

Integer flows and orientations

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The theory of integer flows was introduced by Tutte as a generalization of map-colouring problems. This chapter is a brief survey of integer flows, including their extensions: circular flows, modulo orientations, group connectivity, and an update of recent progress on Tutte's flow conjectures.

1. Introduction

The concept of an integer flow was introduced by Tutte [55], [56] as a generalization of map-colouring problems (see Theorem 1.1). This chapter is a brief survey of integer flow theory; for further study in this area, see Zhang [63].

Let $G = (V, E)$ be a graph. Given an orientation D of $E(G)$, we denote the resulting directed graph by $D(G)$, and for each vertex $v \in V(G)$, let $E^+(v)$ and $E^-(v)$ be the sets of arcs of $D(G)$ with their tails and heads (respectively) at v .

Let G be a graph, let D be an orientation of G , let Γ be an abelian group (an additive group with 0 as the identity) and let $f : E(G) \rightarrow \Gamma$ be a mapping. Then the ordered pair (D, f) is called a *flow* (or *group Γ -flow*) of G if

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e),$$

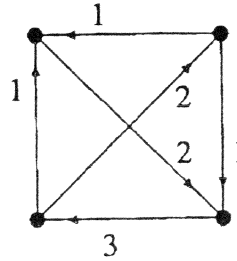


Fig. 1. A nowhere-zero 4-flow of the complete graph K_4 .

for each vertex $v \in V(G)$. In this chapter, we are interested in finite abelian groups, infinite groups \mathbb{Z} (the set of integers), \mathbb{Q} (the set of rational numbers), \mathbb{R} (the set of real numbers) and \mathbb{Z}_k (the cyclic group of order k).

Let (D, f) be a Γ -flow of a graph G and let k be an integer. Then (D, f) is called an *integer flow* if $\Gamma = \mathbb{Z}$, and an integer flow is a *k-flow* if $|f(e)| < k$ for each edge $e \in E(G)$. (D, f) is a *mod-k-flow* if $f : E(G) \rightarrow \mathbb{Z}$ is such that

$$\sum_{e \in E^+(v)} f(e) \equiv \sum_{e \in E^-(v)} f(e) \pmod{k},$$

for each $v \in V(G)$ – that is, (D, f) is a group \mathbb{Z}_k -flow.

The *support* $\text{supp}(f)$ of a Γ -flow (D, f) is the set of all edges of G with $f(e) \neq 0$. A flow (D, f) is a *nowhere-zero flow* if $\text{supp}(f) = E(G)$. An example of a nowhere-zero 4-flow is given in Fig. 1.

Flow-colouring duality

The following theorem of Tutte [56] indicates the important relation between map colouring and integer flows, and motivates the study of the theory of integer flows.

Theorem 1.1 *Let G be a planar bridgeless graph. Then G is k -face-colourable if and only if G admits a nowhere-zero k -flow.*

Note that the ‘only if’ part of Theorem 1.1 holds not only for planar graphs, but also for all graphs embeddable on some orientable surface. Tutte [56] also proved the following result.

Theorem 1.2 *Let G be a bridgeless graph with a closed 2-cell embedding on some orientable surface. If G is k -face-colourable, then G admits a nowhere-zero k -flow.*

Tutte’s conjectures

The following conjectures are the most famous in the theory of integer flows. They were proposed by Tutte ([56], [59] and Problem 48 in [7]).

Conjecture A (5-flow conjecture) *Every bridgeless graph admits a nowhere-zero 5-flow.*

Conjecture B (4-flow conjecture) *Every bridgeless graph containing no subdivision of the Petersen graph admits a nowhere-zero 4-flow.*

Conjecture C (3-flow conjecture) *Every bridgeless graph containing no 3-edge-cut admits a nowhere-zero 3-flow.*

These well-known conjectures are motivated by the following map-colouring theorems of Heawood [23], Appel and Haken [2], [3] and Grötzsch [21].

Theorem 1.3 (The five-colour theorem) *Every bridgeless planar graph is 5-face-colourable.*

Theorem 1.4 (The four-colour theorem) *Every bridgeless planar graph is 4-face-colourable.*

Theorem 1.5 (The three-colour theorem) *Every bridgeless planar graph without a 3-edge-cut is 3-face colourable.*

Although six decades have passed and some significant and important approaches have been made toward these conjectures, they remain essentially open.

2. Basic properties

In this section, we introduce some basic definitions and properties concerning integer flows.

Equivalence of k -flows

The following fundamental theorem is due to Tutte [55], [56].

Theorem 2.1 *Let G be a graph, let k be a positive integer, and let Γ be an abelian group of order k . Then the following statements are equivalent:*

- (1) G admits a nowhere-zero integer k -flow
- (2) G admits a nowhere-zero mod- k -flow
- (3) G admits a nowhere-zero group Γ -flow.

From Theorem 2.1, all nowhere-zero flows are equivalent, so whenever we say that ‘a graph G admits a nowhere-zero k -flow’, it always means that G admits a nowhere-zero integer k -flow, a nowhere-zero group Γ -flow with $|\Gamma| = k$, or a nowhere-zero mod- k -flow. Note that each definition of a nowhere-zero flow has its own special advantages, depending on the topic being studied.

The equivalence of (1) and (2) is strengthened by a useful technical result (Theorem 2.3). The equivalence of (2) and (3) was originally proved by Tutte by using the flow polynomial technique.

From the definition of an integer flow, the following observations of Tutte [56], [57] are straightforward.

Theorem 2.2 *If a graph G admits a nowhere-zero integer k -flow, then G admits a nowhere-zero integer h -flow, for each $h \geq k$.*

A graph G admits a nowhere-zero 2-flow if and only if the degree of every vertex is even.

Mod- k -flows

In Theorem 2.1, the proof of ‘(a) \Rightarrow (b)’ is trivial, since a nowhere-zero integer k -flow is also a nowhere-zero mod- k -flow. The proof of ‘(b) \Rightarrow (a)’ follows from the following stronger result (see [55]).

Theorem 2.3 *If a graph G admits a mod- k -flow (D, f_a) , then G admits an integer k -flow (D, f_b) for which $f_a(e) \equiv f_b(e) \pmod{k}$ for each edge $e \in E(G)$.*

Note that both flows in Theorem 2.3 correspond to the same orientation D .

Products of flows

The following result has been used in the proofs of some landmark theorems (see Jaeger [26] and Seymour [47]).

Theorem 2.4 *Let G be a graph and let k_1 and k_2 be integers. If G admits a k_1 -flow (D, f_1) and a k_2 -flow (D, f_2) , and if $\text{supp}(f_1) \cup \text{supp}(f_2) = E(G)$, then $(D, k_2f_1 + f_2)$ and $(D, f_1 + k_1f_2)$ are nowhere-zero (k_1k_2) -flows of G .*

This result can be generalized to an ‘if and only if’ result (see Zhang [63]).

Theorem 2.5 *Let G be a graph and let k_1 and k_2 be integers. Then G admits a nowhere-zero (k_1k_2) -flow if and only if G admits a k_1 -flow (D, f_1) and a k_2 -flow (D, f_2) with $\text{supp}(f_1) \cup \text{supp}(f_2) = E(G)$.*

Sums of flows

Results on the sum of flows have been obtained by Little, Tutte and Younger [40].

Theorem 2.6 *For each non-negative integer k -flow (D, f) of a graph G , G admits $k - 1$ non-negative 2-flows (D, f_r) ($r = 1, 2, \dots, k - 1$) with $f = \sum_{r=1}^{k-1} f_r$.*

A directed graph is *even* if the in-degree of each vertex equals its out-degree. Since the support of a non-negative 2-flow is a directed even subgraph with orientation D , the following theorem on directed even subgraph covering is equivalent to Theorem 2.6.

Corollary 2.7 *Let G be a graph and let D be an orientation of G . Then the graph G admits a positive k -flow (D, f) if and only if $D(G)$ contains $k - 1$ directed even subgraphs such that each arc of $D(G)$ appears in at least one of them.*

3. 4-flows

In this section we study properties, open problems, and some partial results related to the following major open problem in flow theory.

Recall Tutte's 4-flow conjecture (Conjecture **B**), that every bridgeless graph containing no subdivision of the Petersen graph admits a nowhere-zero 4-flow, and (from Theorem 1.4) that every bridgeless planar graph admits a nowhere-zero 4-flow.

It is well known that, for a cubic graph G , the graph G admits a nowhere-zero 4-flow if and only if G is 3-edge-colourable. Tutte [60] also conjectured, that every bridgeless cubic graph containing no subdivision of the Petersen graph admits a nowhere-zero 4-flow.

Jaeger [28] asked whether the latter conjecture is equivalent to Conjecture **B**. It was eventually proved by Robertson, Sanders, Seymour and Thomas [52], [48], while Conjecture **B** remains open.

Theorem 3.1 *Every bridgeless cubic graph containing no subdivision of the Petersen graph is 3-edge-colourable, and thus admits a nowhere-zero 4-flow.*

The proof of this will consist of a series of papers (see [52]).

The following theorem for highly connected graphs was proved by Jaeger [26].

Theorem 3.2 *Every 4-edge-connected graph admits a nowhere-zero 4-flow.*

4. 3-flows

Recall that a major open problem in integer flow theory is Conjecture **C** (Tutte's 3-flow conjecture), which is a generalization of Grötzsch's 3-colouring theorem (Theorem 1.5) for planar graphs.

The following result was observed by Tutte [55].

Theorem 4.1 *Let G be a cubic graph. Then G admits a nowhere-zero 3-flow if and only if G is bipartite.*

A weak version of Conjecture **C** was proposed by Jaeger [26], that there is an integer h for which every h -edge-connected graph admits a nowhere-zero 3-flow. Some early partial results on this weak conjecture can be found in Lai and Zhang [39] and Alon, Linial and Meshulam [1]. It was recently proved by Thomassen [54] with $h = 8$. Thomassen's theorem [54] was further improved by Lovász, Thomassen, Wu and Zhang [41].

Theorem 4.2 *Every 6-edge-connected graph admits a nowhere-zero 3-flow.*

For embedded graphs, we recall Grötzsch's theorem (Theorem 1.5) that every 4-edge-connected planar graph is 3-face colourable, and so admits a nowhere-zero 3-flow. The following generalizations can be viewed as partial results for the 3-flow conjecture; they are due, respectively, to Grünbaum [22], Steinberg and Younger [51] and Thomassen [53].

Theorem 4.3 *Every bridgeless planar graph containing at most three 3-edge-cuts is 3-face-colourable and so admits a nowhere-zero 3-flow.*

Theorem 4.4 *Every 2-edge-connected graph with at most one 3-edge-cut that can be embedded in the projective plane admits a nowhere-zero 3-flow.*

Theorem 4.5 *Let G be a graph embedded in the torus such that all contractible cycles are of length at least 5. Then G is 3-vertex-colourable.*

Tutte's 3-flow conjecture was originally proposed for graphs with no 1-edge-cut and no 3-edge-cut. It was pointed out in [26], [28] and [47] that no 2-edge-cut exists in any smallest counter-example to some well-known flow conjectures, including Conjecture C.

A graph G is λ_o -odd edge-connected if every odd edge-cut of G has at least λ_o edges. Theorem 4.2 has a stronger version for odd edge-connectivity, as proved by Lovász, Thomassen, Wu and Zhang [41].

Theorem 4.6 *Every 7-odd edge-connected graph admits a nowhere-zero 3-flow.*

For any smallest counter-example to Conjecture C, the following proposition was obtained by applying a vertex-splitting lemma in [64].

Theorem 4.7 *Any smallest counter-example to the 3-flow conjecture is 5-regular and 5-odd edge-connected.*

Kochol [32] also proved that it suffices to prove the 3-flow conjecture for 5-edge-connected graphs.

Theorem 4.8 *The following statements are equivalent.*

- *Every 4-edge-connected graph admits a nowhere-zero 3-flow.*
- *Every 5-edge-connected graph admits a nowhere-zero 3-flow.*

Unlike k -flows with $k > 3$, nowhere-zero 3-flows can be viewed as a modulo orientation problem. An orientation D of a graph G is called a *modulo 3-orientation* if $|E^+(v)| \equiv |E^-(v)| \pmod{3}$, for each $v \in V(G)$.

The following observation appeared in [62] and [51].

Theorem 4.9 *A bridgeless graph G admits a nowhere-zero 3-flow if and only if G has a modulo 3-orientation D .*

This observation is further generalized in Section 7.

5. 5-flows

The main topic of this section is the 5-flow conjecture (Conjecture A) of Tutte [56]. When Tutte introduced the concept of an integer flow, he first conjectured that there is an integer k for which every bridgeless graph admits a nowhere-zero k -flow, and pointed out that $k \geq 5$, since the Petersen graph does not admit a nowhere-zero 4-flow. This conjecture was proved independently by Jaeger [26] and Kilpatrick [30] with $k = 8$ (the 8-flow theorem). The best approach to it is currently the 6-flow theorem of Seymour [47].

Theorem 5.1 *Every bridgeless graph admits a nowhere-zero 6-flow.*

The approaches in the proofs of the 8-flow and 6-flow theorems were different. The 8-flow theorem was proved by applying a theorem of Tutte and Nash-Williams [58], [43] and showing the existence of three edge-disjoint spanning trees in $2G$ for a 3-edge-connected graph G . The 6-flow theorem was proved by showing that every bridgeless graph admits two flows (D, f_1) and (D, f_2) , where one is a 2-flow and the other is a 3-flow with $\text{supp}(f_1) \cup \text{supp}(f_2) = E(G)$. The 5-flow conjecture is still open.

For graphs embedded on some surface, the dual of vertex-colouring is the flow problem. Thus, Heawood's colouring result (Theorem 1.3) implies the 5-flow conjecture for certain families of embedded graphs.

The following result of Möller, Carstens and Brinkmann [42] and Fouquet [18] is related to the 5-colour theorem and the 5-flow conjecture.

Theorem 5.2 *Every bridgeless graph embeddable in an orientable surface of genus $g \leq 2$, or in a non-orientable surface of genus $g \leq 4$, admits a nowhere-zero 5-flow.*

The 5-flow conjecture has been proved for some special families of graphs. The following result was proved by Jaeger [25].

Theorem 5.3 *Let e be an edge of a graph G . If G admits a nowhere-zero 4-flow and $G - e$ is bridgeless, then $G - e$ admits a nowhere-zero 5-flow.*

Let H be a spanning even subgraph of a graph G . The *oddness* of H is the number of components of H containing an odd number of odd-degree vertices of G . The *oddness* of G is the minimum of the oddnesses of all spanning even subgraphs of G . It is straightforward to show that G admits a nowhere-zero 4-flow if and only if the oddness of G is 0.

Special cases of such graphs $G - e$ in Theorem 5.3 are graphs containing a Hamiltonian path, and (in general) graphs of oddness at most 2. On the other hand, Celmins [11] observed a similar result in the opposite direction.

Theorem 5.4 *If a bridgeless graph G has an edge e for which $G - e$ admits a nowhere-zero 4-flow, then G admits a nowhere-zero 5-flow.*

Both results have been further extended to the deletion of more than one edge by Steffen [49], [50], under some conditions of cyclic edge-connectivity.

For vertex-deletions, Gerards and Seymour (personal communication, 1995) proved the 5-flow conjecture for apex graphs. (An *apex graph* is one in which at least one vertex is adjacent to all the others.)

Theorem 5.5 *Every apex graph admits a nowhere-zero 5-flow.*

By applying a lemma in [46], that every cubic graph with girth at least 6 must have a Petersen minor, Kochol [31] further generalized the above theorem.

Theorem 5.6 *Every Petersen minor-free cubic graph admits a nowhere-zero 5-flow.*

A smallest counter-example to the 5-flow conjecture must be cubic and not 3-edge-colourable. The following further result was proved by Kochol [34].

Theorem 5.7 *A smallest counter-example G to the 5-flow conjecture has girth at least 11 and cyclic edge-connectivity at least 6.*

The first result was proved by using a computer-aided search, which extended the girth results in some earlier articles.

Modulo 5-orientations

An 8-flow is obtained as the product of three 2-flows (see Jaeger [26]), while a 6-flow is obtained as the product of a 2-flow and a 3-flow (see Seymour [47]). But 8 and 6 are composite numbers, while 5 is a prime number. So what can we do for 5-flows? Certainly they cannot be the product of smaller flows. Various approaches have been proposed, such as orientable 5-even subgraph double covers (see Archdeacon [4] and Jaeger [28]), modulo 5-orientations (see Jaeger [28]) and bipartizing matching (see Fleischner [17]).

The modulo 5-orientation was proposed by Jaeger [28] as an approach to the 5-flow conjecture (see Section 7 for a detailed discussion on modulo orientation).

Jaeger proved [28] that the following conjecture implies the 5-flow conjecture.

Conjecture D *Every 9-edge-connected graph has a modulo 5-orientation.*

A partial result concerning this conjecture is a result of Lovász, Thomassen, Wu and Zhang [41].

Theorem 5.8 *Every 12-edge-connected graph has a modulo 5-orientation.*

6. Bounded orientations and circular flows

Let $\{A, B\}$ be a partition of the vertex-set $V(G)$, and let D be an orientation of G . The set of arcs of $D(G)$ with tails in A and heads in B is denoted by $[A, B]_D$, or simply by $[A, B]$ if no confusion arises.

The following is a revised version of a result by Hoffman (see [6]).

Theorem 6.1 *Let G be a bridgeless graph, let D be an orientation of G and let a and b be two positive integers with $a \leq b$. Then the following statements are equivalent:*

- (a) $a/b \leq |[A, B]_D|/|[B, A]_D| \leq b/a$, for every edge-cut (A, B) of G ;
- (b) G admits a nowhere-zero integer flow (D, f_1) such that $a \leq f_1(e) \leq b$ for each $e \in E(G)$;
- (c) G admits a nowhere-zero real-valued flow (D, f_2) such that $a \leq f_2(e) \leq b$ for each $e \in E(G)$.

Corollary 6.2 *A graph G admits a nowhere-zero k -flow if and only if G has an orientation D for which*

$$\frac{1}{k-1} \leq \frac{|[A, B]_D|}{|[B, A]_D|} \leq k-1,$$

for every edge-cut (A, B) of G .

Let k and d be two integers with $0 < d \leq \frac{1}{2}k$. An integer flow (D, f) of a graph G is a *circular k/d -flow* if $f : E(G) \rightarrow \{\pm d, \pm(d+1), \dots, \pm(k-d)\} \cup \{0\}$. The concept of a circular flow, introduced by Goddyn, Tarsi and Zhang [20], is a generalization of integer flows and is a dual version of the circular colouring problem (see Zhu [65] for a comprehensive survey of this area).

Theorem 6.3 *Let G be a bridgeless graph, let D be an orientation of G and let $k, d \in \mathbb{Z}^+$ and $q \in \mathbb{Q}^+$ be such that $q = k/d \geq 2$. Then the following statements are equivalent:*

- (a) G admits a positive circular k/d -flow (D, f_1) ;
- (b) G admits a rational-valued flow (D, f_2) such that $f_2 : E(G) \rightarrow [1, q-1]$;
- (c)

$$q-1 = \frac{k-d}{d} \geq \frac{|[U, V(G)-U]_D|}{|[V(G)-U, U]_D|} \geq \frac{d}{k-d} = \frac{1}{q-1},$$

for each non-empty proper subset $U \subset V(G)$.

An immediate corollary of Theorem 6.3, proved in [20], is the following result, analogous to Theorem 2.2.

Theorem 6.4 *Let G be a graph and let $p \in \mathbb{Q}^+$. If G admits a nowhere-zero circular p -flow, then G admits a nowhere-zero circular q -flow for every $q \in \mathbb{Q}^+$ with $q \geq p$.*

For a bridgeless graph G , the *flow index* $\varphi(G)$ is the smallest rational number ℓ for which G admits a nowhere-zero circular ℓ -flow. It is natural to ask, for any given rational number ℓ , whether there is a graph G for which the flow index $\varphi(G)$ of the graph is precisely ℓ . The following result of Pan and Zhu [44] answers this question.

Theorem 6.5 *For every rational number ℓ in the interval $[2, 5]$, there is a graph G for which $\varphi(G) = \ell$.*

Tutte's Theorem 1.1 states the relation between integer flows and face-colouring for planar graphs. The following result of DeVos, Goddyn, Mohar, Vertigan and Zhu [13] extends this theorem to locally planar graphs.

Theorem 6.6 *For a given orientable surface Σ and positive number ε , there is a function $f(\Sigma, \varepsilon)$ such that, for every graph G embedded on Σ with edge-width at least $f(\Sigma, \varepsilon)$, if the dual graph G^* admits a nowhere-zero circular ℓ -flow, then the graph G is circular $(\ell + \varepsilon)$ -vertex-colourable.*

Together with Theorem 1.2, Theorem 6.6 provides a close relationship between face-colouring and flow index for 'locally planar' graphs (graphs with sufficiently large edge-width).

7. Modulo orientations and $(2 + 1/t)$ -flows

As with 3-flows and mod-3-orientations (Theorem 4.9), a circular $(2 + 1/t)$ -flow can also be considered as a modulo $(2t + 1)$ -orientation for each positive integer t .

Let k be an odd integer. An orientation D of a graph G is called a *modulo k -orientation* if $d_D^+(v) \equiv d_D^-(v) \pmod{k}$, for every $v \in V(G)$. The following result is due to Jaeger [27].

Theorem 7.1 *Let G be a graph and let t be a positive integer. Then G has a modulo $(2t + 1)$ -orientation if and only if G admits a nowhere-zero circular $(2 + 1/t)$ -flow.*

The following conjecture was proposed by Jaeger [27] (see also [28] and [63]).

Conjecture E *Let G be a graph and let $k (\geq 3)$ be an odd integer. If G is $(2k - 2)$ -edge-connected, then G has a modulo k -orientation.*

The 3-flow conjecture (by Theorem 4.9) and Conjecture D are special cases of Conjecture E (for $k = 3$ and 5).

A weak version of this conjecture, the $(2 + \varepsilon)$ -flow conjecture, was proposed by Seymour (personal communication 1999) and Galluccio, Goddyn and Hell [19] as an analogue to Jaeger's weak-3-flow conjecture [26]. It was proved recently by Thomassen [54] and improved by Lovász, Thomassen, Wu and Zhang [41].

Theorem 7.2 *Let G be a graph and let $k = 2t + 1 \geq 3$ be an odd integer. Then G has a modulo k -orientation if G is $(3k - 3)$ -edge-connected, and so admits a nowhere-zero circular $(2 + 1/t)$ -flow.*

As we discussed in Section 4, odd edge-connectivity plays an important role for flows and modulo orientations. The following conjecture was proposed in [64] as a refinement of Conjecture E and Theorem 7.2 for $(2 + 1/t)$ -flows (modulo $(2t + 1)$ -orientations).

Conjecture F *For each positive integer t , every graph with odd edge-connectivity at least $4t + 1$ admits a nowhere-zero circular $(2 + 1/t)$ -flow.*

A weak version [64] of Conjecture **F** has now been solved by Lovász, Thomassen, Wu and Zhang [41].

Theorem 7.3 *For each positive integer t , every $(6t + 1)$ -odd edge-connected graph admits a nowhere-zero circular $(2 + 1/t)$ -flow.*

This result relaxes the edge-connectivity requirements in Theorem 7.2. Theorem 4.6 is a special case of this general result.

8. Contractible configurations

Contraction is one of the most useful and powerful operations in the inductive study of graph theory, if the resulting graph preserves a given graph property. In this section, we introduce *contractible configurations* and *collapsible graphs*; contractions of such graphs preserve some properties, such as integer flows.

Let \mathcal{P} be a graph-theoretic property. A graph H is a *contractible configuration* of \mathcal{P} if, for each supergraph G of H , G/H has property \mathcal{P} if and only if G does.

Group connectivity

Group connectivity was introduced by Jaeger, Linial, Payan and Tarsi [29] as a generalization of integer flows. Let G be a graph, let Γ be an abelian group, and let $\beta : V(G) \rightarrow \Gamma$. Then β is called a *boundary* if it has *zero-sum* – that is, if $\sum_{v \in V(G)} \beta(v) = 0$. The graph G is Γ -*connected* if, for each boundary β , there are an orientation D_β and a nowhere-zero weight f_β of $E(G)$ such that

$$\sum_{e \in E_{D_\beta}^+(v)} f_\beta(e) - \sum_{e \in E_{D_\beta}^-(v)} f_\beta(e) = \beta(v), \quad (1)$$

for each vertex $v \in V(G)$.

By the definition of group connectivity, we easily deduce the following result.

Theorem 8.1 *If H is Γ -connected, then H is a contractible configuration for Γ -flow.*

As for Tutte's flow conjectures, several open problems were proposed by Jaeger, Linial, Payan and Tarsi [29].

Conjecture **G**

Every 5-edge connected graph is \mathbb{Z}_3 -connected.

Every 3-edge connected graph is \mathbb{Z}_5 -connected.

Note that the 5-edge-connectivity is sharp for the first of these conjectures, since some 4-edge-connected counter-examples were discovered in [29] and [37]. The following theorem of Jaeger, Linial, Payan and Tarsi [29] gives some partial results.

Theorem 8.2

Every 3-edge-connected graph is Γ -connected for every group Γ of order at least 6.
 Every 4-edge-connected graph is Γ -connected for every group Γ of order at least 4.

The first part of Conjecture **G** was verified by Lai and Li [38] for planar graphs, and by Lovász, Thomassen, Wu and Zhang [41] for 6-edge-connected graphs.

Short cycles

Analogous to the girth studies of small counter-examples to Tutte's conjectures, shorter cycles are Γ -connected for larger groups Γ . The following observation was due to Jaeger, Linial, Payan and Tarsi [29].

Theorem 8.3 Every cycle of length at most $r - 1$ is Γ -connected for every abelian group Γ of order at least r .

Note that a cycle C of length r is not Γ -connected if $|\Gamma| \leq r$: one can easily see that a constant boundary $\beta = 1$ is a 'bad' boundary – that is, there is no f_β satisfying equation (1).

From Theorem 8.3, we know that a digon is a contractible configuration for the 3-flow problem, and we also see that a triangle is not a contractible configuration for 3-flows. However, some graphs with many triangles (for example, even wheels) can be contractible configurations for the 3-flow problem.

A graph G is *triangularly connected* if, for each pair of edges e and f of G , there are triangles T_1, T_2, \dots, T_r such that $e \in E(T_1), f \in E(T_r)$, and $E(T_i) \cap E(T_{i+1}) \neq \emptyset$, for each $i = 1, 2, \dots, r - 1$.

Let \mathcal{WF} be the subfamily of triangularly connected graphs constructed (recursively) as follows:

- the triangle is a member of \mathcal{WF}
- odd wheels are members of \mathcal{WF}
- for a pair of \mathcal{WF} -graphs G_1 and G_2 , let $e_i \in E(G_i)$; a new \mathcal{WF} -graph G is constructed from G_1 and G_2 by merging e_1 and e_2 into a single edge e (see Fig. 2)

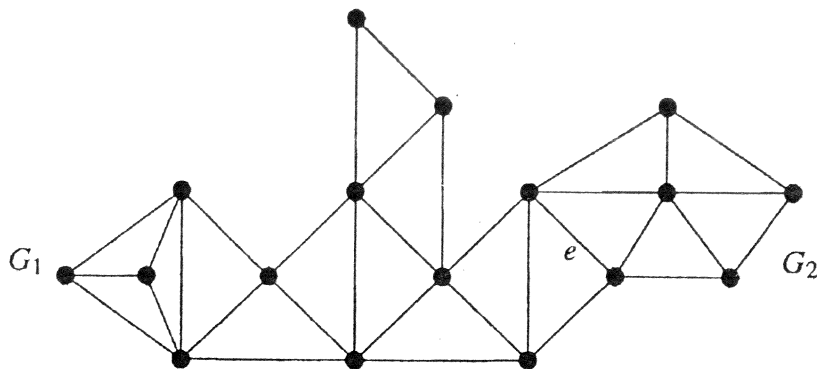


Fig. 2. A \mathcal{WF} -graph.

The following result of Fan, Lai, Xu, Zhang and Zhou [16] characterizes all triangularly connected graphs that are \mathbb{Z}_3 -connected.

Theorem 8.4 *A triangularly connected graph is \mathbb{Z}_3 -connected if and only if it is not a member of \mathcal{WF} .*

This result yields 3-flows for many families of dense graphs, such as locally connected graphs [36], squares of graphs [15], [61], triangulations of embedded graphs [5] and certain products of graphs [24].

A noticeable feature for 4-flows or their contractible configurations is that, although a 4-cycle is not \mathbb{Z}_4 -connected, it is a contractible configuration for 4-flows. Catlin [9] proved the following result.

Theorem 8.5 *Let G be a graph and let C be a cycle of length 4. Then G admits a nowhere-zero 4-flow if and only if G/C admits a nowhere-zero 4-flow – that is, 4-cycles are contractible configurations for the 4-flow problem.*

Collapsible graphs

‘Collapsible graphs’, introduced by Catlin [8], are also contractible configurations for the 4-flow problem, due to the close relationship between supereulerian graphs and 4-flows.

A graph H is *collapsible* if, for each $X \subseteq V(H)$ of even order, H has a connected spanning subgraph H_X for which $X = O(H_X)$, the set of all odd-degree vertices in H_X .

The following result appeared in Lai [35].

Theorem 8.6 *Collapsible graphs are Γ -connected for every abelian group Γ of order 4.*

For more information on collapsible graphs, see Catlin’s survey [10] and its supplement [12].

Group structure

For group-flow problems, the structure of the group makes no difference to the existence of nowhere-zero k -flows (see Theorem 2.1) if k , the order of the group, is fixed. However, the situation seems different for group connectivity problems. For example, it remains open as to whether \mathbb{Z}_4 -connectivity is equivalent to $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ -connectivity. The following conjecture is due to Jaeger, Linial, Payan and Tarsi [29].

Conjecture H *A graph G is \mathbb{Z}_4 -connected if and only if it is $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ -connected.*

Modulo orientations with boundaries

Beyond group connectivity, which mainly targets *integer-valued* k -flow problems for $k = 3, 4, \dots$, the investigation of contractible configurations has been extended to circular $(2 + 1/t)$ -flows with $t \geq 1$ (see [64], [37] and [54]).

Let G be a graph and let k be an odd integer. A mapping $\beta : V(G) \rightarrow \mathbb{Z}_k$ is called a \mathbb{Z}_k -boundary of G if $\sum_{v \in V(G)} \beta(v) \equiv 0 \pmod{k}$. An orientation D of G is called a β -orientation of G if $d_D^+(v) - d_D^-(v) \equiv \beta(v) \pmod{k}$.

Theorem 8.7 *Let G be a graph and let $k = 2t + 1$ be an odd integer. Then the following statements are equivalent:*

- G has a β -orientation for each \mathbb{Z}_k -boundary β of G
- G is a contractible configuration for modulo k -orientation
- G is a contractible configuration for a circular $(2 + 1/t)$ -flow.

Just as short cycles are contractible configurations for integer flows, so parallel edges are contractible configurations for modulo orientation (see [64]).

Theorem 8.8 *Let $k \geq 3$ be an odd integer. Then the parallel edge $(k - 1)K_2$ is a contractible configuration for modulo k -orientation.*

The following conjecture was proposed by Lai [37].

Conjecture I *If G is a $(2k - 1)$ -edge-connected graph, $k \geq 3$ is an odd integer and β is a \mathbb{Z}_k -boundary of G , then G has a β -orientation.*

A weak version [37] of Conjecture I was proved recently by Thomassen [54] and further improved by Lovász, Thomassen, Wu and Zhang [41].

Theorem 8.9 *Let G be a graph and let $k \geq 3$ be an odd integer. If G is $(3k - 3)$ -edge-connected, then G has a β -orientation for every \mathbb{Z}_k -boundary β , so every $(3k - 3)$ -connected graph is a contractible configuration for modulo k -orientation.*

9. Related problems

Together with the 5-flow conjecture, the cycle double cover conjecture and the Berge–Fulkerson conjecture are major open problems for snark graphs (cubic graphs with chromatic index 4). These conjectures have some important common properties: they all hold for 3-edge-colourable cubic graphs, but remain open for snarks, and investigations of one problem may be related to others, but little is known about how closely they are related.

Cycle double cover

In this section, we present some relations between flow problems and cycle double cover problems. In [55], Tutte proved the following results for cubic graphs. They were generalized and reformulated by Jaeger and Seymour.

Theorem 9.1 *For a graph G , the following statements are equivalent:*

- G admits a nowhere-zero 4-flow

- G has a 3-even subgraph double cover
- G has a 4-even subgraph double cover.

Preissmann [45] and Celmins [11] proposed the following conjecture.

Conjecture J *Every bridgeless graph has a 5-even-subgraph double cover.*

However, we have no knowledge yet about the relationship between 5-even-subgraph double covers and 5-flows.

An early result of Tutte [55] about orientable cycle double covers is closely related to integer flows.

Theorem 9.2 *Let $r = 3$ or 4 and let G be a graph. Then G admits a nowhere-zero r -flow if and only if G has an orientable r -even-subgraph double cover.*

A conjecture of Archdeacon [4] and Jaeger [28] analogous to Conjecture J is as follows.

Conjecture K *Every bridgeless graph has an orientable 5-even-subgraph double cover.*

It is evident that Conjecture K is stronger than Conjecture J, and they both seem to have some relationship with the 5-flow conjecture. We know that Conjecture K implies the 5-flow conjecture, but we do not know whether they are equivalent.

Flow double covering

The following conjecture was made by Zhu (unpublished notes, 2005).

Conjecture L *Let G be a bridgeless graph with an orientation D , and let $r = 2, 3, 4$ or 5 . The graph G admits a set of r -flows $(D, f_1), (D, f_2), \dots, (D, f_{1-r})$ for which each edge is covered by the supports of precisely two of these r -flows.*

Note that, for $r = 2$, Conjecture L is equivalent to Conjecture J, and for $r = 5$, Conjecture L is equivalent to the 5-flow conjecture.

Cycle space minors

The following concepts and conjecture were introduced by Jaeger [28].

Let G_1 and G_2 be bridgeless graphs. We write $G_1 \leq_C G_2$ if G_1 has a subdivision H and there is a bijection $\varphi : E(H) \rightarrow E(G_2)$ such that, for each even subgraph C of H , $\varphi(C)$ is an even subgraph in G_2 .

In the following, the graph with one vertex and one loop is denoted by L , the graph with two vertices and three parallel edges is denoted by $3K_2$, and the Petersen graph is denoted by P_{10} . Jaeger [28] observed that $3K_2 \leq_C G$ if and only if G admits a nowhere-zero 4-flow.

Let \mathcal{G} be the set of all bridgeless graphs. A graph G is called a *cycle space minor* if G is a minimal member of \mathcal{G} under the order \leq_C . The cycle space of a graph is denoted by $\mathcal{C}(G)$ and the rank of $\mathcal{C}(G)$ is denoted by $r_C(G)$; the ranks of the cycle spaces of L , $3K_2$ and P_{10} are 1, 2 and 6, respectively. It is not hard to prove that there is no cycle space minor M with $r_C(M) = 3, 4$ or 5 . Is there any cycle space minor with rank higher than 6? The following conjecture of Jaeger [28] implies many cycle cover conjectures.

Conjecture M *The only three cycle space minors are L , $3K_2$ and P_{10} .*

Conjecture M implies the cycle double cover conjecture, the 5-even subgraph double cover conjecture and the Berge–Fulkerson conjecture. Since the mapping φ preserves even subgraphs but not orientations, the cycle space minor problem is mainly related only to 2^t -flows, where t is a positive integer.

The concept of cycle space minor was further extended by DeVos, Nešetřil and Raspaud [14], to the case where a generalized graph mapping preserves both flow-values and orientations.

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