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ON THE DIFFERENCE BETWEEN DYNAMIC CHROMATIC NUMBER AND CHROMATIC NUMBER OF GRAPHS WITHOUT SOME SUBGRAPHS

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Abstract

For integers $k, r > 0$, a (k, r)-colouring of a graph G is a proper colouring on the vertices of G by k colours such that every vertex v of degree $d(v)$ is adjacent to vertices with at least $\min\{d(v), r\}$ different colours. The dynamic chromatic number, denoted by $\chi_2(G)$, is the smallest integer k for which a graph G has a $(k, 2)$-colouring. In this paper, we prove a sufficient condition for a $K_{1,3}$-free graph G with $\chi_2(G) = \chi(G)$. Also, we give some upper bounds for $\chi_2(G) - \chi(G)$ of $K_{1,4}$-free graphs and graphs without even cycles.

1. Introduction

All graphs in this paper are finite, undirected, and simple. We follow the notation and terminology of [2]. Thus for a graph G, $\Delta(G)$ and $\chi(G)$ denote the maximum degree and the chromatic number of G. If vertices u and v are connected in G, the distance between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G. For $u \in V(G)$, let $N_i(u) = \{v : v \in V(G), d_G(u, v) = i$. We use the symbol $N(v)$ to denote $N_1(v)$, and $d(v) = |N(v)|$.

For an integer $k > 0$, a proper k-colouring of a graph G is a map $c : V(G) \mapsto \{1, 2, \cdots, k\}$ such that if $u, v \in V(G)$ are adjacent vertices in G, then $c(u) \neq c(v)$. The smallest k such that G has a proper k-colouring is the chromatic number of G, denoted by $\chi(G)$. A proper vertex k-colouring of a graph G is called dynamic if for every vertex v with degree at least 2, the neighbours of v receive at least two different colours. The smallest integer k such that G has a dynamic k-colouring is called the dynamic chromatic number of G and denoted by $\chi_2(G)$.

The concept of dynamic colouring of graphs was first introduced in [9] and [6]. Later in [5], it is called conditional colouring. Lately, it has been studied extensively by several authors in [3, 4, 6, 7]. Obviously, $\chi(G) \leq \chi_2(G)$. It was shown in [8] that the difference between the chromatic number and the dynamic chromatic number can be arbitrarily large. However, it was conjectured in [9] that for regular graphs the difference is at most 2. In [7], it has been proved that the computational complexity of $\chi_2(G)$ for a 3-regular graphs is an NP-complete problem. It is an interesting problem to investigate the optimal upper bound for $\chi_2(G) - \chi(G)$. In this paper, we present some bounds for $\chi_2(G) - \chi(G)$ of graphs without some subgraphs.

2. $K_{1,3}$-free Graphs and $K_{1,4}$-free Graphs

A graph G is $K_{1,r}$-free if G does not have an induced subgraphs isomorphic to $K_{1,r}$. In [5], it was proved that if G is a connected and $K_{1,3}$-free, then $\chi_2(G) \leq \chi(G) + 2$, and the equality holds if and only if G is a cycle of length 5 or of even length not a multiple of 3. We will prove a sufficient condition for a $K_{1,3}$-free graph G with $\chi_2(G) = \chi(G)$.

Let c be a proper vertex colouring of a graph G, we denote the vertex set which receives the colour i by $c^{-1}(i)$. We denote the set of colours which appear in the vertex set V by $c(V)$. If for a vertex v with degree at least 2, $|c(N(v))| = 1$, then v is called a bad vertex, otherwise, it is called a good vertex. If a vertex v satisfies $d(u) = 2$, $N(u) = \{v_1, v_2\}$ and $d(v_1) \geq 3$, $d(v_2) \geq 3$, we call u the unique middle vertex.

Lemma 2.1. *Let G be a graph.*

(i) *If* $G[N(u)]$ *has an edge, then u is good.*

(ii) *If G is* $K_{1,r}$*-free, then every vertices of degree at least r is good.*

The proof of Lemma 2.1 is trivial.

Theorem 2.2. *Let G be a $K_{1,3}$-free graph with $\chi(G) \geq 4$, and there is no unique middle vertex in G, then $\chi_2(G) = \chi(G)$.*

Proof. Let G be a $K_{1,3}$-free graph with $\chi(G) \geq 4$, and there is no unique middle vertex in G. Suppose G has a proper vertex colouring $c : V(G) \mapsto \{1, 2, \dots, \chi(G)\}$.

Because G is a $K_{1,3}$-free graph, then by Lemma 2.1 all the vertices we should consider are the vertices with degree 2.

Suppose $d(u) = 2$.

Case 1. u is contained in a triangle. Then by Lemma 2.1, it is a good vertex.

Case 2. u is not in any triangle of G, then u is in a deduced subgraph P of G which is a path, and the degree of the end point of P is not 2.

Suppose the endpoint of P is $v_1, v_2, c(v_1) = c_1, c(v_2) = c_2$. Because there is no unique middle point vertex in G, the inner vertices of P are more than two.

Subcase 2.1. There are two inner vertices in P since $\chi(G) \geq 4$, then there are two colours c_3, c_4, which are different from c_1, c_2 can be used at these vertices, then we have done.

Subcase 2.2. There are $3n (n \in N^+)$ inner vertices in P, we can colour the vertices in P as

$$c_1 \underbrace{c_2 c_3 c_4 c_2 c_3 c_4 \cdots c_2 c_3 c_4}_{3n} c_2.$$

Subcase 2.3. There are $3n + 1 (n \in N^+)$ inner vertices in P, we can colour the vertices in P as

$$c_1 \underbrace{c_2 c_3 c_4 c_2 c_3 c_4 \cdots c_2 c_3 c_4}_{3n} c_1 c_2.$$

Subcase 2.4. There are $3n + 2(n \in N^+)$ inner vertices in P, we can colour the vertices in P as

$$c_1 \underbrace{c_2 c_3 c_4 c_2 c_3 c_4 \cdots c_2 c_3 c_4}_{3n} c_1 c_3 c_2.$$

Finally, we keep the colours of all the other vertices, then we have a dynamic colouring of G with $\chi(G)$ colours.

□

The condition that there is no unique middle vertex in G is necessary. In fact, we can find a graph with $\chi(G) \geq 4$, which contains a unique middle vertex satisfying $\chi_2(G) = \chi(G) + 1$, see Figure 1.

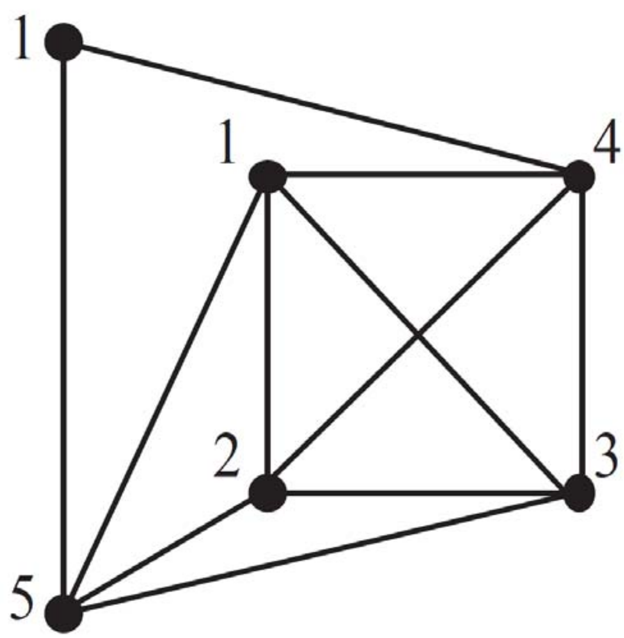

Figure 1. Graph for the note of Theorem 2.2.

Lemma 2.3 ([6]). *For a connected graph G if $\Delta(G) \leq 3$, then $\chi_2(G) \leq 4$ unless $G = C_5$, in which case $\chi_2(C_5) = 5$, and if $\Delta(G) \geq 4$, then $\chi_2(G) \leq \Delta(G) + 1$.*

Theorem 2.4. *Let G be a $K_{1,4}$-free graph, then $\chi_2(G) \leq 2\chi(G)$.*

Proof. Let G be a counterexample with fewest number of vertices.

If $\chi(G) = 1$, it is trivial. If $\chi(G) = 2$, G is $K_{1,4}$-free, then $\Delta(G) \leq 3$. By Lemma 2.3, $\chi_2(G) \leq \Delta + 1 = 4$.

Let $\chi(G) \geq 3$. Let $c' : V(G) \mapsto \{1, 2, \cdots, \chi(G)\}$ is a proper colouring of G. Because G is a counterexample and $K_{1,4}$-free, then there is a bad vertex u in G and $d(u) \leq 3$.

Let v is a neighbour of u, then every colour received by vertices in $N(v) \setminus \{u\}$ can not appears more than twice. Otherwise, there will be a $K_{1,4}$, so $d(v) \leq (\chi(G) - 1) \times 2 + 1 = 2\chi(G) - 1$.

Let $c : V(G) \setminus \{u\} \mapsto \{1, 2, \cdots, 2\chi(G)\}$ is a dynamic colouring of $G - u$. Because G is a counterexample, then $|c(N(u))| = 1$. Without loss of generality, let this colour to be 1. Let v to be a neighbour of u. Because $d(v) \leq 2\chi(G) - 1$, then there is at least one colour missing in $N(v) \setminus \{u\}$. Without loss of generality, let this colour to be $2\chi(G)$.

Case 1. For any vertex w in $N(v) \setminus \{u\}$, there is only one colour assigned to vertices in $N(w) \setminus \{v\}$.

Because $|N(v) \setminus \{u\}| \leq 2\chi(G) - 2$, then there is one colour $i (i \neq 1)$ missing in $N_2(v)$. If i does not appear in $N(v) \setminus \{u\}$, let $\tilde{c}(v) = i$. If i appears in $N(v) \setminus \{u\}$, suppose $c(w_0) = i$. Because c is a 2-hued colouring of $G - u$, then the colour 1 does not appear in $N_2(v)$. Let $\tilde{c}(w_0) = 1$, $\tilde{c}(v) = i$.

Case 2. There is a vertex w_{i_0} in $N(v) \setminus \{u\}$ so that there are at least two colours appearing in $N(w_{i_0}) \setminus \{v\}$.

Because G is a counterexample and the colour $2\chi(G)$ does not appear in $N(v) \setminus \{u\}$, then there is a vertex w_{i_1} in $N(v)$ so that the vertex in $N(w_{i_1}) \setminus \{v\}$ receive only one colour $2\chi(G)$. If there is no vertex w_{i_2} in

$N(v) \setminus \{u\}$ so that there is only one colour $c(w_{i_1})$ in $c(N(w_{i_2}) \setminus \{v\})$. Let $\tilde{c}(w_{i_1}) = 1$, $\tilde{c}(v) = c(w_{i_1})$. If such vertex w_{i_2} exists, we can get a vertex w_{i_3} so that there is only one colour $c(w_{i_2})$ appears in $c(N(w_{i_3}) \setminus \{v\})$. Continuing this procedure, we can stop at a vertex $w_{i_j} \in N(v) \setminus \{u\}$ such that there is no vertex $w_{i_{j+1}}$ in $N(v) \setminus \{u\}$ having the property that there is only one colour $c(w_{i_j})$ in $c(N(w_{i_{j+1}}) \setminus \{v\})$. Let $\tilde{c}(w_{i_j}) = 1$, $\tilde{c}(v) = c(w_{i_j})$. For any vertex $w \in N(v) \setminus \{u\}$ satisfying $c(w) = c(w_{i_j})$, let $\tilde{c}(w) = 1$.

At last, let $\tilde{c}(v) = c(v)$ for all the other vertices in $G - u$. Because $d(u) \leq 3$, we can easily colour the vertex u and get $\tilde{c}$, which is a dynamic colouring of G with $2\chi(G)$ colours, a contradiction.

□

The upper bound in the Theorem 2.4 is best impossible. We can construct a $K_{1,4}$-free graph G with $\chi_2(G) = 2\chi(G)$ as following. Let G' be a complete r-partite graph, G' has 2 nonadjacent vertices in i-th $(1 \leq i \leq r)$ vertex class and contains all edges joining vertices in distinct class. At last, we get G from G' by joining the two vertices in i-th $(1 \leq i \leq r)$ vertex class by a path of length two. It is easy to see that $\chi(G) = r$ and $\chi_2(G) = 2r$. An example graph G with $\chi(G) = 4$ and $\chi_2(G) = 8$ is shown in Figure 2.

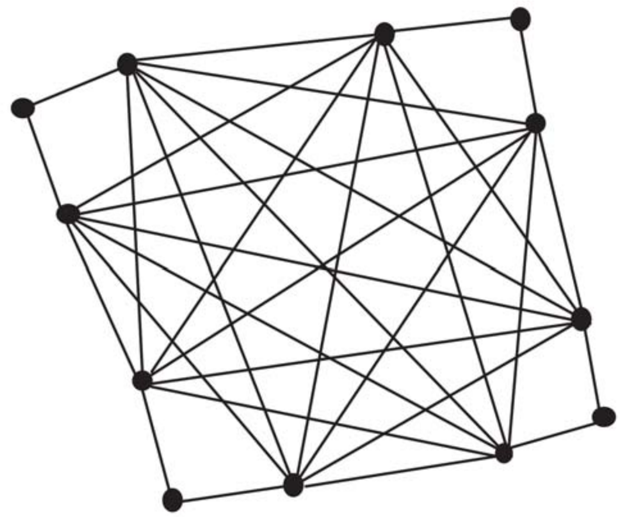

Figure 2. Graph for the note of Theorem 2.4.

3. The Graphs Without Subgraphs $C_{2n}(n \geq 2,\ n \in N^{+})$

Theorem 3.1. *If G does not have a subgraph* $C_{2n}(n \geq 2,\ n \in N^{+})$, *then* $\chi_2(G) \leq 2\chi(G)$.

Proof. Let $c : V(G) \mapsto \{1, 2, \cdots, \chi(G)\}$ to be a proper colouring of G such that every vertex of colour i has at least a neighbour of colour j, for every $j < i$. Clearly, all the bad vertices must be in $c^{-1}(1)$ and $c^{-1}(2)$.

Let F_1 to be a subgraph of G induced by V^1 and the neighbours of V^1, $V^1 = \{v | c(v) = 1,\ v$ is a bad vertex and the neighbours of v are all in $c^{-1}(2)\}$. Let F_2 to be a subgraph of G induced by V^2 and the neighbours of V^2, $V^2 = \{v | c(v) = 2,\ v$ is a bad vertex and the neighbours of v are all in $c^{-1}(1)\}$. Let $F = F_1 \bigcup F_2$, then F is a bipartite graph. We call the vertex set of F which have the colour $1\, V_1$, and we call the other vertex set of $F V_2$. Denote $V_i = \{v | v \in c^{-1}(1), v$ is a bad vertex, the neighbour of v is in $c^{-1}(i)\}$, $3 \leq i \leq \chi(G)$. We call the bipartite graph induced by V_i and the neighbours of V_i F_i, $3 \leq i \leq \chi(G)$.

We construct graphs $G_i(V_i, E_i)$, $1 \leq i \leq \chi(G)$ by the algorithm as following:

Step 0. Let $V := V_i$, $E_i := \varnothing$.

Step 1. Choose a vertex v from V, then choose two vertices v', v'' from $N(v)$. Let $e := (v', v'')$.

Step 2. Let $E_i := E_i \cup \{e\}$, $V := V \setminus \{v\}$. If $V \neq \varnothing$, return Step 1.

Step 3. Let $G_i := G(N(V_i), E_i)$.

For every $1 \leq i \leq \chi(G)$, $G_i(N(V_i), E_i)$ is a bipartite graph. Otherwise, there is an odd cycle in G_i and there is an even cycle in G by the denote of E_i. We change the colour of one part of G_i to a new colour $\chi(G) + i(1 \leq i \leq \chi(G))$ and keep the colours of all the other vertices. We eventually get a dynamic colouring of G with $2\chi(G)$ colours.

□

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