New Sufficient Conditions for s-Hamiltonian Graphs and s-Hamiltonian Connected Graphs

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Abstract: A graph G is s-Hamiltonian if for any $S \subseteq V(G)$ of order at most s, $G - S$ has a Hamiltonian-cycle, and s-Hamiltonian connected if for any $S \subseteq V(G)$ of order at most s, $G - S$ is Hamiltonian-connected. Let $k > 0, s \geq 0$ be two integers. The following are proved in this paper: (1) Let $k \geq s + 2$ and $s \leq n - 3$. If G is a k-connected graph of order n and if $\max\{d(v) : v \in I\} \geq (n+s)/2$ for every independent set I of order $k-s$ such that I has two distinct vertices x, y with $1 \leq |N(x) \cap N(y)| \leq \alpha(G) + s - 1$, then G is s-Hamiltonian. (2) Let $k \geq s + 3$ and $s \leq n - 2$. If G is a k-connected graph of order n and if $\max\{d(v) : v \in I\} \geq (n + s + 1)/2$ for every independent set I of order $k - s - 1$ such that I has two distinct vertices x, y with $1 \leq |N(x) \cap N(y)| \leq \alpha(G) + s$, then G is s-Hamiltonian connected. These extended several former results by Dirac, Ore, Fan and Chen.

Key words: Hamiltonian graph, Hamiltonian-connected graph, s-Hamiltonian graph, s-Hamiltonian connected graph.

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1 Introduction

Graphs considered here are simple and connected. Undefined notations and terminologies here can be found in [1]. For a graph G, we use $V(G)$, $E(G)$, $\delta(G)$ and $\alpha(G)$ to denote its vertex set, edge set, minimal degree and independence number, respectively. If $v \in V(G)$ and H is a subgraph of G, then $N_H(v)$ denotes the set of vertices in H that are adjacent to v in G. Thus, $d_H(v)$, the degree of v relative to H, is $|N_H(v)|$. We also write $d(v)$ for $d_G(v)$ and $N(v)$ for $N_G(v)$. If C and H are subgraphs of G, then $N_C(H) = \cup_{u \in V(H)} N_C(u)$, and $G - C$ denotes the subgraph of G induced by $V(G) - V(C)$. Let $P = x_1 x_2 \cdots x_m$ denote a path of order m. To emphasize the end vertices of the path P, we also say that P is an (x_1, x_m)-path. Define $N_P^+(u) = \{x_{i+1} \in V(P) : x_i \in N_P(u)\}$. So if $x_m \in N_P(u)$, then $|N_P^+(u)| = |N_P(u)| - 1$. Two vertices are consecutive in P if they are the ends of an edge in $E(P)$. Thus, each pair of vertices x_i, x_{i+1} are consecutive in P for any $i \in \{1, \cdots, m-1\}$. When $1 \leq i < j \leq m$, we use $[x_i, x_j]$ to denote the section $x_i x_{i+1} \cdots x_j$ of P and $[x_j, x_i]$ to denote the section $x_j x_{j-1} \cdots x_i$ of P. If there is an (x_1, x_m)-path P^* in G such that $V(P) \subset V(P^*)$ and $|V(P^*)| > |V(P)|$, then we say that P^* extends P. Let $C = x_1 \cdots x_m x_1$ be a cycle. Define $N_C^+(H) = \{x_{i+1} \in V(C) : x_i \in N_C(u)\}$, where the subscriptions are taken by modulo m. Two vertices are consecutive in C if they are the ends of an edge in $E(C)$. If there is a cycle C^* in G such that $V(C) \subset V(C^*)$ and $|V(C^*)| > |V(C)|$, then we say that C^* extends C.

A graph G is *Hamiltonian* if it has a spanning cycle, and *Hamiltonian-connected* if for every pair of distinct vertices $u, v \in V(G)$, G has a spanning (u, v)-path. A graph G is *s-Hamiltonian* if for any $S \subseteq V(G)$ of order at most s, $G - S$ has a Hamiltonian-cycle, and *s-Hamiltonian connected* if for any $S \subseteq V(G)$ of order at most s, $G - S$ is Hamiltonian-connected.

The following sufficient conditions to ensure the existence of a Hamiltonian cycle in a simple graph G of order $n \geq 3$ are well known.

Theorem 1.1 *(Dirac [4]) If $\delta(G) \geq n/2$, then G is Hamiltonian.*

Theorem 1.2 *(Ore [8]) If* $d(u) + d(v) \geq n$ *for each pair of nonadjacent vertices* $u, v \in V(G)$, *then* G *is Hamiltonian.*

Theorem 1.3 *(Fan [6]) If* G *is a 2-connected graph and if* $\max\{d(u), d(v)\} \geq n/2$ *for each pair of vertices* $u, v \in V(G)$ *with* $d(u, v) = 2$, *then* G *is Hamiltonian.*

Theorem 1.4 *(Chen [2]) If* G *is a 2-connected graph and if* $\max\{d(u), d(v)\} \geq n/2$ *for each pair of vertices* $u, v \in V(G)$ *with* $1 \leq |N(u) \cap N(v)| \leq \alpha(G) - 1$, *then* G *is Hamiltonian.*

Theorem 1.5 *(Chen et al [3]) If* G *is a* k*-connected* $(k \geq 2)$ *graph and if* $\max\{d(v) : v \in I\} \geq n/2$ *for every independent set* I *of order* k *such that* I *has two distinct vertices* x, y *with* $d(x, y) = 2$, *then* G *is Hamiltonian.*

Zhao et al recently proved Theorem 1.6 below, which unified and extended the above theorems.

Theorem 1.6 *(Zhao et al [9]) If* G *is a* k*-connected* $(k \geq 2)$ *graph of order* n *and if* $\max\{d(v) : v \in I\} \geq n/2$ *for every independent set* I *of order* k *such that* I *has two distinct vertices* x, y *with* $1 \leq |N(x) \cap N(y)| \leq \alpha(G) - 1$, *then* G *is Hamiltonian.*

In this paper, we shall obtain sufficient conditions for s-Hamiltonian graphs and s-Hamiltonian connected graphs, respectively, as shown below.

Theorem 1.7 *Let* k, s *be two integers with* $k \geq s + 2$ *and* $0 \leq s \leq n - 3$. *If* G *is a* k*-connected graph of order* n *and if* $\max\{d(v) : v \in I\} \geq (n+s)/2$ *for every independent set* I *of order* $k - s$ *such that* I *has two distinct vertices* x, y *with* $1 \leq |N(x) \cap N(y)| \leq \alpha(G) + s - 1$, *then* G *is* s*-Hamiltonian.*

Theorem 1.8 *Let* k, s *be two integers with* $k \geq s + 3$ *and* $0 \leq s \leq n - 2$. *If* G *is a* k*-connected graph of order* n *and if* $\max\{d(v) : v \in I\} \geq (n+s+1)/2$ *for every independent set* I *of order* $k - s - 1$ *such that* I *has two distinct* *vertices x, y with $1 \leq |N(x) \cap N(y)| \leq \alpha(G) + s$, then G is s-Hamiltonian connected.*

Note that Theorem 1.6 is a special case of Theorem 1.7 when $s = 0$. Applying Theorem 1.8 to the case when $s = 0$, we get the following corollary.

Corollary 1.9 *If G is a k-connected ($k \geq 3$) graph of order n and if $max\{d(v) : v \in I\} \geq (n+1)/2$ for every independent set I of order $k-1$ such that I has two distinct vertices x, y with $1 \leq |N(x) \cap N(y)| \leq \alpha(G)$, then G is Hamiltonian-connected.*

The following Lemma 1.10 is very important for the proof of the main theorems. A proof can also be found in [10].

Lemma 1.10 *Let G be a connected graph, $F = x_1 \cdots x_m(x_1)$ be a longest path (or cycle) in G and H be a component of $G - V(F)$. If $x_i, x_j \in N_F(H)$ with $1 \leq i < j < m$, then*
(i) $x_{i+1}x_{j+1} \notin E(G)$;
(ii) $N(x_{i+1}) \cap V(H) = \emptyset$;
(iii) $N_F^+(H) \cup \{x\}$ is an independent set of G, where $x \in V(H)$.

Theorem 1.7 and Theorem 1.8 will be proved in the following two sections, respectively.

2 Proof of Theorem 1.7

Throughout this section, let k, s denote two integers with $k \geq s+2$ and $0 \leq s \leq n-3$.

Lemma 2.1 *[5] Let G be a graph and $P = x_1 \cdots x_n$ be a Hamiltonian-path of G. If $d(x_1) + d(x_n) \geq n$, then G contains a Hamiltonian-cycle.*

Lemma 2.2 *Let G be a k-connected graph of order n, $S \subseteq V(G)$ be a vertex set of order s, $C = x_1 \cdots x_m x_1$ be a cycle of $G - S$ with $|V(C)| < n - s$ and H be a component of $G - S - V(C)$. Then $G - S$ contains a cycle C^* extending C, if one of the following holds:*
(i) there exist two distinct vertices $x_i, x_j \in V(C)$ with $x_{i+1}, x_{j+1} \in N_C^+(H)$ such that $d(x_{i+1}) \geq (n+s)/2$ and $d(x_{j+1}) \geq (n+s)/2$, or
(ii) there exists a vertex $x_{i+1} \in N_C^+(H)$ and a vertex $y \in V(H)$ such that $d(x_{i+1}) \geq (n+s)/2$ and $d(y) \geq (n+s)/2$.

Proof: Since the proof when (ii) holds is similar to the proof when (i) holds, we only present the proof of the lemma assuming (i) holds. Let $x_i', x_j' \in V(H)$ (possibly $x_i' = x_j'$) be such that $x_i'x_i, x_j'x_j \in E(G)$ and let P be an (x_j', x_i')-path in H. Then $G[V(C \cup P)]$ has a Hamiltonian-path $P^* = [x_{i+1}, x_j]P\ [x_i, x_1][x_m, x_{j+1}]$. Let $H' = G - V(S \cup C \cup H)$. If $N_{H'}(x_{i+1}) \cap N_{H'}(x_{j+1}) \neq \emptyset$, let $z \in N_{H'}(x_{i+1}) \cap N_{H'}(x_{j+1})$ and then $G - S$ has a cycle $C^* = z[x_{i+1}, x_j]P\ [x_i, x_1][x_m, x_{j+1}]z$ extending C. Now suppose that $N_{H'}(x_{i+1}) \cap N_{H'}(x_{j+1}) = \emptyset$ and so $d_{H'}(x_{i+1}) + d_{H'}(x_{j+1}) \leq |V(H')|$. If $N_{H-P}(x_{i+1}) \cup N_{H-P}(x_{j+1}) \neq \emptyset$, without loss of generality, let $y \in N_{H-P}(x_{i+1}) \cup N_{H-P}(x_{j+1})$ and $yx_{i+1} \in E(G)$ and let P'' be an (x_i', y)-path in H. So $G - S$ has a cycle $C^* = x_iP''[x_{i+1}, x_m][x_1, x_i]$ extending C. Now we can suppose that $N_{H-P}(x_{i+1}) \cup N_{H-P}(x_{j+1}) = \emptyset$ and so $d_{H-P}(x_{i+1}) + d_{H-P}(x_{j+1}) = 0$. By (i) of Lemma 2.2, both $d(x_{i+1}) \geq (n+s)/2$ and $d(x_{j+1}) \geq (n+s)/2$. Thus,

$$\begin{aligned} d_{P^*}(x_{i+1}) + d_{P^*}(x_{j+1}) &= d(x_{i+1}) + d(x_{j+1}) \\ &\quad -(d_{S \cup H' \cup (H-P)}(x_{i+1}) + d_{S \cup H' \cup (H-P)}(x_{j+1})) \\ &\geq n + s - 2s - |V(H')| \geq |V(P^*)|. \end{aligned}$$

By Lemma 2.1, $G[V(C \cup P)]$ contains a Hamiltonian-cycle C^* extending C. □

Lemma 2.3 *Suppose that G satisfies the hypothesis of Theorem 1.7. Let $S \subseteq V(G)$ be a vertex set with $|S| = s' \leq s$, $C = x_1 \cdots x_m x_1$ be a longest cycle of $G - S$ with $|V(C)| < n - s'$ and H be a component of $G - S - V(C)$. Then*

(i) $|N_C(H)| \geq k - s$;
(ii) if $x \in V(H), x_i \in V(C)$ *are such that* $xx_i \in E(G)$, *then* $1 \leq |N(x) \cap N(x_{i+1})| \leq \alpha(G) + s - 1$;
(iii) $d(x) \geq (n+s)/2$ *for each* $x \in V(H)$ *with* $|N_C(x)| \geq 1$.

Proof: (i) Since $C = x_1 \cdots x_m x_1$ is a longest cycle of $G - S$ with $|V(C)| < n - s'$, it follows that $H \neq \emptyset$ and $V(C) - N_C(H) \neq \emptyset$. By the facts that $N_C(H) \cup S$ separates H and $G - H - (S \cup N_C(H))$ and that G is k-connected, we have $|N_C(H)| + |S| \geq k$ and so $|N_C(H)| \geq k - s' \geq k - s$.

(ii) By Lemma 1.10 (iii), $N_C^+(H) \cup \{x\}$ is an independent set and so $|N_C(H)| = |N_C^+(H)| \leq \alpha(G) - 1$. It follows that $1 \leq |N(x) \cap N(x_{i+1})| \leq |N_C(H) \cup S| \leq \alpha(G) + s' - 1 \leq \alpha(G) + s - 1$.

(iii) Suppose, to the contrary, that there exists an $x \in V(H)$ with $|N_C(x)| \geq 1$ and with $d(x) < (n+s)/2$. Let $x_i \in N_C(x)$. By Lemma 1.10 (iii) and by the fact that $|N_C^+(H)| = |N_C(H)| \geq k - s$, G has an independent set $J = J' \cup \{x\}$ of order $k - s$ with $x_{i+1} \in J' \subseteq N_C^+(H)$. By (ii), $1 \leq |N(x) \cap N(x_{i+1})| \leq \alpha(G) + s - 1$. Hence by the hypothesis of Theorem 1.7 and by the fact that $d(x) < (n+s)/2$, there must exist an $x_{l+1} \in J'$ satisfying $d(x_{l+1}) \geq (n+s)/2$. By (i), $|N_C^+(H)| = |N_C(H)| \geq k - s \geq 2$, and so there exists an $x_{j+1} \in N_C^+(H) - \{x_{l+1}\}$. Since $x_{j+1} \in N_C^+(H)$, $x_j \in N_C(H)$ and we may assume $y \in V(H)$ with $yx_j \in E(G)$ (possible $y = x$). By (ii), we have $1 \leq |N(y) \cap N(x_{j+1})| \leq \alpha(G) + s - 1$. Similarly, G has an independent set $J_1 = J_1' \cup \{y\}$ of order $k - s$, where $x_{j+1} \in J_1' \subseteq N_C^+(H) - \{x_{l+1}\}$. By the hypothesis of Theorem 1.7, there exists a $z \in J_1$ such that $d(z) \geq (n+s)/2$. Consequently, either $z \in N_C^+(H)$, whence by Lemma 2.2 (i), $G - S$ has a cycle C^* extending C; or $z = y$, whence by Lemma 2.2 (ii), $G - S$ has a cycle C^* extending C. In either case, a contradiction to the assumption that C is a longest cycle of $G - S$ is obtained. □

Proof of Theorem 1.7 Let G be a graph satisfying the hypothesis of Theorem 1.7. Suppose, to the contrary, that G is not s-Hamiltonian. Then there exists a vertex set $S \subseteq V(G)$ with $|S| = s' \leq s$ such that $G - S$ does not have a Hamiltonian-cycle. By the fact that $k - s' \geq k - s \geq 2$, $G - S$

is 2-connected. We may assume that

$$C = x_1 \cdots x_m x_1 \text{ is a longest cycle in } G - S. \tag{1}$$

Then $|V(C)| < n - s'$. Let H be a component of $G - S - V(C)$. By Lemma 2.3 (i), we have $|N_C(H)| \geq k - s \geq 2$. Choose $x_i, x_j \in N_C(H)$ to be such that

$$X \cap N_C(H) = \emptyset, \text{ and } |X| \text{ is minimum,} \tag{2}$$

where $X = \{x_{i+1}, \cdots, x_{j-1}\}$. Then $|X| > 0$. Otherwise, there exist $y_i, y_{i+1} \in V(H)$ such that $x_i y_i \in E(G), x_{i+1}y_{i+1} \in E(G)$ (y_i and y_{i+1} might be the same vertex). Let $P_H[y_i, y_{i+1}]$ be a (y_i, y_{i+1})-path in H. Then $C^* = [x_1, x_i]P_H[y_i, y_i+1][x_{i+1}, x_m]x_1$ is a cycle extending C, contrary to (1). By Lemma 2.3 (iii), for each vertex $x \in V(H)$ with $|N_C(x)| \geq 1$, $d(x) \geq (n+s)/2$. Since $N(x) \cup \{x\} \subseteq V(H) \cup N_C(H) \cup S$ for each $x \in V(H)$, $|V(H)| + |N_C(H)| + |S| \geq (n+s)/2 + 1$, and then

$$|V(H)| + |N_C(H)| \geq \frac{n - s'}{2} + 1. \tag{3}$$

Claim 1. $G - S - V(C)$ has only one component $H = G - S - V(C)$ and $|X| < |V(H)|$.

Proof. Suppose, to the contrary, that $G - S - V(C)$ has at least two components. Assume that H is the component with the smallest order and let $H^* = G - S - V(C \cup H)$. Since $|V(H)|$ is minimized, $|V(H)| \leq |V(H^*)|$. It follows by (3) and $|N_C(H)| \geq 2$ that

$$\begin{aligned} |X| &\leq \frac{|V(C)| - |N_C(H)|}{|N_C(H)|} = \frac{n - |V(H^*)| - s' - (|V(H)| + |N_C(H)|)}{|N_C(H)|} \\ &\leq \frac{(n - s')/2 - 1 - |V(H^*)|}{|N_C(H)|} \leq \frac{|V(H)| + |N_C(H)| - 2 - |V(H^*)|}{|N_C(H)|} \\ &= \frac{|V(H)| - |V(H^*)|}{|N_C(H)|} + \frac{|N_C(H)| - 2}{|N_C(H)|}. \end{aligned} \tag{4}$$

Then as $|V(H)| \leq |V(H^*)|$, (4) implies $|X| < 1$, contrary to the fact that $|X| > 0$. Hence, H is the only component of $G - S - V(C)$. Since $|N_C(H)| \geq 2$, we have that $|X| < |V(H)|$. □

Choose $x_i', x_j' \in V(H)$ with $x_i x_i' \in E(G), x_j x_j' \in E(G)$ to be such that $|V(P')|$ is as large as possible, where P' is an (x_i', x_j')-path in H. Then

$C' = [x_1, x_i]P'[x_j, x_m]x_1$ is a cycle such that

$$V(C) \setminus X \subseteq V(C') \text{ and } |V(C')| \text{ is maximized.} \tag{5}$$

By (5), C' is a longest path containing $V(C)\setminus X$ and so by applying Lemma 2.3 and the argument on C to C', it follows that $G-S-V(C')$ has only one component H' and that $H' = G[X \cup V(H-P')]$. By (2) and the fact that $|X| > 0$, $H - P' = \emptyset$. Otherwise, H' is connected while $G[X \cup (H - P')]$ is disconnected, a contradiction. Therefore P' is a path of order $|V(H)|$. By the fact that $|X| < |V(H)|$, we have $|V(C')| = |V(C)| - |X| + |V(H)| > |V(C)|$, contrary to (1). This completes the proof of Theorem 1.7. □

3 Proof of Theorem 1.8

Lemma 3.1 *Let G be a graph and $P = x_1 \cdots x_n$ be a Hamiltonian-path of G. If $d(x_1) + d(x_n) \geq n + 1$, then for any edge $e = x_i x_{i+1} \in E(P)$, G has a Hamiltonian-cycle C such that $e \in E(C)$.*

Proof: Let $T = \{x_j | \; x_1 x_{j+1} \in E, x_{j+1} \in V(P)\}$. Then

$$|T \cap N(x_n)| = |T| + |N(x_n)| - |T \cup N(x_n)| \geq n + 1 - (n-1) = 2.$$

That means there exists $x_j \in T \cap N(x_n) - \{x_i\}$, and so G has a Hamiltonian-cycle $C = [x_1, x_j][x_n, x_{j+1}]x_1$. Clearly, $E(P) - \{x_j x_{j+1}\} \subseteq E(C)$, and so $e = x_i x_{i+1} \in E(C)$. Thus the lemma holds. □

Lemma 3.2 *Let G be a k-connected graph of order n, $S \subseteq V(G)$ be a vertex set with $|S| = s' \leq s$, $P = x_1 \cdots x_m$ be a path of $G - S$ with $|V(P)| < n - s$ and H be a component of $G - S - V(P)$. Then $G - S$ contains a path P^* extending P, if one of the following holds:*
(i) there exist two distinct vertices $x_i, x_j \in V(P)$ with x_{i+1}, x_{j+1} in $N_P^+(H)$ such that $d(x_{i+1}) \geq (n+s+1)/2$ and $d(x_{j+1}) \geq (n+s+1)/2$, or
(ii) there exists a vertex $x_{i+1} \in N_P^+(H)$ and a vertex $y \in V(H)$ such that $d(x_{i+1}) \geq (n+s+1)/2$ and $d(y) \geq (n+s+1)/2$.

Proof: Since the proof when (ii) holds is similar to the proof when (i) holds, we shall only present the proof of the Lemma 3.2 assuming (i) holds. Let $x_i', x_j' \in V(H)$ with $x_i'x_i, x_j'x_j \in E(G)$ and let P' be an (x_j', x_i')-path in H. Define G_1 to be the graph obtained from G by adding a new edge x_1x_m if $x_1x_m \notin E(G)$ and to be G if $x_1x_m \in E(G)$. Then we have an (x_{i+1}, x_{j+1})-path $P_1 = [x_{i+1}, x_j]P'[x_i, x_1][x_m, x_{j+1}]$ with $V(P_1) = V(P) \cup V(P')$ in G_1. Moreover, x_1x_m is an edge of P_1. Let $H^* = G - V(S \cup P \cup H)$. If $N_{H^*}(x_{i+1}) \cap N_{H^*}(x_{j+1}) \neq \emptyset$, let $z \in N_{H^*}(x_{i+1}) \cap N_{H^*}(x_{j+1})$ and then $G[V(P_1) \cup \{z\}]$ has a Hamiltonian-cycle C such that $x_1x_m \in E(C)$. Therefore, $C - \{x_1x_m\}$ is an (x_1, x_m)-path in $G - S$ which extends P. Now suppose that $N_{H^*}(x_{i+1}) \cap N_{H^*}(x_{j+1}) = \emptyset$ and so we have $d_{H^*}(x_{i+1}) + d_{H^*}(x_{j+1}) \leq |V(H^*)|$. If $N_{H-P'}(x_{i+1}) \cup N_{H-P'}(x_{j+1}) \neq \emptyset$, without loss of generality, let $y \in N_{H-P'}(x_{i+1}) \cup N_{H-P'}(x_{j+1})$ and $yx_{i+1} \in E(G)$ and let P'' be an (x_i', y)-path in H. So $G - S$ has a path $P^* = [x_1, x_i]P''[x_{i+1}, x_m]$ extending P. Now we can suppose that $N_{H-P'}(x_{i+1}) \cup N_{H-P'}(x_{j+1}) = \emptyset$ and so $d_{H-P'}(x_{i+1}) + d_{H-P'}(x_{j+1}) = 0$. Since $d(x_{i+1}) \geq (n+s+1)/2$ and $d(x_{j+1}) \geq (n+s+1)/2$, we have

$$\begin{aligned} d_{P_1}(x_{i+1}) + d_{P_1}(x_{j+1}) &= d(x_{i+1}) + d(x_{j+1}) \\ &\quad -(d_{S\cup H^*\cup(H-P')}(x_{i+1}) + d_{S\cup H^*\cup(H-P')}(x_{j+1})) \\ &\geq n+s+1-2s-|V(H^*)| \geq |V(P_1)|+1. \end{aligned}$$

By Lemma 3.1, $G_1[V(P_1)]$ contains a Hamiltonian-cycle C such that $x_1x_m \in E(C)$, and then $C - \{x_1x_m\}$ is an (x_1, x_m)-path P^* in $G - S$ extending P. □

By a proof similar to that for Lemma 2.3, we obtain the following lemma.

Lemma 3.3 *Suppose that G satisfies the hypothesis of Theorem 1.8. Let $S \subseteq V(G)$ be a vertex set with $|S| = s' \leq s$, $P = x_1 \cdots x_m$ be a longest path of $G - S$ with $|V(P)| < n - s'$ and H be a component of $G - S - V(P)$. Then*

(i) $|N_P(H)| \geq k - s$;

(ii) if $x \in V(H), x_i \in V(P)$ with $xx_i \in E$, then $1 \leq |N(x) \cap N(x_{i+1})| \leq \alpha(G) + s$;

(iii) $d(x) \geq (n+s+1)/2$ for each $x \in V(H)$ with $|N_P(x)| \geq 1$.

Proof of Theorem 1.8 Let G be a graph satisfying the hypothesis of Theorem 1.8. Suppose, to the contrary, that $G-S$ is not Hamiltonian-connected for some vertex set $S \subseteq V(G)$ with $|S| = s' \leq s$. Then there exists a pair of vertices, say x and y, such that $G-S$ does not have a Hamiltonian (x,y)-path. Since $k-s' \geq k-s \geq 3$, $G-S$ is 3-connected and we can choose

$$P = x_1x_2\cdots x_m \text{ to be a longest } (x,y)\text{-path in } G-S, \tag{6}$$

where $x = x_1, y = x_m$. Then $|V(P)| < n - s'$. Let H be a component of $G-S-V(P)$. By Lemma 3.3 (i), we have $|N_P(H)| \geq k-s \geq 3$. Choose $x_i, x_j \in N_P(H)$ to be such that

$$X \cap N_P(H) = \emptyset \text{ and } |X| \text{ is minimum}, \tag{7}$$

where $X = \{x_{i+1}, \cdots, x_{j-1}\}$. Then $|X| > 0$. Otherwise, there exist $y_i, y_{i+1} \in V(H)$ such that $x_iy_i \in E(G), x_{i+1}y_{i+1} \in E(G)$ (y_i and y_{i+1} might be the same vertex). Let $P_H[y_i, y_i+1]$ be a (y_i, y_{i+1})-path in H. Then $P^* = [x_1, x_i]P_H[y_i, y_{i+1}][x_{i+1}, x_m]$ is an (x_1, x_m)-path extending P, contrary to (6). By Lemma 3.3 (iii), for each vertex $x \in V(H)$ with $|N_C(x)| \geq 1$, $d(x) \geq (n+s+1)/2$. Since for each $x \in V(H)$, $N(x) \cup \{x\} \subseteq V(H) \cup N_P(H) \cup S$,

$$|V(H)| + |N_P(H)| \geq (n-s')/2 + 3/2. \tag{8}$$

By a proof similar to that for the Claim 1 in the proof of Theorem 1.7, we get the following.

Claim 2. $G-S-V(P)$ has only one component $H = G-S-V(P)$ and $|X| < |V(H)|$.

Choose $x_i', x_j' \in V(H)$ with $x_i', x_j' \in V(H)$ to be such that $|V(P')|$ is as large as possible, where P' is an (x_i', x_j')-path in H. Then $P^* = [x_1, x_i]P'[x_j, x_m]$ is a path such that

$$V(P) \setminus X \subseteq V(P^*) \text{ and } |V(P^*)| \text{ is maximized}. \tag{9}$$

By (9), P^* is a longest path containing $V(P)\setminus X$ and so by applying Lemma 3.3 and the argument on P to P^*, it follows that $G-S-V(P^*)$ has only one component H' and that $H' = G[X \cup V(H - P')]$. By (7) and the fact that $|X| > 0$, $H - P' = \emptyset$. Otherwise, H' is connected while $X \cup (H - P')$ is disconnected, a contradiction. Therefore, P' is a path of order $|V(H)|$. By the fact that $|X| < |V(H)|$, we have $|V(P^*)| = |V(P)| - |X| + |V(H)| > |V(P)|$, contrary to (6). This completes the proof of Theorem 1.8. □

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