Hamiltonian graphs involving neighborhood conditions

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Abstract

Let G be a graph on n vertices, δ and α be the minimum degree and independence number of G, respectively. We prove that if G is a 2-connected graph and $|N(x)\cup N(y)| \geq n-\delta-1$ for each pair of nonadjacent vertices x, y with $1 \leq |N(x)\cap N(y)| \leq \alpha - 1$, then G is hamiltonian or $G \in \{G_1, G_2\}$ (see Figure 1.1 and Figure 1.2). As a corollary, if G is a 2-connected graph and $|N(x) \cup N(y)| \geq n - \delta$ for each pair of nonadjacent vertices x, y with $1 \leq |N(x) \cap N(y)| \leq \alpha - 1$, then G is hamiltonian. This result extends former results by Faudree et al ([5]) and Yin ([7]).

keywords: hamiltonian, neighborhood unions, neighborhood intersection, an essential independent set.

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1 Introduction

We shall follow the notation of Bondy and Murty [1] and consider simple graphs only. Let G be a graph on n vertices, δ and α be the *minimum degree* and *independence number* of G, respectively. For any vertex v of G, $d(v) = |N(v)|$ where $N(v)$ denotes the neighborhood of v in G. If A, B are subgraphs of G, we define $N(A) = \bigcup_{v \in V(A)} N(v)$, $N_B(A) = N(A) \cap V(B)$.

Theorem 1.1 *(Dirac, [3]) If $d(u) \geq \frac{n}{2}$ for every vertex u in a graph G, then G is hamiltonian.*

Theorem 1.2 *(Ore, [6]) If $d(u) + d(v) \geq n$ for each pair of nonadjacent vertices u, v in a graph G, then G is hamiltonian.*

Theorem 1.3 *(Fan, [4]) If G is a 2-connected graph and $max\{d(u), d(v)\} \geq \frac{n}{2}$ for each pair of nonadjacent vertices u, v with $d(u, v) = 2$, then G is hamiltonian.*

Theorem 1.4 *(Chen, [2]) If G is a 2-connected graph and $max\{d(u), d(v)\} \geq \frac{n}{2}$ for each pair of nonadjacent vertices $u, v \in V(G)$ with $1 \leq |N(u) \cap N(v)| \leq \alpha - 1$, then G is hamiltonian.*

Theorem 1.5 *(Faudree et al, [5]) If G is a 2-connected graph and $|N(u) \cup N(v)| \geq n - \delta$ for each pair of nonadjacent vertices $u, v \in V(G)$, then G is hamiltonian.*

Theorem 1.6 *(Yin, [7]) If G is a 2-connected graph and $|N(u) \cup N(v)| \geq n - \delta$ for each pair of nonadjacent vertices $u, v \in V(G)$ with $d(u, v) = 2$, then G is hamiltonian.*

Among Theorem 1.1 through 1.3, each of them extends the former theorem. Theorem 1.4 extends 1.3 by changing $d(u, v) = 2$ to $1 \leq |N(u) \cap N(v)| \leq \alpha - 1$ and Theorem 1.6 extends 1.5 by considering only those pairs of vertices with distance 2. Naturally, we ask if Theorem 1.6 can be further improved by changing $d(u,v) = 2$ to $1 \leq |N(u) \cap N(v)| \leq \alpha - 1$. This is proved true stated as Corollary 1.8. In fact, we prove a stronger result as follows.

Theorem 1.7 *If G is a 2-connected graph and $|N(x) \cup N(y)| \geq n - \delta - 1$ for each pair of nonadjacent vertices x, y with $1 \leq |N(x) \cap N(y)| \leq \alpha - 1$, then G is hamiltonian or $G \in \{G_1, G_2\}$ (see Figure 1.1 and Figure 1.2).*

Let G_1 be the graph obtained from $K_{2,3}$ by replacing each of the divalent vertex by a complete graph $K_{\frac{n-2}{3}}$, and denoting the two trivalent vertices by x_i, x_j, respectively, then joining x_i, x_j with every vertex of each $K_{\frac{n-2}{3}}$ and possibly joining x_i and x_j by an edge. Let $G_2 = G^*_{\frac{n-1}{2}} \vee K^C_{\frac{n+1}{2}}$ where $G^*_{\frac{n-1}{2}}$ is a subgraph on $\frac{n-1}{2}$ vertices and $K^C_{\frac{n+1}{2}}$ is the complement of a complete graph $K_{\frac{n+1}{2}}$.

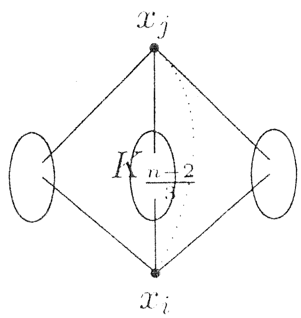

Figure 1.1. $G_1 : n = 3\delta - 1$

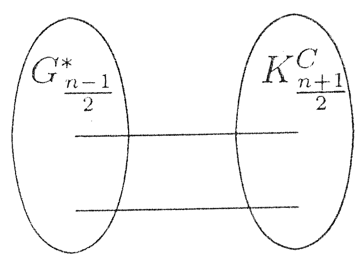

Figure 1.2. $G_2 : n$ is odd

Corollary 1.8 *If G is a 2-connected graph and $|N(x) \cup N(y)| \geq n - \delta$ for each pair of nonadjacent vertices x, y with $1 \leq |N(x) \cap N(y)| \leq \alpha - 1$, then G is hamiltonian.*

Neither G_1 nor G_2 satisfies $|N(x) \cup N(y)| \geq n - \delta$, so Corollary 1.8 follows directly from Theorem 1.7.

2 Longest Cycles

Let C be a cycle of a graph G oriented clockwise with m vertices, denoted $C_m = x_1x_2\cdots x_mx_1$. We let $N^+_{C_m}(u) = \{x_{i+1} : x_i \in N_{C_m}(u)\}$, $N^-_{C_m}(u) = \{x_{i-1} : x_i \in N_{C_m}(u)\}$ and $[x_i, x_j] = \{x_i, x_{i+1}, \cdots, x_j\}$, where the subscripts are taken modulo m. Given two vertices a, b of C, we let $[a, b]$ and $[a, b]^-$ respectively denote the path of C from a to b clockwise and counterclockwise respectively. A cycle C is called a *longest cycle* if there does not exist a longer cycle C^* such that $|V(C)| < |V(C^*)|$. In the following two lemmas, we always assume that G is a 2-connected graph on n vertices, $C_m = x_1x_2\cdots x_mx_1$ is a longest cycle of G, H is a component of $G - C_m$ and x_i, x_j are distinct vertices in $N_{C_m}(H)$.

Lemma 2.1 *Each of the following holds.*
(1) $\{x_{i-1}, x_{i+1}, x_{j-1}, x_{j+1}\} \cap N_{C_m}(H) = \emptyset$.
(2) $x_{i+1}x_{j+1} \notin E(G)$ *and* $x_{i-1}x_{j-1} \notin E(G)$.
(3) If $x_tx_{j+1} \in E(G)$ *for some vertex* $x_t \in [x_{j+2}, x_i]$, *then* $x_{t-1}x_{i+1} \notin E(G)$; *if* $x_tx_{j+1} \in E(G)$ *for some vertex* $x_t \in [x_{i+1}, x_j]$, *then* $x_{t+1}x_{i+1} \notin E(G)$.
(4) If $x_tx_{j+1} \in E(G)$, $x_{t-1} \notin N_{C_m}(H)$.
(5) No vertex of $G - (V(C_m) \cup V(H))$ *is adjacent to both* x_{i+1} *and* x_{j+1}; *if* $x \in V(H)$, *then no vertex of* $G-(V(C_m)\cup V(H))$ *is adjacent to both* x_{i+1} *and* x.
(6) If $x \in V(H)$, *then* $\{x\} \cup N^+_{C_m}(H)$ *must be an independent set.*

Proof. (1), (2) and (5) follow immediately from the assumption that C_m is a longest cycle of G. Since $x_i, x_j \in N_{C_m}(H)$, there exist $x'_i, x'_j \in V(H)$ such that $x_ix'_i, x_jx'_j \in E(G)$.

(3) Suppose that there exists a vertex $x_t \in [x_{j+2}, x_i]$ satisfying $x_tx_{j+1} \in E(G)$. Let P' denote an (x_i, x_j)-path in H. If $x_{t-1}x_{i+1} \in E(G)$, then $x_iP'x_j[x_{j-1}, x_{i+1}]^-[x_{t-1}, x_{j+1}]^-[x_t, x_i]$ is a longer cycle than C_m, contrary to the assumption that C_m is longest. Hence $x_{t-1}x_{i+1} \notin E(G)$. The proof for the second part is similar, and so it is omitted.

By (2), (4) holds.

(6) By Lemma 2.1(1) and (2), $\{x\} \cup N^+_{C_m}(H)$ is an independent set. □

Lemma 2.2 *For* $x \in V(H)$ *and* $x_i \in N_{C_m}(x)$, $1 \leq |N(x_{i+1}) \cap N(x)| \leq \alpha - 1$ *and* $|N(x_{i+1}) \cap N(x_{j+1})| \leq \alpha - 1$.

Proof. We first prove $1 \leq |N(x_{i+1}) \cap N(x)| \leq \alpha - 1$. As $x_i \in N_{C_m}(x)$, $1 = |\{x_i\}| \leq |N(x_{i+1}) \cap N(x)|$. By Lemma 2.1(1) and (5), $N(x_{i+1}) \cap N(x) \subseteq V(C_m)$. Thus by Lemma 2.1(6), $|N(x_{i+1}) \cap N(x)| = |N_{C_m}(x_{i+1}) \cap N_{C_m}(x)| \leq |N_{C_m}(x)| = |N^+_{C_m}(x)| \leq |N^+_{C_m}(H)| \leq \alpha - 1$. Hence $1 \leq |N(x_{i+1}) \cap N(x)| \leq \alpha - 1$.

To prove that $|N(x_{i+1}) \cap N(x_{j+1})| \leq \alpha - 1$, we assume by way of contradiction that $|N(x_{i+1}) \cap N(x_{j+1})| \geq \alpha$. By Lemma 2.1(1) and (5), $N(x_{i+1}) \cap N(x_{j+1}) \subseteq V(C_m)$. Let $N_1 = N(x_{i+1}) \cap N(x_{j+1})$. Then $N_1 \subseteq V(C_m)$ and $|N_1| \geq \alpha$.

Claim 1: $N_1^- \cup \{u\}$ is an independent set of G for any $u \in V(H)$.

Proof of Claim 1. Firstly, by Lemma 2.1(2), $x_{i+1}, x_{j+1} \notin N_1$. And by Lemma 2.1(4), $x_{t-1}u \notin E(G)$ for any vertex $x_t \in N_{[x_{j+2}, x_i]}(x_{j+1}) \cup N_{[x_{i+2}, x_j]}(x_{i+1})$. So $x_{t-1}u \notin E(G)$ for any $x_{t-1} \in N_1^-$ and $u \in V(H)$.

Secondly, if there are two vertices $x_{k-1}, x_{h-1} \in N_1^-$ such that $x_{k-1}x_{h-1} \in E(G)$, we will get contradictions in either of the following two cases. Let $P(H)$ be an (x_i, x_j)-path in H.

Case 1. $x_{k-1} \in [x_{i+1}, x_{j-1}]$ and $x_{h-1} \in [x_{j+1}, x_{i-1}]$, then the cycle

$$C : P(H)[x_{j-1}, x_k]^-[x_{j+1}, x_{h-1}][x_{k-1}, x_{i+1}]^-[x_h, x_i]$$

is a longer cycle than C_m, a contradiction.

Case 2. Either $x_{k-1}, x_{h-1} \in [x_{i+1}, x_{j-1}]$ or $x_{k-1}, x_{h-1} \in [x_{j+1}, x_{i-1}]$. Without loss of generality we assume that $x_{k-1}, x_{h-1} \in [x_{j+1}, x_{i-1}]$ and $x_k \in [x_{j+1}, x_{h-1}]$. Then the cycle

$$C_1 = P(H)[x_j, x_{i+1}]^-[x_k, x_{h-1}][x_{k-1}, x_{j+1}]^-[x_h, x_i]$$

is longer than the longest cycle C_m, a contradiction.

Hence Claim 1 holds. So $N_1^- \cup \{u\}$ is an independent set of G, and $|N_1^- \cup \{u\}| \geq \alpha+1$, contrary to the fact that α is the independent number of G. □

3 Preliminary Lemmas

Note that Lemmas 2.1 and 2.2 in Section 2 do not need the condition in Theorem 1.7 that $|N(x) \cup N(y)| \geq n - \delta - 1$ for each pair of nonadjacent vertices x, y with $1 \leq |N(x) \cap N(y)| \leq \alpha - 1$. In Lemmas 3.1 and 3.2, we always assume that G is a 2-connected graph on n vertices and $|N(x) \cup N(y)| \geq n - \delta - 1$ for each pair of nonadjacent vertices x, y with $1 \leq |N(x) \cap N(y)| \leq \alpha - 1$. Also, let C_m be a longest cycle in G, H be a component of $G - C_m$, and $x_{i+1}, x_{j+1} \in N^+_{C_m}(H)$ with $[x_{i+1}, x_{j-1}] \cap N_{C_m}(H) = \emptyset$.

Lemma 3.1 $1 \leq |N(x_{i+1}) \cap N(_{j+1})| \leq \alpha - 1$.

Proof By Lemma 2.2 that $|N(x_{i+1}) \cap N(_{j+1})| \leq \alpha - 1$, it suffices to show that $1 \leq |N(x_{i+1}) \cap N(_{j+1})|$, or $d(x_{i+1}, x_{j+1}) = 2$. Suppose that $d(x_{i+1}, x_{j+1}) \neq 2$. Choose $x \in V(H)$ such that $xx_i \in E(G)$.

Claim 2: There exists a vertex $u \in N_{[x_{j+2}, x_i]}(x_{j+1})$ such that $u \notin N^-_{[x_{j+2}, x_i]}(x_{j+1})$ and $u \notin N(x_{i+1}) \cup N(x)$.

Proof of Claim 2. Since $d(x_{i+1}, x_{j+1}) \neq 2$, $x_i x_{j+1} \notin E(G)$. And as $x_{j+2}x_{j+1} \in E(G)$, let $x_h \in N_{[x_{j+2}, x_{i-1}]}(x_{j+1})$ such that $N_{[x_{h+1}, x_i]}(x_{j+1}) = \emptyset$. As $d(x_{i+1}, x_{j+1}) \neq 2$ and $x_h x_{j+1} \in E(G)$, $x_h x_{i+1} \notin E(G)$. If $xx_h \notin E(G)$, then $u = x_h$ satisfies Claim 2; if $xx_h \in E(G)$, then $x_h \neq x_{i-1}$ and $xx_{h+1} \notin E(G)$ by Lemma 2.1(1), and $x_{h+1}x_{i+1} \notin E(G)$ by Lemma 2.1(2), and so $u = x_{h+1}$ satisfies Claim 2. □

Claim 3: If $N_{[x_{i+1}, x_{j-2}]}(x_{j+1}) \neq \emptyset$, then there exists $v \in N_{[x_{i+1}, x_{j-2}]}(x_{j+1})$ such that $v \notin N^+_{[x_{i+1}, x_{j-2}]}(x_{j+1})$ and $v \notin N(x_{i+1}) \cup N(x)$.

Proof of Claim 3. As $x_{i+2}x_{j+1} \notin E(G)$ by $d(x_{i+1}, x_{j+1}) \neq 2$ and $N_{[x_{i+1}, x_{j-2}]}(x_{j+1}) \neq \emptyset$, let $x_l \in N_{[x_{i+1}, x_{j-2}]}(x_{j+1})$ with $N_{[x_{i+1}, x_{l-1}]}(x_{j+1})$

$= \emptyset$. Then $x_{i+1}x_l \notin E(G)$ by $d(x_{i+1}, x_{j+1}) \neq 2$. Since $[x_{i+1}, x_{j-1}] \cap N_{C_m}(H) = \emptyset$, $xx_l \notin E(G)$. So $v = x_l$ satisfies Claim 3. $\square$

By Lemma 2.1(3), $N^-_{[x_{j+2},x_i]}(x_{j+1}) \cap (N(x_{i+1}) \cup N(x)) = \emptyset$; by Lemma 2.1(4) and $[x_{i+1}, x_{j-1}] \cap N_{C_m}(H) = \emptyset$, $N^+_{[x_{i+1},x_{j-2}]}(x_{j+1}) \cap (N(x_{i+1}) \cup N(x)) = \emptyset$; by Lemma 2.1(5), $N_{G-C_m-V(H)}(x_{j+1}) \cap (N(x_{i+1}) \cup N(x)) = \emptyset$; by Claims 2 and 3, $\{u, v\} \cap (N(x_{i+1}) \cup N(x)) = \emptyset$ and $u \notin N^-_{[x_{j+2},x_i]}(x_{j+1})$, $v \notin N^+_{[x_{i+1},x_{j-2}]}(x_{j+1})$; by Lemma 2.1(1), $N_H(x_{j+1}) = \emptyset$ and so $N_{G-C_m-V(H)}(x_{j+1}) = N_{G-C_m}(x_{j+1})$. Hence

$$|N(x_{i+1}) \cup N(x)| \leq |V(G)| - |N^-_{[x_{j+2},x_i]}(x_{j+1}) \cup N^+_{[x_{i+1},x_{j-2}]}(x_{j+1}) \cup N_{G-C_m-V(H)}(x_{j+1})| - |\{x_{i+1}, x, u, v\}| = |V(G)| - (|N_{[x_{j+2},x_i]}(x_{j+1})| + |N_{[x_{i+1},x_{j-2}]}(x_{j+1})|) - |N_{G-C_m-V(H)}(x_{j+1})| - |\{x_{i+1}, x, u, v\}| = |V(G)| - |N_{C_m}(x_{j+1}) - \{x_j, x_{j-1}\}| - |N_{G-C_m}(x_{j+1})| - |\{x_{i+1}, x, u, v\}| = n - d(x_{j+1}) - 2.$$

Together with $1 \leq |N(x_{i+1}) \cap N(x)| \leq \alpha - 1$ by Lemma 2.2, it is contrary to the condition that $|N(x) \cup N(y)| \geq n - \delta - 1$. $\square$

Lemma 3.2 *Let $h = |V(H)|$, $k = |N_{C_m}(H)|$. Each of the following holds.*

(1) $h + k = \delta(G) + 1$.

(2) For every $v \in V(H)$, $N(v) = (V(H) \setminus \{v\}) \cup N_{C_m}(H)$.

(3) H is a complete subgraph.

Proof For every vertex v in H, $N(v) \subseteq (V(H) \setminus \{v\}) \cup N_{C_m}(H)$, so $h(h - 1 + k) \geq \sum_{v \in V(H)} d(v) \geq h\delta(G)$, which implies that $h + k \geq \delta(G) + 1$. Let $x_{i+1}, x_{j+1} \in N^+_{C_m}(H)$ with $[x_{i+1}, x_{j-1}] \cap N_{C_m}(H) = \emptyset$. By Lemma 3.1, $1 \leq |N(x_{i+1}) \cap N(x_{j+1})| \leq \alpha - 1$. Then

$$n - \delta(G) - 1 \leq |N(x_{i+1}) \cup N(x_{j+1})| \leq |V(G)| - |N^+_{C_m}(H)| - |V(H)| \leq n - (h + k) \leq n - \delta(G) - 1.$$

Thus $h + k = \delta(G) + 1$, H is complete and $N(v) = (V(H) \setminus \{v\}) \cup N_{C_m}(H)$. $\square$

Lemma 3.3 *$G - C_m$ has only one component.*

Proof Suppose that $G - C_m$ has components H_1, H_2. Let $h_i = |V(H_i)|, k_i = |N_{C_m}(H_i)| (i = 1, 2)$. Then we claim that $k_1 = k_2$. Suppose that $k_1 > k_2 \geq 2$. Let $x_i, x_j, x_t \in N_{C_m}(H_1)$ with $[x_{i+1}, x_{j-1}] \cap N_{C_m}(H) = \emptyset$. Then by Lemma 2.1(5) $|\{x_{i+1}, x_{j+1}, x_{t+1}\} \cap N_{C_m}(H_2)| \leq 1$. Assume that $x_{i+1}, x_{j+1} \notin N_{C_m}(H_2)$. Then $|N(x_{i+1}) \cup N(x_{j+1})| \leq |V(G)| - |V(H_1)| - |N^+_{C_m}(H_1)| - |V(H_2)| \leq n - h_1 - k_1 - 1 = n - \delta - 2$ by Lemma 3.2(1), together with Lemma 3.1, we get a contradiction.

By Lemma 2.1(5), we may assume that $N(x_{i+1}) \cap V(H_2) = \emptyset$. Let $x \in V(H_1)$ with $xx_i \in E(G)$. Then by Lemma 3.2(1) $|N(x_{i+1}) \cup N(x)| \leq |V(G)| - |N^+_{C_m}(H_1)| - |V(H_2)| - |\{x\}| \leq n - k_1 - h_2 - 1 = n - k_2 - h_2 - 1 = n - \delta - 2$, a contradiction. □

By Lemma 3.3, we may assume that $G - C_m = H$ in the following lemma and section.

Lemma 3.4 *Let $V_i = [x_{i+1}, x_{j-1}]$ with $V_i \cap N_{C_m}(H) = \emptyset$. Then $|V_i| = h$ where $|V(H)| = h$.*

Proof Clearly, $|V_i| \geq h$. Suppose that $|V_i| \geq h + 1$. We claim that $N(x_{i-1}) \cap \{x_{i+1}, x_{i+2}, \cdots, x_{i+h}\} \neq \emptyset$. Otherwise, take $x \in V(H)$ with $xx_i \in E(G)$. Then by Lemma 3.2(1) $|N(x) \cup N(x_{i-1})| \leq |V(G)| - |N^-_{C_m}(H)| - |\{x, x_{i+1}, x_{i+2}, \cdots, x_{i+h}\}| \leq n - k - h - 1 = n - \delta - 2$ where $k = |N^-_{C_m}(H)|$, a contradiction.

Let $x_{i-1}x_{i+s} \in E(G)$ for some $s \in \{1, 2, \cdots, h\}$. As C_m is a longest cycle of G, we have $|[x_{i+s+1}, x_{j-1}]| \geq h$ and by a similar proof as above, $N(x_{j+1}) \cap [x_{i+s+1}, x_{j-1}] \neq \emptyset$. Then let x_t be the first vertex in $[x_{i+s+1}, x_{j-1}]$ with $x_{j+1}x_t \in E(G)$ and $x' \in V(H)$ with $x_jx' \in E(G)$. Then $|[x_{i+s+1}, x_{t-1}]| \geq h$, and $|N(x') \cup N(x_{j+1})| \leq |V(G)| - |N^+_{C_m}(H)| - |[x_{i+s+1}, x_{t-1}]| - |\{x'\}| \leq n - k - h - 1 = n - \delta - 2$, a contradiction. □

4 The Proof of Theorem 1.7

By way of contradiction we assume that G is not hamiltonian. Then let $C_m : x_1x_2 \cdots x_mx_1$ be a longest cycle in G and $H = G - C_m$ by

Lemma 3.3. We consider the following cases.

Case 1. $|N_{C_m}(H)| \geq 3$.

Claim 4: (1) $|V(H)| = 1$.

(2) $\delta(G) \geq 3$.

Proof of Claim 4. (1) By way of contradiction we assume that $|V(H)| \geq 2$. Since $|N_{C_m}(H)| \geq 3$, there are vertices $x_i, x_j, x_h \in N_{C_m}(H)$ $(i < j < h)$ such that $([x_{i+1}, x_{j-1}] \cup [x_{j+1}, x_{h-1}]) \cap N_{C_m}(H) = \emptyset$. By Lemma 3.1, $1 \leq |N(x_{i+1}) \cap N(x_{j+1})| \leq \alpha - 1$.

By Lemma 3.2(3), $G[H]$ is complete. Let $x'_h, x'_i, x'_j \in V(H)$ such that $x_h x'_h, x_i x'_i, x_j x'_j \in E(G)$. So there exist an (x'_h, x'_i)-path and (x'_h, x'_j)-path in $V(H)$ each of which passes through all vertices of $V(H)$, denoted by $P(H)$ and $P'(H)$ respectively. Then $x_{h+2}x_{i+1} \notin E(G)$, $x_{h+2}x_{j+1} \notin E(G)$ otherwise $x_h P(H)[x_i, x_{h+2}]^-[x_{i+1}, x_h]$ or $x_h P'(H)[x_j, x_{h+2}]^-[x_{j+1}, x_h]$ is a longer cycle than C_m (only x_{h+1} is missing, but $|V(P(H))| = |V(P'(H))| = |V(H)| \geq 2$), a contradiction. Let $u \in V(H)$. then $|N(x_{i+1}) \cup N(x_{j+1})| \leq |V(G)| - |N^+_{C_m}(u)| - |N_H(u) \cup \{u\}| - |\{x_{h+2}\}| \leq n - \delta - 2$, a contradiction. So $|V(H)| = 1$. Therefore Claim 4(1) is established.

(2) If $\delta(G) \leq 2$, then $|N(x_{i+1}) \cup N(x_{j+1})| \leq |V(G)| - |N^+_{C_m}(H)| - |V(H)| \leq n - 3 - 1 \leq n - \delta - 2$, a contradiction.□

By Claim 4(1), we may assume that $V(G - C_m) = \{u\}$. By Lemma 3.4, every vertex in $\{x_1, x_3, \cdots, x_{2k-1}\}$ or $\{x_2, x_4, \cdots, x_{2k} = x_m\}$ is adjacent to u. Without loss of generality, assume $\{x_1, x_3, \cdots, x_{2k-1}\} = N(u)$. We choose another longest cycle $C_1 : x_1 u [x_3, x_m] x_1$. Then $G - C_1 = x_2$. By Claim 4(2) that $\delta(G) \geq 3$, we have that $|N_{C_1}(x_2)| \geq 3$. Using a similar argument, we get $\{x_1, x_3, \cdots, x_{2k-1}\} = N(x_2)$. Similarly, $\{x_1, x_3, \cdots, x_{2k-1}\} = N(x_4) = \cdots = N(x_{2k})$. Together with Claim 4(1) and Lemma 2.1(2), the graph is $G = G^*_{\frac{n-1}{2}} \bigvee K^C_{\frac{n+1}{2}}$ and $n = 2k + 1$ (see Figure 1.2) where $G_{\frac{n-1}{2}}$ is a subgraph on $\frac{n-1}{2}$ vertices and $K^C_{\frac{n+1}{2}}$ is the complement of a complete graph $K_{\frac{n+1}{2}}$.

Case 2. $|N_{C_m}(H)| = 2$. Assume that $N_{C_m}(H) = \{x_i, x_j\} (i < j)$.

Claim 5: Let $V_1 = [x_{i+1}, x_{j-1}], V_2 = [x_{j+1}, x_{i-1}]$. Then for every vertex $v \in V_1 \cup V_2$, $vx_i \in E(G)$ and $vx_j \in E(G)$.

Proof of Claim 5. Let $x'_j, x'_i \in V(H)$, $x_j x'_j, x_i x'_i \in E(G)$ and $P(H)$ be an (x'_j, x'_i)-path passing through all vertices in H by Lemma 3.2(3).

Let $C'_m = [x_i, x_j]P(H)x_i$ be a cycle of G. By Lemma 3.4, $|V_1| = |V_2| = |V(H)| = h$. So $|C'_m| = |V_1|+2+|V(H)| = |V_2|+2+|V(H)| = |V_2| + 2 + |V_1| = |C_m|$, C'_m is a longest cycle of G and by Lemma 3.3, $G - C'_m = V_2$. If $|N_{C'_m}(V_2)| \geq 3$, then by Claim 4(1), $|V_2| = 1$ and so $1 \geq \delta - 1$, or $\delta \leq 2$, contrary to Claim 4(2). So we assume that $|N_{C'_m}(V_2)| = 2$. By Lemma 3.2(2) and (3), every vertex $v \in V_2$, $vx_i, vx_j \in E(G)$, and $G[V_2]$ is complete. Similarly $G[V_1]$ is complete. □

By Lemma 3.2(3), $G[V_i](i = 1, 2)$ and H are complete subgraphs. By Claim 5, we obtain the graph G_1 (see Figure 1.1) and note that there may be an edge joining x_i and x_j. □

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K_2. Let G^* denote the graph obtained by contracting the two copies of G in $\alpha(G)$. Then 4-cycle is the spanning eulerian subgraph of G^*. Hence by Theorem 1, for any $\alpha \in S_{|V(G)|}$, $\alpha(G)$ is supereulerian. But $G \notin \mathscr{F}$ since for any even subset $S \subseteq V(G)$, whenever S does not contain the vertex of degree one, G cannot have a spanning connected subgraph with $O(H) = S$.

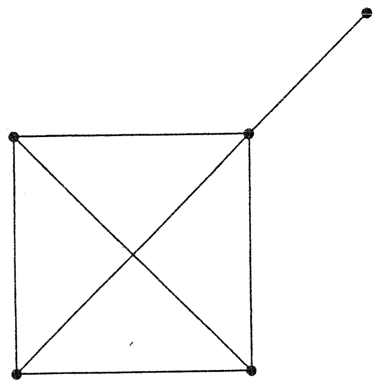

Figure 1: Graph G

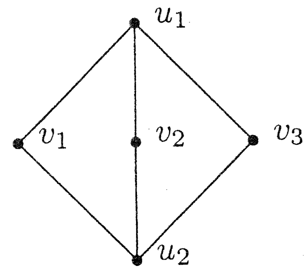

Figure 2: Graph $K_{2,3}$

Lemma 1 $K_{2,t} \in \mathscr{F}_1$, where $t \geq 3$ is an odd integer.

Proof: Let (X, Y) be a bipartition of $K_{2,t}$, where $X = \{u_1, u_2\}$ and $Y = \{v_1, v_2, \cdots, v_t\}$(As an example, $K_{2,3}$ is shown in Fig. 2). To show $K_{2,t} \in \mathscr{F}_1$, we only need to show for arbitrary distinct two vertices $u, v \in V(K_{2,t})$, $K_{2,t}$ has a spanning eulerian subgraph H with $O(H) = \{u, v\}$.

Case 1 $u, v \in X$

Let $u = u_1$ and $v = u_2$. Since t is an odd integer, $K_{2,t}$ is a spanning eulerian subgraph which odd vertex set is $\{u, v\}$.

Case 2 $u, v \in Y$

Let $u = v_i$ and $v = v_j$, $1 \leq i \leq 2$, $1 \leq j \leq t$. Since $i \neq j$, $K_{2,t} - u_1v_i - u_2v_j$ is a spanning eulerian subgraph which odd vertex set is $\{u, v\}$.

Case 3 $u \in X, v \in Y$

Let $u = u_i$ and $v = v_j$, then $K_{2,t} - u_iv_j$ is a spanning eulerian subgraph which odd vertex set is $\{u, v\}$, $1 \leq i \leq 2$, $1 \leq j \leq t$.

Case 4 $v \in X, u \in Y$

The result is obtained similarly.□

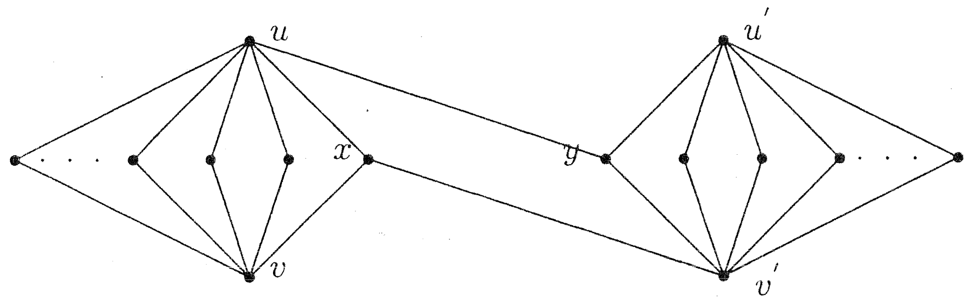

Figure 3: One case of $\alpha(G)^*$

Theorem 4 Let G be a graph with $|V(G)| \geq 2$ and $F(G) \leq 2$. If G has at most one cut edge, then for any $\alpha \in S_{|V(G)|}$, $\alpha(G)$ is supereulerian.

Proof: Let $\alpha \in S_{|V(G)|}$ be a permutation. Let G' be the reduction of G. $\alpha(G)^*$ denotes the graph obtained by contracting the two copies of G in $\alpha(G)$. By Theorem 1, $\alpha(G) \in \mathscr{SL}$ if and only if $\alpha(G)^* \in \mathscr{SL}$. By Theorem 2, G' must be K_1 or K_2 or $K_{2,t}$ for some integer $t \geq 1$. G has at most one cut edge implies $t \geq 2$.

Let G_x and G_y be two copies of G, G_x' and G_y' the reductions of G_x and G_y, respectively. For every $v \in V(G')$, let H_v denote the preimage of v.

Case 1 G' is K_1.

Since 2-cycle is collapsible, then for any $\alpha \in S_{|V(G)|}$, $\alpha(G) \in \mathscr{CL}$ by corollary 1. Thus $\alpha(G)$ is supereulerian.

Case 2 G' is K_2.

Thus 4-cycle is the spanning eulerian subgraph of $\alpha(G)^*$. Hence by Theorem 1, for any $\alpha \in S_{|V(G)|}$, $\alpha(G)$ is supereulerian.

Case 3 G' is $K_{2,t}$ for some odd integer $t \geq 3$.

We choose vertex $u \in V(G_x')$ such that for every $v \in V(G_x')$, $|V(H_u)| \geq |V(H_v)|$. Select vertex $v \in V(G_x')$ such that $v \neq u$. Thus we can pick two distinct vertices $u' \in V(G_y')$ and $v' \in V(G_y')$. There exist four vertices $x_1 \in V(H_u)$, $x_2 \in V(H_v)$, $y_1 \in V(H_{u'})$, $y_2 \in V(H_{v'})$, such that $\alpha(x_1) = y_1$ and $\alpha(x_2) = y_2$. By Lemma 1, $G' \in \mathscr{F}_1$, then there exists an open eulerian trail L_x in G_x' whose origin is u and whose terminus is v. Similarly in G_y' there exists an open eulerian trail L_y whose origin is u' and whose termi-