^1$, and so $i_1^2 < i_1^1$ or $i_1^2 \in \{i_1^1, i_2^1\}$.

If $i_1^2 < i_1^1$, then $i_2^1 - i_1^1 \geq 2$. Since $(i_2^2 - i_2^1) + (i_3^2 - i_3^1) \geq 4$, it follows that $l - s \geq i_3^2 - i_1^2 \geq 8$. By Claim 5(i), $i_1^2 = 0$, $i_1^1 = 1$, $i_2^2 = 5$, $i_3^1 = 6$ and $i_3^2 = 8$. Hence $v_e v_0 u_2 P^-[v_5, v_1]u_1 v_6 v_7 v_{e'}$ is longer than P.

If $i_1^2 = i_1^1$, then by $girth(G(e, e')) \geq 4$ and by Claim 3, we must have $i_2^1 \geq 3$. It follows by $l - s \leq 8$ and by $(i_2^2 - i_2^1) + (i_3^2 - i_3^1) \geq 4$, we must have $i_1^1 \in \{0, 1\}$, and $(i_2^1, i_2^2, i_3^1, i_3^2) \in \{(3, 5, 6, 8), (3, 6, 7, 8)\}$. If $i_1^1 = 0$, then $P[v_e, v_{i_3^1}]u_1 v_0 u_2 v_{i_3^2} v_{e'}$ is longer than P. If $i_1^1 = 1$, then $P[v_e, v_{i_2^1}]u_1 P^-[v_{i_3^1}, v_{i_2^2}]u_2 v_8 v_{e'}$ is longer than P. This excludes this subcase.

Case 5(vi).1C. $i_3^1 < i_2^2$.

Then $i_1^1 < i_2^1 < i_3^1 < i_2^2 < i_3^2$. By Claim 5(ii), $i_2^1 \geq i_1^2$. If $i_2^1 > i_1^2 > i_1^1$, then As $P[v_e, v_{i_1^1}]u_1 P^-[v_{i_3^1}, v_{i_1^2}]u_2 P[v_{i_2^2}, v_{e'}]$ is not longer than P, we have $(i_1^2 - i_1^1) + (i_2^2 - i_3^1) \geq 4$. By $girth(G(e, e')) \geq 4$, $i_3^2 - i_2^2 \geq 2$ and $i_3^1 - i_2^1 \geq 2$, it follows $i_3^2 - i_1^1 \geq 9$, contrary to Claim 5(i). If $i_1^2 < i_1^1$, then by $girth(G(e, e')) \geq 4$, we must have $i_3^2 - i_1^2 \geq 8$, and so by Claim 5(i), we have $i_3^2 = 8$. By Claim 3, $i_2^2 \leq 5$. which forces $i_3^2 - i_1^2 \geq 9$, contrary to Claim 5(i).

Hence $i_1^2 \in \{i_1^1, i_2^1\}$. If $i_2^1 = i_1^1 = 1$, then by $girth(G(e, e')) \geq 4$, we have $i_3^2 = 8$. By Claim 3, $i_2^2 \leq 5$, which forces $i_3^2 - i_1^2 \geq 9$, contrary to Claim 5(i). If $i_2^1 = i_1^1 = 0$, then by Claim 3 and by $girth(G(e, e')) \geq 4$, we have $i_3^1 \in \{5, 6\}$ and $i_2^2 \in \{6, 7\}$. Thus $P[v_e, v_{i_3^1}]u_1 v_0 u_2 P[v_{i_2^2}, v_{e'}]$ is longer than P. If $i_1^2 = i_2^1$, then by $girth(G(e, e')) \geq 4$, we have $i_3^2 - i_1^1 \geq 7$. As $P[v_e, v_{i_1^1}]u_1 P^-[v_{i_3^1}, v_{i_2^1}]u_2 P[v_{i_2^2}, v_{e'}]$ is not longer than P, we have $(i_2^1 - i_1^1) + (i_2^2 - i_3^1) \geq 4$, forcing $i_3^2 - i_1^1 \geq 8$. By Claim 5(i), we have $(i_2^2, i_3^2) = (7, 8)$, and so $P[v_e, v_7]u_2 v_8 v_{e'}$ is longer than P. This excludes this subcase and completes the proof of Case 5(vi).1.

Case 5(vi).2. $i_3^1 = i_3^2$. Thus $i_3^1 = i_3^2 > i_2^2 > i_2^1 > i_1^1$.

By Claim 3, we cannot have $i_1^1 < i_1^2 < i_2^1$. By Claim 5(iv) and (v), $i_1^2 \notin \{i_1^1, i_2^1\}$. Hence we have $i_1^2 < i_1^1$. Then either $v_e v_0 u_2 P^-[v_{i_2^2}, v_{i_1^1}]u_1 P[v_{i_3^2}, v_{e'}]$ (if $i_1^2 = 0$) or $P[v_e, v_{i_1^2}]u_2 P^-[v_{i_2^2}, v_{i_1^1}]u_1 P[v_{i_3^2}, v_{e'}]$ (if $i_1^2 > 0$) is longer than P. This proves this case.

Case 5(vi).3. $i_3^1 = i_2^2$. Thus $i_3^2 > i_3^1 = i_2^2 > i_2^1 > i_1^1$.

By $girth(G(e,e')) \geq 4$, $i_3^2 - i_1^1 \geq 6$. By Claim 5(iv), $i_1^1 \neq i_1^2$. Assume first that $i_1^2 < i_1^1$, and so $i_3^2 - i_1^2 \geq 7$, implying $i_3^2 \in \{7,8\}$. Note that when $i_3^2 \in \{7,8\}$, we must have $i_3^1 - i_1^1 = 4$, and so by Theorem ??(v) and (vi), adding any edge to

$$G(e,e')[V(P[v_1^1, v_3^1])] \text{ will result in a non reduced subgraph.} \quad (4.9)$$

As $i_3^2 \in \{7,8\}$, by Claim 3 and by $girth(G(e,e')) \geq 4$, we must have $(i_1^2, i_1^1, i_2^1, i_2^2) = (0,1,3,5)$. By (4.9) and (4.8), $x_4 \notin \{v_1, v_2, v_3, v_4, v_5, v_6\}$. Table 11D shows that for any other values of x_4, a contradiction to (4.1) can always be found, which completes the proof of this case.

x_4	i_3^2	$(v_e, v_{e'})$-path longer than P
v_0		$v_e v_0 P^-[v_4, v_1] u_1 P[v_5, v_{e'}]$
v_7 or v_8	7, 8	$v_e v_0 u_2 v_5 u_1 P[v_1, v_4] v_{i_3^2} v_{e'}$

Table 11D: Claim 5(vi).3 with $i_3^2 \in \{7,8\}$ and $(i_1^2, i_1^1, i_2^1, i_2^2) = (0,1,3,5)$.

Claim 6. $s = |\{v_{i_1^1}, v_{i_2^1}, v_{i_3^1}\} \cap \{v_{i_1^2}, v_{i_2^2}, v_{i_3^2}\}| \geq 2$.

If $s < 2$ then by Claim 5(i), $s = 1$. Without loss of generality, we assume that $i_1^1 \leq i_1^2$. Hence by Claim 5, we have $i_1^1 < i_1^2 < i_2^2 = i_2^1 \leq i_3^1$. By Claim 5(iii), we have $i_1^1 < i_1^2$ and $i_3^1 < i_3^2$. By (4.2), $i_3^1 - i_1^2 \geq 4$. It follows that $P[v_e, v_{i_1^1}] u_1 P^-[v_{i_3^1}, v_{i_1^2}] u_2 P[v_{i_3^2}, v_{e'}]$ is longer than P, contrary to (4.1). Claim 6 is justified.

Claim 7. $s = |\{v_{i_1^1}, v_{i_2^1}, v_{i_3^1}\} \cap \{v_{i_1^2}, v_{i_2^2}, v_{i_3^2}\}| = 3$.

By contradiction and Claim 6, assume that $s = 2$. Without loss of generality, we assume that $i_1^1 \leq i_1^2$. Hence by Claim 5, we have $i_1^1 \leq i_1^2 \leq i_2^2 = i_2^1 \leq i_3^1$, and so $s = 2$ implies, in addition to $i_2^1 = i_2^2$, either $i_1^1 = i_1^2$ or $i_3^1 = i_3^2$. By symmetry, it suffices to assume that both $i_1^1 = i_1^2$ and $i_2^1 = i_2^2$ to find contradictions.

Hence we have $i_1^1 = i_1^2 < i_2^2 = i_2^1 < i_3^1$. By symmetry, we may assume that $i_3^1 < i_3^2$. As $P[v_e, v_{i_1^1}] u_1\ P^-[v_{i_3^1}, v_{i_2^1}]\ u_2 P[v_{i_3^2}, v_{e'}]$ is not longer than P, $(i_3^2 - i_3^1) + (i_2^1 - i_1^1) \geq 4$, and so by (4.1) and (4.2), $i_3^2 - i_1^1 \geq 6$. Hence $i_2^1 \geq 3$, $i_1^1 \leq 1$ or $i_3^2 \geq 7$. If $v_1^1 = 0$, then by (4.1), $i_3^1 \geq 5$, and so $P[v_e, v_{i_3^1}] u_1 v_0 u_2 v_{i_3^2} v_{e'}$ is longer than P. This forces that $i_1^1 = 1$ and $i_3^1 - i_2^1 = 2$, and so $(i_2^1, i_3^1, i_3^2) \in \{(4,6,7), (4,6,8), (3,5,7), (3,5,8)\}$.

Assume first that $(i_2^1, i_3^1, i_3^2) \in \{(3,5,7), (3,5,8)\}$. By (4.8), $x_4 \notin \{v_2, v_3, v_4, v_5\}$. If $v_1 v_4 \in E(G)$, then by Lemma 2.2(i), $G[V(P[v_1, v_5]) \cup \{u_2\}]$ is not reduced. Hence when $x_4 = v_1$, $x_4' \in V(J)$. By $\kappa'(G) \geq 3$, we can choose $x_4 \in N_G(x'_4) - \{v_1, v_4\}$. Thus we may assume that $x_4 \neq v_1$ as well. If $u_2v_7, x_4v_7 \in E(G)$, then by Lemma 2.2(i), $G[V(P[v_3, v_7]) \cup \{v_1, u_1, u_2\}]$ is not reduced. Hence when $i_3^2 = 7$ and $x_4 = v_1$, $x'_4 \in V(J)$. Table 12A indicates that a contradiction can always be found.

x_4	$(v_e, v_{e'})$-path longer than P
v_0	$v_e v_0 P^-[v_1, v_4]u_1 P[v_5, v_{e'}]$ or $v_e v_0 x'_4 P^-[v_1, v_4]u_1 P[v_5, v_{e'}]$
v_6	$P[v_e, v_3]u_1 v_5 v_4 P[v_6, v_{e'}]$ or $P[v_e, v_3]u_1 v_5 v_4 x'_4 P[v_6, v_{e'}]$
v_7	$P[v_e, v_3]u_1 v_5 v_4 x'_4 P[v_5, v_{e'}]$, if $i_3^2 = 7$.
v_7	$v_e v_1 u_1 v_5 v_6 v_7 v_4 v_3 u_2 v_8 v_{e'}$ or $v_e v_1 u_1 v_5 v_6 v_7 x'_4 v_4 v_3 u_2 v_8 v_{e'}$, if $i_3^2 = 8$.
v_8	$P[v_e, v_3]u_2 P^-[v_4, v_7]v_8 v_{e'}$ or $P[v_e, v_3]u_2 P^-[v_4, v_7]x'_4 v_8 v_{e'}$, if $i_3^2 = 7$.
v_8	$P[v_e, v_3]u_2 v_8 P[v_4, v_{e'}]$ or $P[v_e, v_3]u_2 v_8 x'_4 P[v_4, v_{e'}]$, if $i_3^2 = 8$.

Table 12A:

Claim 7, with $i_1^1 = i_1^2$, $i_2^1 = i_2^2$ and $(i_2^1, i_3^1, i_3^2) \in \{(3,5,7), (3,5,8)\}$.

Hence $(i_2^1, i_3^1, i_3^2) \in \{(4,6,7), (4,6,8)\}$. By (4.8), $x_5 \notin \{v_3, v_4, v_5, v_6\}$. If $v_1 v_v \in E(G)$, then by Lemma 2.2(i), $G[\{v_1, v_4, v_5, v_6, u_1, u_2\}]$ is not reduced. Hence when $x_5 = v_1$, $x'_5 \in V(J)$. By $\kappa'(G) \geq 3$, we can choose $x_5 \in N_G(x'_5) - \{v_1, v_5\}$. Thus we may assume that $x_5 \neq v_1$ as well. Table 12B indicates that a contradiction can always be found. This completes the proof for Claim 7.

x_5	$(v_e, v_{e'})$-path longer than P
v_0	$v_e v_0 P^-[v_5, v_1]u_1 P[v_6, v_{e'}]$ or $v_e v_0 x'_5 P^-[v_5, v_1]u_1 P[v_6, v_{e'}]$
v_2	$v_e v_1 u_1 v_4 v_3 v_2 P[v_5, v_{e'}]$ or $v_e v_1 u_1 v_4 v_3 v_2 x'_5 P[v_5, v_{e'}]$
v_7 or v_8	$P[v_e, v_4]u_1 v_6 v_5 x_5 v_{e'}$ or $P[v_e, v_4]u_1 v_6 v_5 x'_5 x_5 v_{e'}$

Table 12B: Claim 7, with $i_1^1 = i_1^2$, $i_2^1 = i_2^2$ and $(i_2^1, i_3^1, i_3^2) \in \{(4,6,7), (4,6,8)\}$.

Claim 8. $|V(G(e, e')) - V(P) \cup \{v_0, v_c\}| = 1$.

By contradiction, we assume that $u_1, u_2 \in V(G(e, e')) - V(P) \cup \{v_0, v_c\}$. By Claim 7, $|N_G(u_1) \cap N_G(u_2)| = 3$, and so $i_j^1 = i_j^2$ for $1 \leq j \leq 3$. If $i_1^2 = i_1^1 + 2$ or $i_3^1 = i_2^1 + 2$, then $G(e, e')$ has a collapsible subgraph $K_{3,3}^-$, contrary to (4.2). Therefore, we must have both $i_2^1 \geq i_1^1 + 3$ and $i_3^1 \geq i_2^1 + 3$. Since $G(e, e')$ does not contain a collapsible subgraph $K_{3,3}$, by Claim 7, $V(G) = V(P) \cup \{v_0, v_c, u_1, u_2\}$.

Since $V(G) = V(P) \cup \{v_0, v_c, u_1, u_2\}$. $x_3 = x_3'$. By (4.8), $x_3 \notin \{v_1, v_2, v_3, v_4, v_5\}$. If $x_3 = v_7$, then $v_3v_7 \in E(G)$, and so by Lemma 2.2(i), $G[\{v_1, v_2, v_3, v_4, v_7, u_1, u_2\}]$ is not reduced. Thus $x_3 \neq v_7$ and so $x_3 \in \{v_0, v_6, v_8\}$. It follows that $v_ev_0v_3v_2v_1u_1P[v_4, v_{e'}]$ (if $x_3 = v_0$) or $P[v_e, v_3]v_6v_5v_4u_1v_7v_{e'}$ (if $x_3 = v_6$), or $v_ev_1u_1P^-[v_7, v_3]v_8v_{e'}$ (if $x_3 = v_8$) is longer than P.

Thus either $i_1^1 = 0$ or $i_3^1 = c$. Without loss of generality, we assume that $i_1^1 = 0$. As neither $v_ev_0u_1P[v_{i_2^1}, v_{e'}]$ nor $P[v_e, v_{i_2^1}]u_1v_0u_2P[v_{i_3^1}, v_{e'}]$ is longer than P, we must have $i_2^1 = 3$. By symmetry, that $i_3^1 = c = 8$ implies $i_2^1 = 5 > 3$, and so $i_1^1 = 1$ implies $i_3^1 = 7$. Hence $(i_1^1, i_2^1, i_3^1) \in \{(0,3,6), (0,3,7)\}$. If $(i_1^1, i_2^1, i_3^1) = (0,3,6)$, then $v_ev_1v_2v_3u_2v_0u_1P[v_6, v_{e'}]$ is longer than P. Therefore $(i_1^1, i_2^1, i_3^1) = (0,3,7)$.

By (4.8), $x_4 \notin \{v_2, v_3, v_4, v_5, v_6\}$. Since $G(e, e')$ cannot have a $K_{3,3}^-$, $x_4 \notin \{v_0, v_7\}$, and so $x_4 \in \{v_1, v_8\}$. Hence $v_ev_0u_1v_3v_2v_1P[v_4, v_{e'}]$ (if $x_4 = v_1$) or $P[v_e, v_3]u_1P^-[v_7, v_4]v_8v_{e'}$ (if $x_4 = v_8$) is longer than P, contrary to (4.1). This proves Claim 8.

Define a new graph L_z from $G(e, e')$ by adding a new vertex z and new edges zv_e and $zv_{e'}$. Then $G(e, e')$ has a spanning $(v_e, v_{e'})$-trail if and only if L_z is supereulerian. If $c \leq 7$, or if $c = 8$ and $v_0 = v_c$, then by Claim 8, $|V(L_z)| \leq 12$. As L_z has exactly one edge cut of size 2, it follows by Lemma 2.2(iv) that either L is supereulerian, whence $G(e, e')$ has a spanning $(v_e, v_{e'})$-trail: or $L_z = P(10)(e)$, whence $G = W_8$. In either case, a contradiction to the assumptions of Theorem 3.1 is found. Hence we will assume that $c = 8$ and that $v_0 \neq v_c$.

By Claim 8, we denote $V(J) = \{u\}$. By Claim 7, $N_G(u) \subseteq V(P) \cup \{v_0, v_c\}$. Let $N_G(u) = \{v_{i_1}, v_{i_2}, v_{i_3} \ldots\}$. By Claim 4, we may assume $i_1 < i_2 < i_3$. By (4.2), both $i_2 > i_1 + 2$ and $i_3 \geq i_2 + 2$, and if $i_1 = 0$, then $i_2 > 2$, and if $i_3 = c$, then $i_2 < c - 2$. Therefore, the possibilities of (i_1, i_2, i_3) can be listed below:

i_1	i_2	(i_1, i_2, i_3)	Symmetric case	i_1	i_2	(i_1, i_2, i_3)	Symmetric case
0	3	(0, 3, 5)	(3,5,8)	1	3	(1, 3, 5)	(3,5,7)
0	3	(0, 3, 6)	(2,5,8)	1	3	(1, 3, 6)	(2,5,7)
0	3	(0, 3, 7)	(1,5,8)	1	3	(1, 3, 7)	(1,5,7)
0	3	(0, 3, 8)	(0,5,8)	1	4	(1, 4, 6)	(2,4,7)
0	4	(0, 4, 6)	(2,4,8)	1	4	(1, 4, 7)	(1,4,7)
0	4	(0, 4, 7)	(1,4,8)	2	4	(2, 4, 6)	(2,4,6)
0	4	(0, 4, 8)	(0,4,8)				
0	5	(0, 5, 7)	(1,3,8)				

Table 13: Possibilities of (i_1, i_2, i_3).

We shall show that in each of these cases of (i_1, i_2, i_3), either a longer $(v_e, v_{e'})$-path is found or a nontrivial collapsible subgraph of $G(e, e')$ is found, leading to contractions to (4.2). For each i with $1 \le i \le 7$, denote $x_i (= x_i') = v_{s_i}$. By (4.8),

$$s_1 \in \{4, 5, 6, 7, 8\}, s_2 \in \{0, 5, 6, 7, 8\}, s_3 \in \{0, 6, 7, 8\}$$

$$s_4 \in \{0, 1, 7, 8\}, s_5 \in \{0, 1, 2, 8\}, s_6 \in \{0, 1, 2, 3, 8\}, s_7 \in \{0, 1, 2, 3, 4\} \quad (4.10)$$

Claim 9. If $i_1 = 0$, then each the following statements holds.
(i) Let $t \in \{i_2, i_3\}$. Then $v_0v_{t-1}, v_0v_{t-2} \notin E(G)$, and $v_0v_{t+1}, v_0v_{t+2}, v_1v_{t+1}, v_1v_{t+2} \notin E(G)$ if $t+1, t+2 \le 8$.
(ii) If $i_3 \neq 8$, then $v_1v_8 \notin E(G)$.

As $i_3 \ge 5$, Table 14 proves Claim 9.

	Edge in $E(G)$	$(v_e, v_{e'})$-path longer than P	
(i)	$v_0v_{t-1} \in E(G)$	$P[v_e, v_{t-1}]v_0uP[v_t, v_{e'}]$	
	$v_0v_{t-2} \in E(G)$	$P[v_e, v_{t-2}]v_0uP[v_t, v_{e'}]$	
	$v_0v_{t+1} \in E(G)$	$P[v_e, v_t]uv_0P[v_{t+1}, v_{e'}]$	$t+1 \le 8$
	$v_0v_{t+2} \in E(G)$	$P[v_e, v_t]uv_0P[v_{t+2}, v_{e'}]$	$t+2 \le 8$
	$v_1v_{t+1} \in E(G)$	$v_ev_0uP^-[v_t, v_1]P[v_{t+1}, v_{e'}]$	$t+1 \le 8$
	$v_1v_{t+2} \in E(G)$	$v_ev_0uP^-[v_t, v_1]P[v_{t+2}, v_{e'}]$	$t+2 \le 8$
(ii)	$v_1v_8 \in E(G)$	$v_ev_0u\, P^-[v_{i_3}, v_1]v_8v_{e'}$	

Table 14: Proof of Claim 9.

Claim 10. If $i_1 = 0$, then $i_2 = 3$.

If $i_2 \ge 4$, then $(i_1, i_2, i_3) \in \{(0,4,6), (0,4,7), (0,4,8), (0,5,7)\}$. By (4.10), $s_5 \in \{0, 1, 2, 8\}$. Hence $P[v_e, v_4]uv_0P[v_5, v_{e'}]$ (if $v_0v_5 \in E(G)$), or $v_ev_0uP^-[v_4, v_1]P[v_5, v_{e'}]$ (if $v_1v_5 \in E(G)$), or $v_ev_0uv_4v_3v_2P[v_5, v_{e'}]$ (if $v_2v_5 \in E(G)$) is longer than P. Thus $v_5v_8 \in E(G)$. If $i_3 \in \{6, 7\}$, then $P[v_e, v_4]uv_{i_3}P^-[v_{i_3}, v_5]v_8v_{e'}$ is longer than P. Hence $(i_1, i_2, i_3) = (0, 4, 8)$. In this case, $P[v_e, v_4]uv_8P[v_5, v_{e'}]$ is longer than P, contrary to (4.1), and so Claim 10 holds.

Claim 11. Both $i_1 \ge 1$ and $i_3 \le c - 1$. Furthermore, when $i_1 = 1$, each of the following holds.
(i) Let $t \in \{i_2, i_3\}$. Then $x_0v_{t-1} \notin E(G)$.

(ii) If $i_2 \geq 4$, then $v_0 v_{i_2-2} \notin E(G)$. If $i_3 \geq i_2 + 3$, then $v_0 v_{i_3-2} \notin E(G)$.

By symmetry, we assume that $i_1 = 0$. By Claim 10, $(i_1, i_2, i_3) \in \{(0,3,5), (0,3,6), (0,3,7), (0,3,8)\}$. By (4.10), $s_1 \in \{4,5,6,7,8\}$. By Claim 9, $(i_1, i_2, i_3) \neq (0,3,5)$.

If $(i_1, i_2, i_3) = (0,3,6)$, then by Claim 9, $v_1 v_6 \in E(G)$, and so $v_2 v_6 \notin E(G)$. By (4.10), $s_2 \in \{5,7,8\}$.

If $(i_1, i_2, i_3) = (0,3,8)$, then by (4.10) and Claim 9, $s_1 \in \{6,7,8\}$. If $s_1 = 6$, then $s_2 \neq 6$ and so by (4.10), and Claim 9, $s_2 \in \{5,7,8\}$.

If $(i_1, i_2, i_3) = (0,3,7)$, then by (4.10) and Claim 9, $s_4 \in \{7,8\}$; and if $s_4 = 7$, $s_5 \in \{2,8\}$. Similarly, When $v_4 v_7, v_2 v_5 \in E(G)$, $s_1 \in \{6,7\}$.

With these analysis, Table 15 proves Claim 11.

Cases	(i_1, i_2, i_3)	Edge in $E(G)$	$(v_e, v_{e'})$-path longer than P	Conclusions
$i_1 \geq 1$ $i_3 \leq c-1$	(0,3,6)	$v_2 v_5 \in E(G)$	$v_e v_1 v_2 v_5 v_4 v_3 uP[v_6, v_{e'}]$	
		$v_2 v_{s_2} \in E(G)$	$v_e v_0 u P^-[v_6, v_2] v_{s_2} v_{e'}$	$s_2 \in \{7,8\}$ $s_2 \in \{7,8\}$
	(0,3,8)	$v_1 v_7 \in E(G)$	$v_e v_1 P^-[v_7, v_3] u v_8 v_{e'}$	
		$v_1 v_8 \in E(G)$	$v_e v_0 u v_8 P[v_1, v_{e'}]$	$s_1 = 6$, and so $s_2 \in \{5,7,8\}$
		$v_2 v_5 \in E(G)$	$v_e v_0 u v_3 v_4 v_5 v_2 v_1 v_6 v_7 v_{e'}$	
		$v_2 v_{s_2} \in E(G)$	$v_e v_0 u P[v_3, v_6] v_1 v_2 v_{s_2} v_{e'}$	$s_2 \in \{7,8\}$
	(0,3,7)	$v_4 v_8 \in E(G)$	$P[v_e, v_3] u P^-[v_7, v_4] v_8 v_{e'}$	$v_4 v_7 \in E(G)$, and so $s_5 \in \{2,8\}$
		$v_4 v_7$, $v_5 v_8 \in E(G)$	$P[v_e, v_4] v_7 v_6 v_5 v_8 v_{e'}$	$v_2 v_5 \in E(G)$, and so $s_1 \in \{6,7\}$
		$v_2 v_5$, $v_1 v_6 \in E(G)$	$v_e v_0 u v_3 v_4 v_5 v_2 v_1 P[v_6, v_{e'}]$	
		$v_2 v_5$, $v_1 v_7 \in E(G)$	$v_e v_0 u v_3 v_4 v_5 v_2 v_1 v_7 v_{e'}$	
$i_1 = 1$ $t \in \{i_2, i_3\}$		$x_0 v_{t-1} \in E(G)$	$v_e v_0 P^-[v_{t-1}, v_1] u P[v_t, v_{e'}]$	
$i_1 = 1$ $i_2 \geq 4$		$v_0 v_{i_2-2} \in E(G)$	$v_e v_0 P^-[v_{i_2-2}, v_1] u P[v_{i_2}, v_{e'}]$	
$i_1 = 1$ $i_3 \geq i_2 + 3$		$v_0 v_{i_3-2} \in E(G)$	$v_e v_0 P^-[v_{i_3-2}, v_1] u P[v_{i_3}, v_{e'}]$	

Table 15: Proof of Claim 11.

Claim 12. $i_1 \geq 2$ and $i_3 \leq c-2$.

By contradiction, by symmetry and Claim 11, we assume that $i_1 = 1$ and so $(i_1, i_2, i_3) \in \{(1,3,5), (1,3,6), (1,3,7), (1,4,6), (1,4,7)\}$.

Case 12.1. $(i_1, i_2, i_3) = (1,3,5)$.

By (4.2), $G(e,e')[\{v_1, v_2, \cdots, v_5, u\}] \not\cong K_{3,3}^-$, and so $s_1 \neq 4$ and $s_2 \neq 5, 6 \notin E(G)$. By (4.10) and Claim 11, $s_2 \in \{0,7,8\}$. If $v_2v_8 \in E(G)$, then $P' = v_ev_1uv_3 \cdots v_7v_{e'}$ satisfies $|V(P')| = |V(P)|$ and $N_G(v_2) \cap V(P') = \{v_8, v_3, v_1\}$, and so P' violates applying Claim 11 (when P is replaced by P'). Hence $v_2v_7 \in E(G)$.

Note that $v_4v_8, v_6v_8, v_0v_6 \notin E(G)$, as otherwise, $v_ev_1uv_3v_2v_7v_6v_5v_4v_8v_{e'}$ or $v_ev_1v_5v_4v_3v_2v_7v_6v_8v_{e'}$ or $v_ev_0v_6v_5v_4v_3uv_1v_2v_7v_{e'}$ would be longer than P. Hence by (4.10) and Claim 11, $s_4 \in \{1,7\}$ and $s_6 \in \{1,2,3\}$. Let $H_1 = G[V(P[v_1, v_7] \cup \{u\}]$. As $s_4 \in \{1,7\}$ and $s_6 \in \{1,2,3\}$, by Theorem 2.1(vi) and (v), $F(H_1) \leq 2(8) - 12 - 2 = 2$, and so H_1 is not reduced, contrary to (4.2). This excludes Case 12.1.

Case 12.2. $(i_1, i_2, i_3) = (1,3,6)$.

By (4.10) and Claim 11, $s_2 \in \{5,6,7,8\}$. Note that $v_2v_5, v_2v_7, v_2v_8 \notin E(G)$, as otherwise, $v_ev_1v_2v_5v_4v_3uv_6v_7v_{e'}$, or $v_ev_1uP^-[v_6, v_2]v_7v_{e'}$, or $v_ev_1u P^-[v_6, v_2]v_8v_{e'}$ would be longer than P. This implies $v_2v_6 \in E(G)$. By (4.10) and Claim 11, $s_5 \in \{1,8\}$. If $v_5v_8 \in E(G)$, then $v_ev_1uv_6P[v_2, v_5]v_8v_{e'}$ would be longer than P, and so $v_5v_1 \in E(G)$. By Lemma 2.2(i), with $v_2v_6, v_5v_1 \in E(G)$, $G[(e,e')[\{v_1, v_3, v_6, u, v_2, v_5\}]$ is not reduced, contrary to (4.2). This excludes Case 12.2.

Case 12.3. $(i_1, i_2, i_3) = (1,3,7)$.

First note that $v_2v_6 \notin E(G)$, as otherwise $v_ev_1v_2v_6v_5v_4v_3uv_7v_{e'}$ is longer than P. Hence by (4.10), Claim 11, and symmetry, $s_2 \in \{5,7\}$. But then $v_6v_8 \notin E(G)$, as otherwise, either $v_ev_1v_2v_5v_4v_3uv_7v_6v_8v_{e'}$ (if $s_2 = 5$) or $v_ev_1uv_7P[v_2, v_6]v_8v_{e'}$ (if $s_2 = 7$) is longer than P. By Claim 11 and as $s_6 \neq 8$, $s_6 \in \{1,3\}$. As $G(e,e')[\{v_1, v_3, v_7, u, v_6, v_2\} \not\cong K_{3,3}^-$, we have $s_2 \neq 7$, and so $s_2 = 5$.

As $s_2 = 5$, $v_0v_4 \notin E(G)$, as otherwise, $v_ev_0v_4v_3uv_1v_2P[v_5, v_{e'}]$ is longer than P. By Claim 11, $s_4 \in \{1,7\}$. As $s_2 = 5$ and $s_4 \in \{1,7\}$, by Lemma 2.2(i), $G(e,e')[V(P[v_1, v_7]) \cup \{u\}]$ is not reduced, contrary to (4.2). This proves Case 12.3.

Case 12.4. $(i_1, i_2, i_3) = (1,4,6)$.

By (4.10) and Claim 11, $s_2 \in \{5,6,7,8\}$. Note that $v_2v_5, v_2v_7, v_2v_8 \notin E(G)$ as otherwise, $v_ev_1uv_4v_3v_2P[v_5, v_{e'}]$, or $v_ev_1uP^-[v_2, v_6]v_7v_{e'}$, $v_ev_1uP^-[v_6, v_2]v_8v_{e'}$ is longer than P. Thus $s_2 = 6$. By (4.10), Claim 11, and symmetry, $s_5 = 1$. Thus by Lemma 2.2(i), $G(e,e')[\{v_1, v_4, v_6, u, v_5, v_2\}]$ is not reduced, contrary to (4.2). This proves Case 12.4.

Case 12.5. $(i_1, i_2, i_3) = (1, 4, 7)$.

Note that $v_2v_5, v_3v_6 \notin E(G)$, as otherwise $v_ev_1uv_4v_3v_2P[v_5, v_{e'}]$ or $P[v_e, v_3]v_6v_5v_4uv_7v_{e'}$ is longer than P. By (4.10), Claim 11, and by symmetry, we may assume that $s_2 \in \{6, 7\}$, $s_3 = 7$ and $s_5 = 1$. By Lemma 2.2(i), $G(e,e')[V(P[v_1, v_7]) \cup \{u\}]$ is not reduced, contrary to (4.2). This precludes Case 12.5, and so Claim 12 holds.

By Claim 12, $(i_1, i_2, i_3) = (2, 4, 6)$. By (4.10) and Claim 11, $s_3 \in \{0, 6, 7, 8\}$. Note that $v_3v_0, v_3v_7, v_3v_8 \notin E(G)$, as otherwise $v_ev_0v_3v_2uP[v_4, v_{e'}]$ or $v_ev_1v_2uP^-[v_6, v_3]v_7v_{e'}$, or $v_ev_1v_2uP^-[v_6, v_3]v_8v_{e'}$ is longer than P. Hence $s_3 = 6$. By Lemma 2.2(i), $G(e,e')[V(P[v_2, v_6]) \cup \{u\}]$ is not reduced, contrary to (4.2). The proof for Theorem 3.1 is now complete.

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