Graph homomorphism into an odd cycle

Hong-Jian Lai
West Virginia University, Morgantown, WV 26506
EMAIL: hjlai@math.wvu.edu

Bolian Liu*
South China Normal University, Guangzhou, P. R. China
EMAIL: liubl@hsut.scun.edu.cn

Abstract

For integers $n > k \geq 4$, a simple graph with n vertices that contains no odd cycle of length at most $2k-1$ and that has minimum degree exceeding $\frac{2n-1}{2(k+1)}$ can have a homomorphism into the odd cycle of length $2k+1$. As a corollary, such a graph will contain an independent set with at least $\frac{kn}{2k+1}$ vertices.

1. Introduction

In a simple graph with n vertices, let $\alpha(G), \delta(G)$ and $\sigma(G)$ denote the independence number, minimum degree and the length of a shortest odd cycle of G. If u and v are adjacent, then we write $u \sim v$. For a vertex v and a vertex set A in G, let $N(v) = \{u : u \sim v\}$ and $N(A) = \cup_{v \in A} N(v)$. As in [1], the independence of G is $\mu(G) = \frac{\alpha(G)}{n}$. The following is proved in [1].

Theorem 1.1 (Albertson, Chan and Haas, [1]) For $n \geq k \geq 3$, if $\sigma(G) \geq 2k+1$ and $\delta(G) > \frac{n}{k+1}$, then $\mu(G) \geq \frac{k}{2k+1}$.

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*Visiting Scholar at West Virginia University, research is supported by the NSF of China

Theorem 1.1 is best possible for $k = 3$, as shown by the generalized Möbius ladder of order 12. However, no example is known to show the sharpness of Theorem 1.1 for larger values of k. This is the motivation of this short note. The main purpose of this paper is to improve Theorem 1.1, when $k \geq 4$, to the following.

Theorem 1.2 For $n \geq k \geq 4$, if $\sigma(G) \geq 2k+1$ and $\delta(G) > \frac{2n-1}{2(k+1)}$, then $\mu(G) \geq \frac{k}{2k+1}$.

2. The Proof

Let G and H be graphs. A map $f : V(G) \mapsto V(H)$ is a homomorphism if f preserves adjacence. We shall denote an odd cycle of length $2k+1$ by Z_{2k+1}, which also denotes the set of integers modulo $2k+1$. The method used by Albertson et al in [1] is to show that under the hypothesis of Theorem 1.1, there is a homomorphism from G into Z_{2k+1}. The following is then applied.

Theorem 2.1 (Albterson and Collins [2]) If there is a homomorphism from G to H, and if H is vertex transitive, then $\mu(G) \geq \mu(H)$.

Our proof is a refinement of the proof presented in [1]. As in [1], we need:

Lemma 2.2 (Andrásfai, Erdös and Sós, [3]) If G is no bipartite and $\delta(G) \geq \frac{2n}{2k+3}$, then $\sigma(G) \leq 2k+1$.

Since $n \geq k$, $\delta(G) > \frac{2n-1}{2(k+1)}$. By Lemma 2.2 and by the assumption that $\sigma(G) \geq 2k+1$, G has a cycle C of length $2k+1$. Label the vertices of C with elements in Z_{2k+1}. Define

$$D_i = N(i-1) \cap N(i+1), \forall i \in Z_{2k+1}, \text{ and } D = \cup_i D_i; \text{ and}$$

$$L_i = \{v \in V(G) - D \text{ and } N(v) \cap D_i \neq \emptyset\}, \forall i \in Z_{2k+1} \text{ and } L = \cup_i L_i.$$

Lemma 2.3 $V(G) = D \cup L$ is disjoint union.

Proof By the definition of D and L, $D \cap L = \emptyset$. It remains to show that $V(G) - D \subseteq L$. By $\sigma(G) = 2k+1 \geq 9$, $D_i \cap D_j = \emptyset$ whenever $i \neq j$. It follows that

$$\frac{(2k+1)(2n-1)}{2(k+1)} \quad < \quad (2k+1)\delta(G)$$

$$\leq \sum_{i\in Z_{2k+1}} (|D_{i-1}| + |D_{i+1}|) + (n - |D|) = n + |D|.$$

Therefore $|D| > \frac{kn}{k+1} - \frac{2k+1}{2(k+1)}$. If there is a vertex $v \in (V(G) - D) - L$, then $N(v) \cap D = \emptyset$, and so

$$n \geq |N(v) \cup \{v\} \cup D| > \frac{2n-1}{2(k+1)} + 1 + \frac{kn}{k+1} - \frac{2k+1}{2(k+1)} = n,$$

a contradiction. □

Lemma 2.4 If $u \in V(G) - D$, $u \sim p \in D_i$ and $u \sim q \in D_j$, then either $j = i$ or $j = i \pm 2$.

Proof Assume that $i \neq j$. Without loss of generality, we may assume that $1 \leq i < j \leq 2k+1$, and that i and j have distinct parity. Note that $p = i$ and $q = j$ are possible.

If $j - i = 1$, then a 3-cycle or a 5-cycle exists, and so we assume that $j - i \geq 3$. Then $(j-1) - (i+1)$ is odd and so $i+1, i+2, \cdots, j-2, j-1, q, u, p, i+1$ is an odd cycle C' with length $j - 1 - (i+1) + 4$.

If $j + 1 = i$ in Z_{2k+1}, then either $j+1, i+1, p, u, q, j+1$ is a 5-cycle or the subgraph induced by these vertices contains a 3-cycle, contrary to $\sigma(G) \geq 9$. When $j + 1 \neq i$ in Z_{2k+1}, then $C - E(C')$ has at least 4 edges: $(i-1, i), (i, i+1), (j-1, j), (j, j+1)$. Since $|E(C) - E(C')| = 4$ also, we must have $|E(C')| = 2k+1$, by $\sigma(G) = 2k+1$. It follows that $j - 1 - (i+1) + 4 = 2k+1$, or $j = i - 2$ in Z_{2k+1}. □

Lemma 2.5 If $u \in L_i$ and $u \sim v \in L_j$, then $j = i \pm 3$ or $j = i \pm 1$ in Z_{2k+1}.

Proof Assume that $u \sim p \in D_i$ and $v \sim q \in D_j$. Again that $p = i$ and $q = j$ are possible.

If $i = j$, then i, p, u, v, q, i is a 5-cycle, or the subgraph induced by these vertices contains a 3-cycle, contrary to $\sigma(G) \geq 9$. Hence $i \neq j$. Assume that $2k+1 \geq j > i \geq 1$, and that i and j have distinct parity.

Then $i-1, p, u, v, q, j+1, j+2, \cdots, 2k+1, 1, 2, \cdots, i-1$ is an odd cycle C' with length $|E(C')| = (2k+1) - (j+1) + (i-1) + 5$ If $j - 1 = i$ or if $j - 1 = i + 2$, then we are done. Assume that $j - 1 \notin \{i, i+2\}$, then $E(C) - E(C')$ has at least 5 edges: $(i-1, i), (i, i+1), (i+1, i+2), (j-1, j), (j, j+1)$. As $|E(C') - E(C)| = 5$ as well, it follows by $\sigma(G) = 2k+1$ that $|E(C')| = 2k+1$, and so $j = i + 3$. □

Define $L_{i,+} = \{u \in L_i : u \sim v \in L_{i+3} \text{ or } u \in L_{i+2}\}$, $L_{i,-} = \{u \in L_i : u \sim v \in L_{i-3} \text{ or } u \in L_{i-2}\}$, and $L_i^* = L_i - (L_{i,+} \cup L_{i,-}), \forall i \in Z_{2k+1}$.

Lemma 2.6 For each $i \in Z_{2k+1}$, $L_{i,+}$, $L_{i,-}$ and L_i^* are mutually disjoint.

Proof By Lemma 2.5, it suffices to show that if $u \in L_i$ and $v \in L_{i+3}$ and $v' \in L_{i-3}$, then it is impossible for $u \sim v$ and $u \sim v'$ to hold simultaneously.

By contradiction, assume that $u \sim v \sim q \in D_{i+3}$ and $u \sim v' \sim q \in D_{i-3}$. By $\sigma(G) \geq 9$, $v \neq p$ and $v' \neq q$.

By $\sigma(G) = 2k+1 \geq 9$, $i-3 \not\sim i+3$ and $i-4 \not\sim i+4$, (for otherwise G has a 5-cycle or a 7-cycle). It follows that the segment of C from $i-4$ to $i+4$ (denoted by ${}_{i-4}C_{i+4}$) is a path of length 8. Replacing ${}_{i-4}C_{i+4}$ by the path $i-4, p, v', u, v, q, i+4$, one obtains an odd cycle of length $2k+1-8+6 = 2k-1 < 2k+1$, contrary to $\sigma(G) = 2k+1$. □

Lemma 2.7 If $k = 2m$, then $L_i^* = \emptyset$ and $L_i = L_{i,+} \cup L_{i,-}$ is a disjoint union.

Proof It suffices to prove that case when $i = 0$, that is, $L_0 \subseteq L_{0,+} \cup L_{0,-}$. (Note that $2k+1 = 0$ in Z_{2k+1}).

Assume by contradiction that $\exists u \in L_0 - L_{0,+} \cup L_{0,-} = L_0^*$. By Lemmas 2.4 and 2.5, and by $u \notin L_{0,+} \cup L_{0,-}$, $N(u) \subseteq D_0 \cup L_{2k} \cup L_1$, and so $d(u) \leq |D_0| + |L_{2k}| + |L_1|$. For $j = 4t+2, 4t+3$, where $t = 0, 1, 2, \cdots, m-1$, $d(j) \leq |D_{j-1}| + |D_{j+1}| + |L_j|$. It follows that

$$\begin{aligned} n - \frac{1}{2} &< (k+1)\delta(G) \leq d(u) + \sum_{t=0}^{m-1} [d(4t+2) + d(4t+3)] \\ &\leq |D| + |L - L_0| = n - |L_0| \leq n - 1, \end{aligned}$$

a contradiction. □

Lemma 2.8 If $k = 2m+1 \geq 5$, then for any $u \in L_i^*$, $N(u) \subseteq D_i$.

Proof Again it suffices to prove the case when $i = 0$. Assume by contradiction that $u \in L_0 - L_{0,+} \cup L_{0,-}$. As in Lemma 2.7, $d(u) \leq |D_0| + |L_{2k}| + |L_1|$. Note that $d(0) \leq |D_1| + |D_{2k}| + |L_0| + |L_2| + |L_{2k-1}|$, and that for $j = 4t-1, 4t$, where $t = 1, 2, \cdots, m$, $d(j) \leq |D_{j-1}| + |D_{j+1}| + |L_j|$. Hence by $k \geq 5$,

$$\begin{aligned} n - \frac{1}{2} &< (k+1)\delta(G) \leq d(0) + d(u) + \sum_{t=1}^{m} [d(4t-1) + d(4t)] \\ &\leq |D_0| + |L_1| + |L_{2k}| + |D_1| + |D_{2k}| + |L_0| + |L_2| + |L_{2k-1}| \end{aligned}$$

$$+\sum_{t=1}^{m}[|D_{4t-2}|+|D_{4t}|+|L_{4t-1}|+|D_{4t-1}|+|D_{4t+1}|+|L_{4t}|]$$

$$\leq \quad |D|+(|L|-|L_1|-|L_{2k}|-\sum_{t=1}^{m-1}(|L_{4t+1}|+|L_{4t+2}|)\leq n.$$

It follows that $L_1 = L_{2k} = \emptyset$. By Lemma 2.5, and by the assumption that $u \in L_0^*$, $N(u) \subseteq D_0$. □

Lemma 2.9 Define $f : V(G) \mapsto Z_{2k+1}$ by $f(D_i \cup L_{i-1,+} \cup L_{i+1,-} \cup L_{i-1}^*) = \{i\}$ for each $i \in Z_{2k+1}$. Then f is well-defined and is a homomorphism.

Proof Note that if $D_i \cap D_j \neq \emptyset$, then G has an odd cycle shorter than C. Therefore by Lemma 2.8,

$$(D_i \cup L_{i-1}^*) \cap (D_j \cup L_{j-1}^*) = \emptyset, \text{ whenever } i \neq j. \tag{1}$$

By (1), we conclude that

the restriction of f to $D \cup L^*$ is well-defined and is a homomorphism. (2)

To show that f is well-defined and is a homomorphism, it suffices to show, by (2), that

(A) if $u \in L - L^*$, then there is only one way to determine $f(u)$; and

(B) if $v \sim u \in L - L^*$, then $f(v) \sim f(u)$ in Z_{2k+1}.

Without loss of generality, assume that $u \in L_{i-1,+}$. Then $u \notin L_{i-3}$ and $u \not\sim v \in L_{i-4}$.

By Lemma 2.4, it is possible that $u \in L_{i+1}$. If $u \in L_{i+1,+}$, then G would have an odd cycle shorter than C, a contradiction. Therefore, $u \in L_{i+1,-}$, and so (A) follows.

Assume that $u \sim v$. Then by Lemma 2.5, $v \in L_j$, where $j = i+2, i$ or $i-2$. (That $u \in L_{i-1,+}$ implies that $v \notin L_{i-3}$.)

If $v \in L_{i+2}$, then $v \in L_{i+2,-}$, and so $f(v) = i+1$.

If $v \in L_i$, then either $v \in L_i^*$, whence $f(v) = i+1$; or $v \in L_{i,+}$, whence $f(v) = i+1$; or $v \in L_{i,-}$, whence $f(v) = i-1$. In any case, (B) follows.

The proof of Theorem 1.2 now follows from Lemma 2.9 and Theorem 2.1.

References

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