Group connectivity of certain graphs

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Abstract

Let G be an undirected graph, A be an (additive) Abelian group and $A^* = A - \{0\}$. A graph G is A-connected if G has an orientation such that for every function $b : V(G) \mapsto A$ satisfying $\sum_{v \in V(G)} b(v) = 0$, there is a function $f : E(G) \mapsto A^*$ such that at each vertex $v \in V(G)$ the net flow out of v equals $b(v)$. We investigate the group connectivity number $\Lambda_g(G) = \min\{n : G$ is A-connected for every Abelian group with $|A| \geq n\}$ for complete bipartite graphs, chordal graphs, and biwheels.

1. Introduction

Graphs in this paper are finite and may have loops and multiple edges. Terms and notation not defined here are from [1]. Throughout the paper, $\mathbf{Z}_n$ denotes the cyclic group of order n, for some integer $n \geq 2$.

Let $D = D(G)$ be an orientation of an undirected graph G. If an edge $e \in E(G)$ is directed from a vertex u to a vertex v, then let *tail*$(e) = u$ and *head*$(e) = v$. For a vertex $v \in V(G)$, let

$$E_D^+(v) = \{e \in E(D) : v = \text{tail}(e)\} \text{ and}$$
$$E_D^-(v) = \{e \in E(D) : v = \text{head}(e)\}.$$

The subscript D may be omitted when $D(G)$ is understood from the context.

Let A denote a nontrivial (additive) Abelian group with identity 0, and $A^* = A - \{0\}$. Let $F(G, A)$ denote the set of all functions from $E(G)$ to A, and $F^*(G, A)$ denote the set of all functions from $E(G)$ to A^*. Unless otherwise stated, we shall adopt the following convention: if $X \subseteq E(G)$

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and $f : X \mapsto A$ is a function, then we regard f as a function $f : E(G) \mapsto A$ where $f(e) = 0$ for all $e \in E(G) - X$.

Given a function $f \in F(G, A)$, let $\partial f : V(G) \mapsto A$ be given by

$$\partial f(v) = \sum_{e \in E_D^+(v)} f(e) - \sum_{e \in E_D^-(v)} f(e),$$

where "$\sum$" refers to the addition in A.

A function $b : V(G) \mapsto A$ is called an *A-valued zero-sum function on* G if $\sum_{v \in V(G)} b(v) = 0$. The set of all A-valued zero-sum functions on G is denoted by $Z(G, A)$. Given $b \in Z(G, A)$ and an orientation D of G, a function $f \in F^*(G, A)$ is an *(A, b)-nowhere-zero flow* ((A, b)-NZF) if $\partial f = b$. A graph G is *A-connected* if G has an orientation D such that for any $b \in Z(G, A)$, G has an (A, b)-NZF. For an Abelian group A, let $\langle A \rangle$ be the family of graphs that are A-connected. The concept of A-connectivity was introduced by Jaeger, et al. in [6]. A concept similar to group connectivity was independently introduced in [7], with a different motivation from [6].

It is observed in [6] that the property $G \in \langle A \rangle$ is independent of the orientation of G: If $D(G)$ and f satisfy the condition for G to be A-connected, then for an orientation D' of G that reverses the direction of an edge e, replace $f(e)$ with $-f(e)$. Thus, A-connectivity is a property of an undirected graph whose definition assumes an arbitrary orientation.

An *A-nowhere-zero flow* (abbreviated as *A-NZF*) in G is an $(A, 0)$-NZF; thus, each A-connected graph admits an A-NZF. Nowhere-zero flows were introduced by Tutte [14] and have been studied extensively; for a survey see [5]. A graph that admits an A-NZF is necessarily 2-edge-connected (bridgeless) (see [15]).

Tutte [5] conjectured that every 4-edge-connected graph admits a $\mathbf{Z}_3$-nowhere-zero flow and Jaeger, et al. [6] conjectured that every 5-edge-connected graph is $\mathbf{Z}_3$-connected. For more on the literature on nowhere-zero flow problems, see Tutte [14], Jaeger [5] and Zhang [15].

For a 2-edge-connected graph G, the *group connectivity number* of G is defined as

$$\Lambda_g(G) = \min\{k : G \text{ is } A\text{-connected for every Abelian group with } |A| \geq k\}.$$

We show that if G is 2-edge-connected, then $\Lambda_g(G)$ exists as a finite number. We also investigate the group connectivity number for certain families of graphs and determine the corresponding best possible upper bounds.

2. Preliminaries

In this section we present some of known results that we use in our proofs.

Let G be a graph. For a subset $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the two ends of each edge e in X and deleting e. Note that even when G is a simple graph, the contraction G/X may have loops and multiple edges. For convenience, we write G/e for $G/\{e\}$, where $e \in E(G)$. If H is a subgraph of G, then we write G/H for $G/E(H)$.

Proposition 2.1 (Lai [9]) Let A be an Abelian group. Then $\langle A \rangle$ satisfies each of the following:

(C1) $K_1 \in \langle A \rangle$.
(C2) If $G \in \langle A \rangle$ and $e \in E(G)$, then $G/e \in \langle A \rangle$.
(C3) If H is a subgraph of G, $H \in \langle A \rangle$, and $G/H \in \langle A \rangle$, then $G \in \langle A \rangle$.

Lemma 2.2 (Jaeger, et al. [6], Lai [9]) Let A be an Abelian group and C_n denote a cycle on $n \geq 1$ vertices. Then $C_n \in \langle A \rangle$ if and only if $|A| \geq n+1$.

Lemma 2.3 (Jaeger, et al. [6]) Let G be a connected graph with n vertices and m edges. Then $\Lambda_g(G) = 2$ if and only if $n = 1$ (and so G has m loops).

Let $O(G) = \{$odd degree vertices of $G\}$. A graph G is *collapsible* if for any subset $R \subseteq V(G)$ with $|R| \equiv 0 \pmod 2$, G has a spanning connected subgraph Γ_R such that $O(\Gamma_R) = R$.

Theorem 2.4 (Catlin [2]) Suppose that graph G is one edge short of having two edge-disjoint spanning trees. Then G is collapsible if and only if $\kappa'(G) \geq 2$.

Lemma 2.5 (Lai [8]) Let G be a collapsible graph and let A be an Abelian group with $|A| = 4$. Then $G \in \langle A \rangle$.

Lemma 2.6 (Lai [10]) Let A be an Abelian group with $|A| \geq 3$, and S be a connected spanning subgraph of graph G. If, for each $e \in E(S)$, G has a subgraph $H_e \in \langle A \rangle$ with $e \in E(H_e)$, then $G \in \langle A \rangle$.

We will sometimes apply Lemma 2.6 with $S = G$.

A *wheel* W_n is a graph obtained by joining a cycle with n vertices and

K_1. The vertex of K_1 is called the *center of* W_n.

Lemma 2.7 (Lai, Xu and Zhang [11])
(1) $W_{2n} \in \langle \mathbf{Z}_3 \rangle$.
(2) Let $G \cong W_{2n+1}$ and $b \in Z(G, \mathbf{Z}_3)$. Then there exists a $(\mathbf{Z}_3, b)$-NZF $f \in F^*(G, \mathbf{Z}_3)$ if and only if $b \neq 0$.

Lemma 2.8 $\Lambda_g(W_{2n}) = 3$ for $n \geq 1$.

Proof. Since every edge of W_{2n} lies in a C_3, it follows from Lemma 2.2 and Lemma 2.6 that $W_{2n} \in \langle A \rangle$ for any Abelian group A with $|A| \geq 4$. Furthermore, by Lemma 2.7, we know that $W_{2n} \in \langle \mathbf{Z}_3 \rangle$. □

Proposition 2.9 If G is a 2-edge-connected graph, then $\Lambda_g(G)$ exists as a finite number.

Proof. Since G is 2-edge-connected, every edge of G must be in a cycle. Since G is finite, there exists an integer $k > 0$ such that every edge of G lies in a cycle of length at most $k - 1$. By Lemmas 2.2 and 2.6, $\Lambda_g(G) \leq k$.
□

3. Reduction methods

Let G be a graph and $v \in V(G)$. Let $E_G(v) = \{e_1, e_2, \cdots, e_d\}$ denote the set of edges in G that are incident with v, where d is the degree of v in G. Suppose that $d \geq 3$, and that for $i = 1, 2$, e_i is incident with v and v_i such that $v_1 \neq v_2$. We define $G_\Delta\{e_1, e_2\}$ to be the graph obtained from $G - \{e_1, e_2\}$ by adding a new edge e joining v_1 and v_2 (see Figure 3.1). We also say that $G_\Delta\{e_1, e_2\}$ is obtained by *splitting* v *with respect to the edges* e_1 *and* e_2.

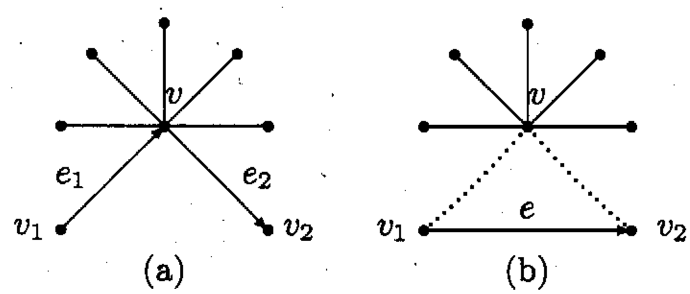

Figure 3.1: Vertex splitting

Theorem 3.1 Let A be an Abelian group. If $G_\Delta\{e_1, e_2\} \in \langle A \rangle$, then $G \in \langle A \rangle$. Hence, $\Lambda_g(G) \le \Lambda_g(G_\Delta\{e_1, e_2\})$.

Proof. For any $b \in Z(G, A)$, since $V(G) = V(G_\Delta\{e_1, e_2\})$, we can view $b \in Z(G_\Delta\{e_1, e_2\}, A)$ as well. Suppose there is a function $f \in F^*(G_\Delta\{e_1, e_2\}, A)$ such that $\partial f = b$. Then we can assign the value $f(e)$ to the edges e_1 and e_2 in G (see Figure 3.1(a)); the values of $\partial f(v)$, $\partial f(v_1)$, and $\partial f(v_2)$ will be the same in G as in $G_\Delta\{e_1, e_2\}$. Thus, when $G_\Delta\{e_1, e_2\}$ is A-connected, so is G. □

Theorem 3.2 Let A be an Abelian group, and H be a connected subgraph of 2-edge-connected graph G. If $G \in \langle A \rangle$, then $G/H \in \langle A \rangle$. Hence, $\Lambda_g(G/H) \le \Lambda_g(G)$.

Proof. Fix an Abelian group A with $|A| \ge \Lambda_g(G)$. Let $b' \in Z(G/H, A)$, and v_H be the vertex of G/H onto which H is contracted. Fix a vertex $v_0 \in V(H)$. Define $b : V(G) \mapsto A$ as follows:

$$b(z) = \begin{cases} b'(z) & \text{if } z \in V(G) - V(H) \\ b'(v_H) & \text{if } z = v_0 \\ 0 & \text{if } z \in V(H) - \{v_0\} \end{cases}$$

Then

$$\sum_{z \in V(G)} b(z) = \sum_{z \in V(G/H)} b'(z) = 0,$$

and so $b \in Z(G, A)$.

Since $|A| \ge \Lambda_g(G)$, there is a function $f \in F^*(G, A)$ such that $\partial f = b$. Let $A_G(H) = \{z \in V(H) : z \text{ is incident with an edge in } E(G) - E(H)\}$. Let f' be the restriction of f on $E(G) - E(H)$. Then at v_H,

$$\begin{aligned} \partial f'(v_H) &= \sum_{e \in E^+_{G/H}(v_H)} f'(e) - \sum_{e \in E^-_{G/H}(v_H)} f'(e) \\ &= \sum_{v \in A_G(H)} \left(\sum_{e \in E^+_G(v)} f(e) - \sum_{e \in E^-_G(v)} f(e) \right) \\ &= \sum_{v \in A_G(H)} \partial f(v). \end{aligned}$$

Since $\partial f = b$, $b(v_0) = b'(v_H)$, and $b(z) = 0$ for all $z \in V(H) - \{v_0\}$, we have

$$\partial f'(v_H) = \sum_{v \in A_G(H)} \partial f(v) = \sum_{v \in V(H)} \partial f(v) = \partial f(v_0) = b'(v_H).$$

Furthermore, for any $z \in V(G/H) - \{v_H\}$, $\partial f'(z) = \partial f(z) = b(z) = b'(z)$. Hence, $\partial f' = b'$, and f' is an (A, b')-NZF of G/H. □

Theorem 3.3 If H is a 2-edge-connected subgraph of a 2-edged-connected graph G, then $\Lambda_g(G) \leq \max(\Lambda_g(H), \Lambda_g(G/H))$.

Proof. Let A be an Abelian group with $|A| \geq \max(\Lambda_g(H), \Lambda_g(G/H))$. Then $H \in \langle A \rangle$ and $G/H \in \langle A \rangle$. By Proposition 2.1(C3), $G \in \langle A \rangle$ also. Therefore, $\Lambda_g(G) \leq \max(\Lambda_g(H), \Lambda_g(G/H))$. Note that if $\Lambda_g(H) \leq \Lambda_g(G/H)$, then, by Theorem 3.2, $\Lambda_g(G) = \Lambda_g(G/H)$. □

4. Complete graphs and complete bipartite graphs

For a graph G, let $\lambda_g(G)$ be the smallest positive integer k such that for any Abelian group A with $|A| \geq k$, G has an A-NZF. Shahmohamad ([12, 13]) investigated the value of $\lambda_g(G)$ for several classes of graphs.

Proposition 4.1 (Shahmohamad [12, 13]) Let l, m and n be positive integers.
(i) If $l \geq 3$ is odd, then $\lambda_g(K_l) = 2$.
(ii) If $l \geq 6$ is even, then $\lambda_g(K_l) = 3$.
(iii) $\lambda_g(K_4) = 4$.
(iv) If both m and n are even, then $\lambda_g(K_{m,n}) = 2$.
(v) If m and n are not both even, then $\lambda_g(K_{m,n}) = 3$.

In this section we determine the group connectivity number for complete graphs and complete bipartite graphs.

Proposition 4.2 Let $n \geq 3$ be an integer. Then

$$\Lambda_g(K_n) = \begin{cases} 4 & \text{if } 3 \leq n \leq 4 \\ 3 & \text{if } n \geq 5 \end{cases}.$$

Proof. By Lemma 2.2, $\Lambda_g(K_3) = 4$. Let A be an Abelian group with $|A| \geq 4$. Since every edge of K_n lies in a 3-cycle, which is in $\langle A \rangle$ by

Lemma 2.2, it follows by Lemma 2.6 that $K_n \in \langle A \rangle$. Thus, $\Lambda_g(K_n) \leq 4$ for $n \geq 4$. It is well known that K_4 does not have a $\mathbf{Z}_3$-NZF, and so $\Lambda_g(K_4) = 4$.

Now suppose $n \geq 5$, and let A be an Abelian group with $|A| \geq 3$. Since every edge of K_n lies in a subgraph isomorphic to W_4, by Lemmas 2.6 and 2.8, $K_n \in \langle A \rangle$. By Lemma 2.2, $\Lambda_g(K_n) \neq 2$. □

Lemma 4.3 Let H be a graph on 2 vertices with $n \geq 2$ edges joining these two vertices. Then $\Lambda_g(H) = 3$.

Proof. Let $E(H) = \{e_1, e_2, \ldots, e_n\}$ with $n \geq 2$, and let C be the 2-cycle in H containing the edges e_1 and e_2. Let A be an Abelian group with $|A| \geq 3$. By Lemma 2.2, $C \in \langle A \rangle$. Since H/C is a single vertex, by Lemma 2.3, $\Lambda_g(H/C) = 2$, and so $H/C \in \langle A \rangle$. By Proposition 2.1(C3), $H \in \langle A \rangle$. □

The following lemma gives an upper bound for $\Lambda_g(K_{m,n})$.

Lemma 4.4 If $n \geq 2$ and $m \geq \max(n, 3)$, then $\Lambda_g(K_{m,n}) \leq \max(\Lambda_g(K_{m-1,n}), 3)$.

Proof. If $n \geq 2$ and $m \geq \max(n, 3)$, the complete bipartite graph $K_{m,n}$ has a subgraph isomorphic to $K_{m-1,n}$, and $K_{m-1,n}$ is 2-edge-connected. $K_{m,n}/K_{m-1,n}$ is a graph with two vertices and $n \geq 2$ edges. By Lemma 4.3, $\Lambda_g(K_{m,n}/K_{m-1,n}) = 3$. Thus, by Theorem 3.3, we have $\Lambda_g(K_{m,n}) \leq \max(\Lambda_g(K_{m-1,n}), 3)$. □

Repeated application of Lemma 4.4 yields the following corollary.

Corollary 4.5 If $n \geq 2$ and $m \geq \max(n, 3)$, then $\Lambda_g(K_{m,n}) \leq \max(\Lambda_g(K_{n,n}), 3)$.

We now state the main result of this section.

Theorem 4.6 Let $m \geq n \geq 2$ be integers. Then

$$\Lambda_g(K_{m,n}) = \begin{cases} 5 & \text{if } n = 2 \\ 4 & \text{if } n = 3 \\ 3 & \text{if } n \geq 4 \end{cases}.$$

Proof. The cases for $n = 2$, $n = 3$ and $n \geq 4$ follow from Lemmas 4.7, 4.9 and 4.10, respectively. □

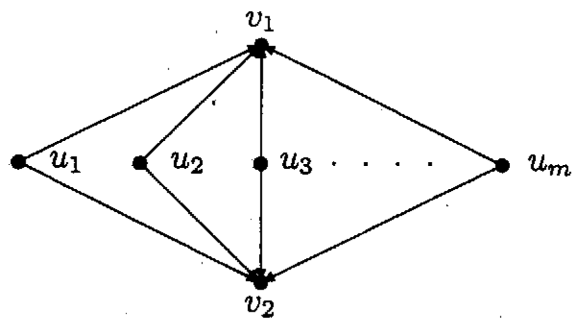

Figure 4.1: $K_{m,2}$

Lemma 4.7 $\Lambda_g(K_{m,2}) = 5$ for any integer $m \geq 2$.

Proof. Note that $K_{2,2}$ is isomorphic to the 4-cycle C_4. By Lemma 2.2, we have

$$\Lambda_g(K_{2,2}) = 5. \tag{1}$$

Then, by Corollary 4.5,

$$\Lambda_g(K_{m,2}) \leq 5, \text{ when } m \geq 3. \tag{2}$$

Next, we show that

$$\Lambda_g(K_{m,2}) > 4, \text{ when } m \geq 2. \tag{3}$$

We prove Inequality (3) by contradiction. Let $A = \{0, a_1, a_2, a_3\}$ be an Abelian group, where a_2 is an element of order 2. By way of contradiction, assume that $K_{m,2} \in \langle A \rangle$. Thus, for each $b \in Z(A, G)$, one can always find $f \in F^*(G, A)$ such that

$$\partial f = b. \tag{4}$$

Using the notation in Figure 4.1, we consider the following function $b : V(G) \mapsto A$ such that $b(u_1) = b(u_2) = \ldots = b(u_m) = a_2$. Orient each edge in this $K_{m,2}$ from a u_i to a v_j. Thus,

$$f(u_i v_1) + f(u_i v_2) = b(u_i) = a_2, \text{ for each } i = 1, 2, \cdots, m. \tag{5}$$

We will discuss the two groups of order 4, $\mathbf{Z}_4$ and $\mathbf{Z}_2 \times \mathbf{Z}_2$, separately.

Case 1: Suppose that $A = \mathbf{Z}_4$. The Equations (5) above each have solutions $f(u_iv_1) = f(u_iv_2) = a_1$ and $f(u_iv_1) = f(u_iv_2) = a_3$. It follows by Equation (4) that

$$b(v_1) = -\sum_{i=1}^{m} f(u_iv_1) = -\sum_{i=1}^{m} f(u_iv_2) = b(v_2). \tag{6}$$

Now if we set $b(v_1) = a_1 \neq b(v_2) = a_3$ when m is even, and set $b(v_1) = 0 \neq b(v_2) = a_2$ when m is odd (in both cases $\sum b(v_i) = 0$ is satisfied), we find a contradiction to Equation (6).

Case 2: Suppose that $A = \mathbf{Z}_2 \times \mathbf{Z}_2$. Then the Equations (5) above each have the solution $\{f(u_iv_1), f(u_iv_2)\} = \{a_1, a_3\}$. Without loss of generality, we may assume that for $1 \leq i \leq k$, $f(u_iv_1) = a_1$, and for $k+1 \leq i \leq m$, $f(u_iv_1) = a_3$. It follows by Equation (4) that

$$b(v_1) = -ka_1 - (m-k)a_3 = ka_2 + ma_3, \tag{7}$$

where we have used the fact that $a_i = -a_i$ $(i = 1, 2, 3)$ and $a_1 + a_3 = a_2$. When m is even, Equation (7) implies that $b(v_1) = ka_2 = a_2$ or 0. If we set $b(v_1) = a_1 = b(v_2)$, we get a contradiction. When m is odd, Equation (7) implies that $b(v_1) = ka_2 + a_3 = a_1$ or a_3. If we set $b(v_1) = 0$ and $b(v_2) = a_2$, we also get a contradiction.

These contradictions imply that no function $f \in F^*(G, A)$ satisfying Equation (4) exists. Thus, Equation (3) must hold. The lemma now follows by combining Equations (1), (2), and (3). □

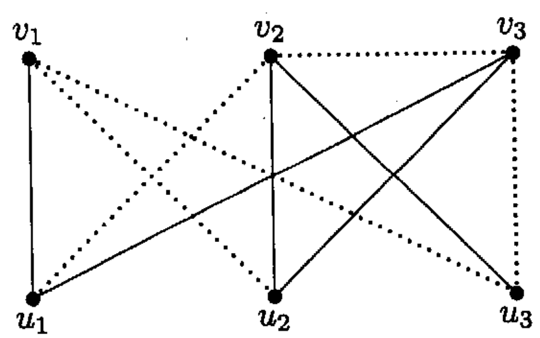

Figure 4.2: $K_{3,3}$ plus an edge

Lemma 4.8 $\Lambda_g(K_{3,3}) \leq 4$.

Proof. By Lemma 4.4 and Lemma 4.7, $\Lambda_g(K_{3,3}) \leq 5$.

$K_{3,3}$ has nine edges, and therefore, does not have two edge-disjoint spanning trees. If we add the edge v_2v_3 to the graph $K_{3,3}$ (as depicted in Figure 4.2), we can find two edge-disjoint spanning trees:

$$T_1 \text{ with } E(T_1) = \{v_1u_1, u_1v_3, v_3u_2, u_2v_2, v_2u_3\}, \text{ and}$$
$$T_2 \text{ with } E(T_2) = \{u_1v_2, v_2v_3, v_3u_3, u_3v_1, v_1u_2\}.$$

Therefore, by Theorem 2.4, $K_{3,3}$ is collapsible. Then, by Lemma 2.5, $\Lambda_g(K_{3,3}) \leq 4$. □

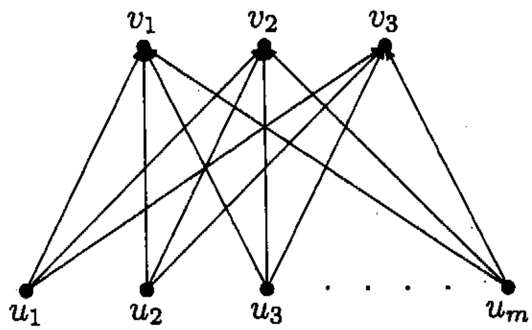

Figure 4.3: $K_{m,3}$

Lemma 4.9 $\Lambda_g(K_{m,3}) = 4$ for any integer $m \geq 3$.

Proof. By Corollary 4.5 and Lemma 4.8, when $m \geq 3$, $\Lambda_g(K_{m,3}) \leq \Lambda_g(K_{3,3}) \leq 4$. We shall show that

$$\Lambda_g(K_{m,3}) > 3, \text{ when } m \geq 3. \tag{8}$$

It suffices to show that $K_{m,3} \notin \langle \mathbf{Z}_3 \rangle$. By way of contradiction, suppose that $K_{m,3} \in \langle \mathbf{Z}_3 \rangle$.

We shall use the notation in Figure 4.3 and denote $\mathbf{Z}_3 = \{0, 1, 2\}$. Consider a function $b : V(K_{m,3}) \mapsto \mathbf{Z}_3$ such that for each $i = 1, 2, \ldots, m$, $b(u_i) = 0$, and $b(v_1) = 0, b(v_2) = 1$ and $b(v_3) = 2$. Then $b \in Z(G, \mathbf{Z}_3)$. Orient each edge in this $K_{m,3}$ from a u_i to a v_j.

Since $K_{m,3}$ is assumed to be in $\langle \mathbf{Z}_3 \rangle$, there must be an $f \in F^*(K_{m,3}, \mathbf{Z}_3)$ such that $\partial f = b$. Then the equality $\partial f = b$ reduces, for each i, to

$$b(u_i) = f(u_iv_1) + f(u_iv_2) + f(u_iv_3) = 0. \tag{9}$$

Note that in $\mathbf{Z}_3$, for each $i = 1, 2, \ldots, m$, Equation (9) has solutions $f(u_iv_1) = f(u_iv_2) = f(u_iv_3) = 1$ and $f(u_iv_1) = f(u_iv_2) = f(u_iv_3) = 2$. In all cases, we have $\partial f(v_1) = \partial f(v_2) = \partial f(v_3)$.

Therefore, as $b = \partial f$, we must have $b(v_1) = b(v_2)$, which is contrary to the fact that $b(v_1) \neq b(v_2)$. This contradiction establishes Equation (8). □

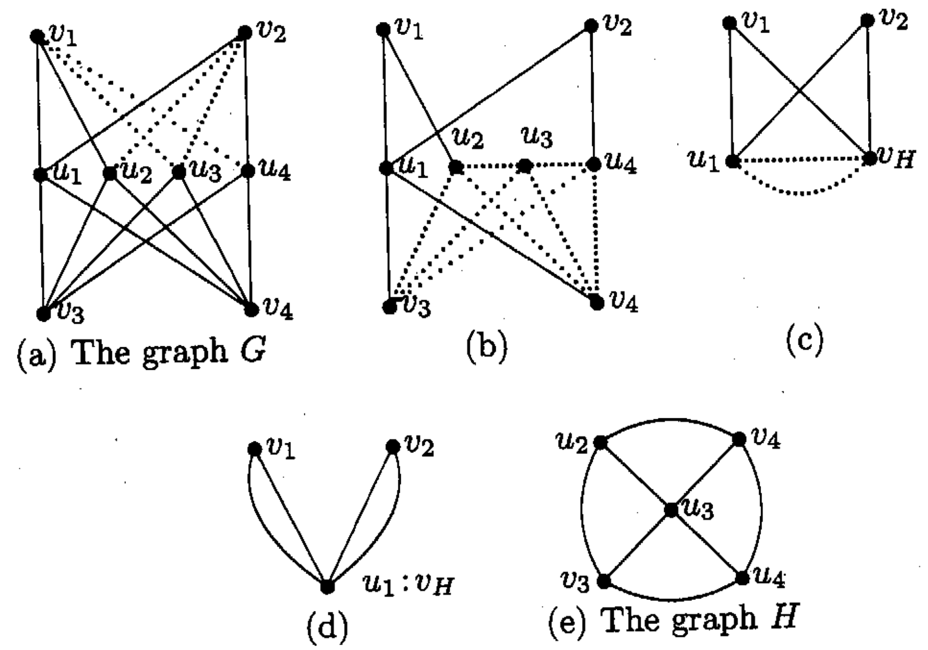

Figure 4.4: Reduction of $K_{4,4}$

Lemma 4.10 $\Lambda_g(K_{m,n}) = 3$ for any integers $m \geq n \geq 4$.

Proof. Suppose that $m \geq n \geq 4$. By Lemma 2.3, it suffices to prove that for any Abelian group A with $|A| \geq 3$, $K_{m,n} \in \langle A \rangle$. Since every edge of $K_{m,n}$ lies in a subgraph isomorphic to $K_{3,3}$, it follows by Lemmas 2.6 and 4.8 that $K_{m,n} \in \langle A \rangle$ whenever $|A| \geq 4$. Thus, it suffices to show that $K_{m,n} \in \langle \mathbf{Z}_3 \rangle$.

We first show that $K_{4,4} \in \langle \mathbf{Z}_3 \rangle$. The process is depicted in Figure 4.4. Using the notation in Figure 4.4, we split v_1 with respect to the edges v_1u_3 and v_1u_4, and split v_2 with respect to the edges v_2u_2 and v_2u_3. The resulting graph, depicted in Figure 4.4(b), contains the subgraph H induced by the vertices $\{u_2, u_3, u_4, v_3, v_4\}$, which is isomorphic to $W_4 \in \langle \mathbf{Z}_3 \rangle$. The graph H is illustrated in Figure 4.4(e).

We contract H to obtain the graph depicted in Figure 4.4(c). By Theorem 3.1, and by Lemma 2.7 and Proposition 2.1(C3), if the graph in Figure 4.4(c) is $\mathbf{Z}_3$-connected, so is $K_{4,4}$. Note that the graph in Figure 4.4(c) contains a 2-cycle. Contract the 2-cycle to obtain the graph depicted in Figure 4.4(d), which can then be seen to be in $\langle \mathbf{Z}_3 \rangle$ by Lemmas 2.2 and 2.6. By Lemma 2.2 and Proposition 2.1(C3), the graph in Figure 4.4(c) is also in $\langle \mathbf{Z}_3 \rangle$, and so $K_{4,4} \in \langle \mathbf{Z}_3 \rangle$, as desired. And hence, by Lemma 2.3, $\Lambda_g(K_{4,4}) = 3$.

It follows by Corollary 4.5 that we have an upper bound for $K_{m,n}$ when $m > n \geq 4$,

$$\Lambda_g(K_{m,n}) \leq \max(\Lambda_g(K_{4,4}), 3) = 3. \tag{10}$$

Therefore, the lemma follows by Lemma 2.3 and Inequality (10). □

For a nontrivial graph G, the *line graph of* G, denoted by $L(G)$, has vertex set $E(G)$, where two vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. Tutte conjectured [5] that every 4-edge-connected graph has an A-NZF, for any Abelian group A with $|A| \geq 3$. In [3], it is shown that to prove this conjecture of Tutte, it suffices to prove the same conjecture restricted to line graphs. As an application, we have the following corollary.

Corollary 4.11 Each of the following hold:
(1) If $G = L(H)$ is the line graph of a connected graph H with minimum degree $\delta(H) \geq 5$, then $\Lambda_g(G) = 3$.
(2) In particular, the line graph of a 5-edge-connected graph is A-connected for any Abelian group A with $|A| \geq 3$.

Proof. Statement (2) follows from (1), so it suffices to prove (1). If H is a connected graph with $\delta(H) \geq 5$, then by the definition of a line graph, every edge of G lies in a subgraph isomorphic to K_5. Thus, by Lemma 2.3, Lemma 2.6, and Proposition 4.2, we have $\Lambda_g(G) = 3$. □

5. Chordal graphs

A graph G is *chordal* if every cycle in G of length greater than 3 possesses a chord. That is, any induced cycle of G has length at most 3. In this section we characterize the 3-connected chordal graphs with $\Lambda_g(G) = 3$. We also characterize the 2-connected and 1-connected chordal graphs with $\Lambda_g(G) = 4$.

If G is a 2-edge-connected chordal graph, then every edge of G lies in a 2-cycle or 3-cycle of G, and so by Lemmas 2.2 and 2.6,

$$\Lambda_g(G) \leq 4. \tag{11}$$

Let G be a graph with $u'v' \in E(G)$ and H be a graph with $uv \in E(H)$. We use $G \oplus H$ to denote a new graph obtained from the disjoint union of

$G - \{u'v'\}$ and H by identifying u' and u and identifying v' and v. This operation is referred to as *attaching G on H over the edge uv.*

Lemma 5.1 (Lai [9]) Let A be an Abelian group of order at least 3. If G is a 4-edge-connected chordal graph, then $G \in \langle A \rangle$.

Theorem 5.2 (Lai [9]) Let G be a 3-edge-connected chordal graph. Then one of the following holds:
(1) G is A-connected, for any Abelian group A with $|A| \geq 3$.
(2) G has a block isomorphic to a K_4.
(3) G has a subgraph G_1 such that $G_1 \notin \langle \mathbf{Z}_3 \rangle$ and $G = G_1 \oplus K_4$.

Lemma 5.3 (DeVos, et al. [4]) Let G_1, G_2 be graphs and let $H = G_1 \oplus G_2$. If neither G_1 nor G_2 is $\mathbf{Z}_3$-connected, then H is not $\mathbf{Z}_3$-connected.

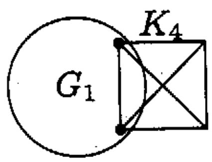

Figure 5.1: $G_1 \oplus K_4$

Theorem 5.4 Let G be a 3-connected chordal graph. Then $\Lambda_g(G) = 3$ if and only if $G \not\cong K_4$.

Proof. By Proposition 4.2 we know that $\Lambda_g(K_4) = 4$. Thus, we assume $G \not\cong K_4$ and show that $\Lambda_g(G) = 3$. Since $3 \leq \kappa(G) \leq \kappa'(G)$, by Lemma 5.1 we need only consider the case when $\kappa(G) = \kappa'(G) = 3$.

If Theorem 5.2(1) holds, we are done. If Theorem 5.2(2) holds, then G has a block isomorphic to K_4 and so G has a cut vertex, contrary to the assumption that $\kappa(G) = 3$. If Theorem 5.2(3) holds, then G has a subgraph G_1 such that $G_1 \notin \langle \mathbf{Z}_3 \rangle$ and $G = G_1 \oplus K_4$ (see Figure 5.1). Thus, G has a vertex cut of size 2, contrary to the assumption that $\kappa(G) = 3$. These contradictions establish the theorem. □

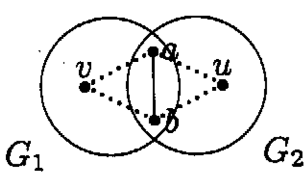

Figure 5.2: A 2-connected chordal graph

Lemma 5.5 Let G be a 2-connected chordal graph and let $V' = \{a, b\}$ be a vertex cut of G. Then $ab \in E(G)$.

Proof. See Figure 5.2. Let G_1 and G_2 be two connected subgraphs of G such that $V(G_1) \cap V(G_2) = V'$, $\min\{|V(G_1)|, |V(G_2)|\} \geq 3$ and $G = G_1 \cup G_2$. Since G is 2-connected, G has a cycle C with $a, b \in V(C)$. As G is chordal, a and b must be adjacent in G. □

A graph G is *triangularly-connected* if it is connected and for every pair $e, f \in E(G)$, there exists a sequence of cycles $C_1, C_2, \dots, C_k$ such that $e \in E(C_1)$, $f \in E(C_k)$, $|E(C_i)| \leq 3$ for $1 \leq i \leq k$, and $E(C_j) \cap E(C_{j+1}) \neq \emptyset$ for $1 \leq j \leq k-1$. We give a sufficient condition for a triangularly-connected graph to be $\mathbf{Z}_3$-connected.

Lemma 5.6 Let G be a triangularly-connected graph. If H is a nontrivial subgraph of G and $H \in \langle \mathbf{Z}_3 \rangle$, then $G \in \langle \mathbf{Z}_3 \rangle$.

Proof. If H is spanning, then the lemma follows trivially from Lemma 2.6. Thus, we assume that H is not a spanning subgraph of G. Since G is triangularly-connected, G/H must contain a 2-cycle. Again, as G is a triangularly-connected graph, we can contract 2-cycles until we obtain a connected graph in which every edge lies in a 2-cycle. Thus, by Lemmas 2.2 and 2.6, this last graph is in $\langle \mathbf{Z}_3 \rangle$, and so by Proposition 2.1(C3), $G \in \langle \mathbf{Z}_3 \rangle$. □

Theorem 5.7 Let G be a 2-connected chordal graph. Then $\Lambda_g(G) = 4$ if and only if $G \in \{K_3, K_4\}$ or G has two subgraphs G_1 and G_2 such that $\Lambda_g(G_1) = \Lambda_g(G_2) = 4$ and $G = G_1 \oplus G_2$.

Proof. By Proposition 4.2, $\Lambda_g(K_3) = \Lambda_g(K_4) = 4$. Now suppose that G has two subgraphs G_1 and G_2 such that $\Lambda_g(G_1) = \Lambda_g(G_2) = 4$ and $G = G_1 \oplus G_2$. Then, by Lemma 5.3 and Inequality (11), $\Lambda_g(G) = 4$.

Conversely, we assume that $\Lambda_g(G) = 4$, but $G \notin \{K_3, K_4\}$. If $\kappa(G) \geq 3$, then by Theorem 5.4, $\Lambda_g(G) = 3$. Hence, G must have a vertex cut $V' = \{a, b\}$. By Lemma 5.5, $ab \in E(G)$.

Therefore, G has two 2-connected chordal subgraphs G_1 and G_2 such that $\min\{|V(G_1)|, |V(G_2)|\} \geq 3$, $V(G_1) \cap V(G_2) = V'$, and $G = G_1 \oplus G_2$. If both $\Lambda_g(G_1) = \Lambda_g(G_2) = 4$, then we are done. Therefore, suppose that $\Lambda_g(G_1) \leq 3$.

Since G is a chordal graph with $\kappa(G) = 2$, any pair of edges is contained in a cycle. Thus, G is a triangularly-connected chordal graph. Since $G_1 \in$

$\langle \mathbf{Z}_3 \rangle$, by Lemma 5.6, $G \in \langle \mathbf{Z}_3 \rangle$. But, $G \in \langle \mathbf{Z}_3 \rangle$ and $\Lambda_g(G) = 4$ is a contradiction. □

Theorem 5.8 Let G be a 2-edge-connected chordal graph that is not 2-connected. Then $\Lambda_g(G) = 4$ if and only if there are subgraphs G_1 and G_2 of G such that $G = G_1 \cup G_2$, $|V(G_1) \cap V(G_2)| = 1$, and $\Lambda_g(G_1) = 4$ or $\Lambda_g(G_2) = 4$.

Proof. First note, by the assumption of Theorem 5.8, G has a cut vertex v. Therefore, G has two 2-edge-connected chordal subgraphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \{v\}$, $\min\{|V(G_1)|, |V(G_2)|\} \geq 3$, and $G = G_1 \cup G_2$.

Let G_1 and G_2 be subgraphs of G such that $G = G_1 \cup G_2$ and $|V(G_1) \cap V(G_2)| = 1$. Note that $G_1 \cong G/G_2$ and $G_2 \cong G/G_1$. Moreover, since G is 2-edge-connected and chordal, G_1 and G_2 are also. Therefore, by Inequality (11), $\Lambda_g(G_i) \leq 4$, for $i \in \{1, 2\}$.

("only if" part) The negation of the conclusion in this case requires that $\Lambda_g(G_1) \leq 3$ and $\Lambda_g(G_2) \leq 3$. Hence, $G_1, G_2 \in \langle \mathbf{Z}_3 \rangle$, and $G/G_2 \cong G_1 \in \langle \mathbf{Z}_3 \rangle$. It follows by Proposition 2.1(C3) and Inequality (11) that $\Lambda_g(G) \leq 3$.

("if" part) If $\Lambda_g(G) \leq 3$ (i.e., $G \in \langle \mathbf{Z}_3 \rangle$), then by Proposition 2.1(C2) $G_1 \cong G/G_2$, $G_2 \cong G/G_1 \in \langle \mathbf{Z}_3 \rangle$. Then, by Inequality (11), $\Lambda_g(G_1) \leq 3$ and $\Lambda_g(G_2) \leq 3$. □

6. Biwheels

In this section we investigate the group connectivity number for biwheels. The *biwheel*, B_n, is the graph obtained by joining a cycle on $n \geq 2$ vertices and K_2 (see Figure 6.1). Shahmohamad [12, 13] gave the following results on minimum flow number of biwheels.

Lemma 6.1 ([12, 13]) Let n be a positive integer.
(1) $\lambda_g(B_{2n+1}) = 2$, for $n \geq 1$.
(2) $\lambda_g(B_{2n}) = 3$, for $n \geq 2$.

We generalize these results to the group connectivity number of biwheels as follows.

Theorem 6.2 $\Lambda_g(B_n) = 3$, for $n \geq 2$.

Proof. Since every edge of B_n lies in a C_3, by Lemma 2.2 and Lemma 2.6

$B_n \in \langle A \rangle$ for any Abelian group A with $|A| \geq 4$. By Lemma 2.3, $\Lambda_g(B_n) \neq 2$. Hence, it suffices to show that $B_n \in \langle \mathbf{Z}_3 \rangle$. We consider two cases.

Case 1: Suppose n is even. By Lemma 2.7 we know that $W_n \in \langle \mathbf{Z}_3 \rangle$. We view W_n as a subgraph of B_n. The subgraph contraction B_n/W_n yields two vertices joined multiple edges, which belongs to $\mathbf{Z}_3$ by Lemma 4.3. Therefore, $B_n \in \langle \mathbf{Z}_3 \rangle$ by Proposition 2.1(C3), and $\Lambda_g(B_n) = 3$.

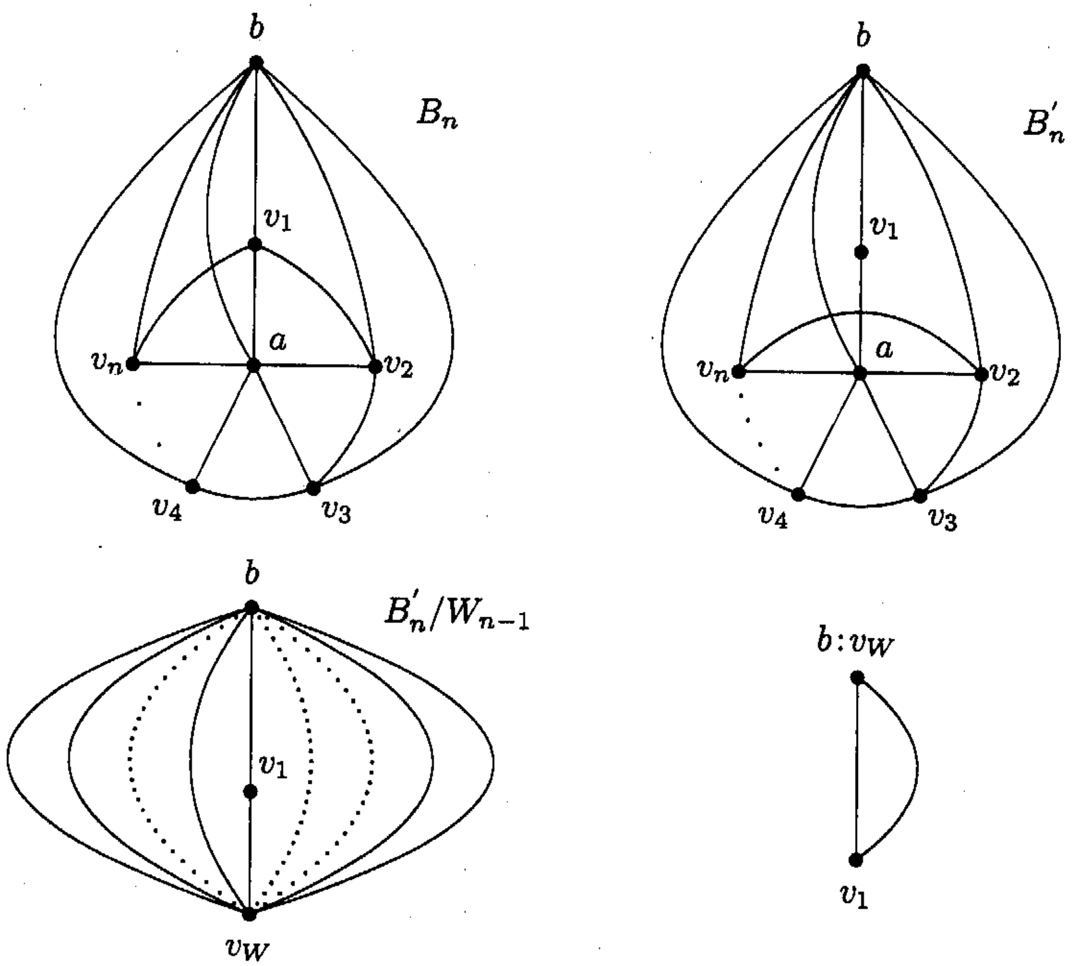

Figure 6.1: Biwheel B_n when n is odd

Case 2: Suppose n is odd. Let B'_n be a graph obtained from B_n by splitting a vertex v_1 on the n-cycle with respect to the two edges on the n-cycle incident with it (see Figure 6.1). By Theorem 3.1, if $B'_n \in \langle \mathbf{Z}_3 \rangle$, then $B_n \in \langle \mathbf{Z}_3 \rangle$.

We now show $B'_n \in \langle \mathbf{Z}_3 \rangle$. Observe that B'_n has an induced subgraph isomorphic to W_{n-1} with center a; we view W_{n-1} as a subgraph of B'_n. By Lemma 2.8, $\Lambda_g(W_{n-1}) = 3$. By Proposition 2.1(C3), we only need to show that $B'_n/W_{n-1} \in \langle \mathbf{Z}_3 \rangle$. We use v_W to label the vertex resulting from contracting W_{n-1}. Since the graph H induced by $\{v_W, b\}$ in B'_n/W_{n-1} has $m \geq 4$ edges joining v_W and b, by Lemma 4.3, $H \in \langle \mathbf{Z}_3 \rangle$. Contracting H produces C_2, and by Lemma 2.2 $C_2 \in \langle \mathbf{Z}_3 \rangle$. It follows by Proposition 2.1(C3) that $B'_n/W_{n-1} \in \langle \mathbf{Z}_3 \rangle$. □

A biwheel is sometimes alternately defined as the join of a cycle on $n \geq 2$ vertices and $K_1 + K_1$, where + is the disjoint union. We note that Theorem 6.2 holds for biwheels thus defined.

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