Connectivity of cycle matroids and bicircular matroids

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ABSTRACT. A unified approach to prove former connectivity results of Tutte, Cunningham, Inukai and Weinberg, Oxley and Wagner.

1 Introduction

We assume familiarity with elementary matroid theory and graph theory. Graphs in this note are finite and have no isolated vertices. The terminology used in this note for matroids and graphs will in general follow Oxley [5] and Bondy and Murty [1], respectively. A *cycle* in a graph G is a 2-regular connected graph. The term *circuit* is reserved for matroids.

This note considers the relationship between the n-connection of a matroid on the edge set of a graph G and the n-connection of the graph G. Such a problem was first studied by Tutte in [9], where Tutte characterized graphs G with Tutte n-connected cycle matroid (to be defined in Section 2) in terms of partitions of the edge set $E(G)$ with certain properties. Simpler proof was later found by Cunningham [2]. Pursuing the relationship between vertex connectivity of a graph G and the corresponding concept in matroids, Cunningham [2], Inukai and Weinberg [3], and Oxley [6] independently discovered the notion of vertical connectivity of a matroid,

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as well as its dual concept (the cyclical connectivity), and characterized graphs G whose cycle matroid $M(G)$ is vertically n-connected (cyclically n-connected, respectively), in a way similar to the Tutte's characterization.

Following the same track, Wagner [10] studied the relationship between the Tutte n-connection of the bicircular matroid (to be defined in Section 2) of a graph G and the Tutte n-connection of G, and found a similar characterization.

The main purposes of this paper are: (1) to investigate properties that are common in different types of n-connection in both the cycle matroid of a nontrivial graph G and the bucircular matroid of G; and (2) to complete the obviously undone job: characterizations of graphs whose bicircular matroids are vertically n-connected and cyclically n-connected, respectively.

The definitions of various type of connections will be given in Section 2. The exact statements of the abovementioned characterizations will be given in Section 4. Some connectivity properties of the cycle matroid and the bicircular matroid of a graph G will be investigated in Section 3. In Section 4, we shall apply the results obtained in Section 3 to present alternative proofs of the abovementioned characterizations, and to prove the characterizations of graphs whose bicircular matroids are vertically n-connected and cyclically n-connected, respectively.

2 Definitions

We will be concerned with partitions of E into two sets, X and $E-X$; thus both X and $E-X$ are assumed non-empty. Let M be a matroid on $E=E(M)$ with rank function r. The *connectivity function*, $k(\cdot)$ of M is

$$k(X)=r(X)+r(E-X)-r(E), \quad \text{for any } X\subseteq E.$$

Let l be an integer. A partition $\{X,E-X\}$ is a *Tutte l-separation* of M if

$$k(X)<l \text{ and } \min\{|X|,|E-X|\}\geq l, \tag{1}$$

a *vertical l-separation* of M if

$$k(X)<l \text{ and } \min\{r(X),r(E-X)\}\geq l, \tag{2}$$

and a *cyclical l-separation* of M if

$$k(X)<l \text{ and each of } M|X \text{ and } M|(E-X) \text{ has a circuit.} \tag{3}$$

An l-separation $\{X,E-X\}$ is *exact* if $k(X)=l-1$.

For a positive integer n, the matroid M is *Tutte n-connected* if for all l, $1\leq l<n$, M has no Tutte l-separation. We define a matroid M to

be *vertically n-connected* and *cyclically n-connected* similarly. A Tutte 2-connected matroid is also called a *connected* matroid.

Let G be a finite connected graph. The *set of vertices of attachment* of the subgraph H in G, is

$$A_G(H) = V(H) \cap V(G[E(G) - E(H)]).$$

Let l be an integer. A partition $\{X, E(G) - X\}$ is a *Tutte l-separation* of G if

$$|A_G(G[X])| \leq l \text{ and } \min\{|X|, |E - X|\} \geq l, \tag{4}$$

a *vertical l-separation* of G if

$$|A_G(G[X])| \leq l \text{ and } \min\{|V(G[X])|, |V(G[E(G) - X])|\} \geq l + 1, \tag{5}$$

and a *cyclical l-separation* of G if

$$|A_G(G[X])| \leq l \text{ and each of } G[X] \text{ and } G[E(G) - X] \text{ has a cycle.} \tag{6}$$

For a positive integer n, the graph G is *Tutte n-connected* if for all l, $1 \leq l < n$, M has no Tutte l-separation. We define a graph G to be *vertically n-connected* and *cyclically n-connected* similarly.

Let G be a graph with $E(G) \neq \emptyset$. The *cycle matroid* of G, denoted $M(G)$, is the matroid on $E(G)$ whose collection of circuits consists of all the cycles of G.

Let G be a graph with $E(G) \neq \emptyset$. Let $D_1(G)$ denote the set of vertices of degree 1 in G. A *bicycle* of G is a connected subgraph H with $|E(H)| = |V(H)| + 1$ and with $D_1(H) = \emptyset$. The *bicircular matroid* of G, denoted $B(G)$, is a matroid on $E(G)$ whose collection of circuits consists of all the bicycles of G. Bicircular matroids were first discovered by Simões-Pereira [7], and have been studied extensively. (See Simões-Pereira [8] for an overview.)

3 Some Properties of $M(G)$ and $B(G)$

Let G be a graph. A component H of G is *acyclic* if H is a tree; otherwise H is *cyclic*. The number of components of G is $\omega(G)$, and the number of acyclic components of G is $\omega_a(G)$. Note that for $X \subseteq E(G)$, the rank function $r(\cdot)$ in the cycle matroid $M(G)$ and the bicircular matroid $B(G)$ can be expressed as follows:

$$r(X) = \begin{cases} |V(G[X])| - \omega(G[X]) & \text{if } M = M(G) \\ |V(G[X])| - \omega_a(G[X]) & \text{if } M = B(G). \end{cases}$$

An edge $e \in E(G)$ is an *end edge* if e is incident with a vertex in $D_1(G)$. If $\{X, E(G) - X\}$ is a partition of $E(G)$, then define $o(X) = \omega(G[X]) + \omega(G[E(G) - X])$. The main result of this section is Theorem 3.8, which shows a property commonly shared by both the cycle matroid $M(G)$ and the bicircular matroid $B(G)$ of a nontrivial graph G. First, we establish some lemmas.

Lemma 3.1. *Let M be a matroid with rank function r. Each of the following holds:*

(i) *If $k(X) = r(X)$ for some $X \subset E(M)$, then $E - X$ contains a basis of M.*

(ii) *If $k(X) = |X|$ for some $X \subset E(M)$, then X is independent and $E - X$ contains a basis of M.*

Proof: (i) The definition of $k(X)$ and the equality $k(X) = r(X)$ give $r(E - X) = r(E)$. Therefore $E - X$ contains a basis of M.

(ii) If $k(X) = r(X) + r(E - X) - r(E) = |X|$, then $r(X) = |X|$ and $r(E - X) = r(E)$. Therefore X is independent and $E - X$ contains a basis of M. □

Lemma 3.2. *Each of the following holds:*

(i) (Proposition 4.1.4 of [5]) *The matroid M is connected if and only if, for every pair of distinct elements of $E(M)$, there is a circuit in M containing both.*

(ii) (Proposition 4.1.8 of [5]) *Let G be a loopless graph without isolated vertices and with $|E(G)| \geq 3$. Then $M(G)$ is connected if and only if G is 2-connected.*

Lemma 3.3. (Proposition 4 of [4]) *Let G be a graph with $|E(G)| \geq 2$ and without isolated vertices, and let $B(G)$ be the bicircular matroid of G. Then $B(G)$ is connected if and only if each of the following holds:*

(i) *G is connected,*

(ii) *G is not a cycle, and*

(iii) *G has no vertices of degree 1.*

Lemma 3.4. *Let M be a matroid with rank function r, and let $\{X, E(M) - X\}$ be a partition of $E(M)$. If there exist nonempty subsets $X_1, X_2, \cdots X_c$ of X such that $r(X) = \sum_{i=1}^{c} r(X_i)$, then each of the following holds.*

(i) $r(X) = r(X_i) + r(X - X_i)$, *for each* i *with* $1 \le i \le c$.

(ii) $k(X_i) \le k(X)$.

Proof: (i) is obvious. By the submodularity of the rank function, $r(E - X_i) \le r(E - X) + r(X - X_i)$. Therefore by (i),

$$\begin{aligned} k(X_i) &= r(X_i) + r(E - X_i) - r(E) \\ &\le r(X_i) + r(X - X_i) + r(E - X) - r(E) = k(X). \end{aligned}$$

□

The following lemma is an immediate consequence of connectivity.

Lemma 3.5. *Let* G *be a connected graph with* $E = E(G)$ *and let* $\{X, E - X\}$ *be a partition of* E *with* $X \ne \emptyset$ *and* $E - X \ne \emptyset$. *If* $X_1 \subset X$ *induces a component* $G[X_1]$ *of* $G[X]$, *then*

$$V(G[X_1]) \cap V(G[E - X]) \ne \emptyset, \text{ and } V(G[X - X_1]) \cap V(G[E - X]) \ne \emptyset.$$

Lemma 3.6. *Let* G *be a connected graph with* $|E(G)| \ge 3$ *and let* M *be either the cycle matroid or the bicircular matroid of* G. *Let* $E = E(G)$ *and let* $\{X, E - X\}$ *be a partition of* $E(G)$. *Suppose that* $X_1, X_2, \cdots, X_c$ *are nonempty subsets of* X *such that* $X = \cup_{i=1}^{c} X_i$, *where* $c = \omega(G[X])$, *and such that* $r(X) = \sum_{i=1}^{c} r(X_i)$. *For a proper subset* $N \subset \{1, 2, \cdots, c\}$, *let* $X_N = \cup_{i \in N} X_i$.

(i) *Suppose that* M *is Tutte (vertically, resp.)* n*-connected, and that* $\{X, E - X\}$ *is a Tutte (vertical, resp.)* n*-separation of* M. *If* $c \ge 2$ *and if for some* $N \subset \{1, 2, \cdots, c\}$, $r(X_N) \ge n$, *then* $\{X_N, E - X_N\}$ *is a Tutte (vertical, resp.)* n*-separation of* M *with* $o(X_N) < o(X)$.

(ii) *Suppose that* M *is Tutte cyclically* n*-connected, and that* $\{X, E - X\}$ *is a cyclical* n*-separation of* M. *If* $c \ge 2$, *if for some* $N \subset \{1, 2, \cdots, c\}$, $r(X_N) \ge n$, *and if* X_N *contains a circuit, then* $\{X_N, E - X_N\}$ *is a cyclical* n*-separation of* M *with* $o(X_N) < o(X)$.

(iii) *Suppose that* M *is Tutte (cyclically, resp.)* n*-connected, and that* $\{X, E - X\}$ *is a Tutte (cyclical, resp.)* n*-separation of* M. *If* $c \ge 2$ *and if for some* $N \subset \{1, 2, \cdots, c\}$, $r(X_N) < |X_N|$, *then* $\{X_N, E - X_N\}$ *is a Tutte (vertical, or cyclical, resp.)* n*-separation of* M *with* $o(X_N) < o(X)$.

(iv) *Suppose that* M *is vertically* n*-connected, and that* $\{X, E - X\}$ *is a vertical* n*-separation of* M. *If* $c \ge 2$ *and if for some* $N \subset \{1, 2, \cdots, c\}$, $k(X_N) < r(X_N))$, *then* $\{X_N, E - X_N\}$ *is a vertical* n*-separation of* M *with* $o(X_N) < o(X)$.

(v) *Suppose that M is Tutte (vertically, resp.) n-connected, and that $\{X, E-X\}$ is an exact Tutte (vertical, resp.) n-separation of M. If $V(G) - (G[E-X]) \neq \emptyset$, then there is Tutte (vertical, resp.) n-separation $\{X', E-X'\}$ of M such that $o(X') = 2$.*

(vi) *Suppose that M is Tutte (vertically, resp.) n-connected, and that $\{X, E-X\}$ is an exact Tutte (vertical, resp.) n-separation of M. If both X and $E-X$ are independent in M, then there is a Tutte (vertical, resp.) n-separation $\{X', E-X'\}$ of M such that $o(X') = 2$.*

Proof: (i) and (ii). Assume $r(X_N) \geq n$. Then $|X_N| \geq r(X_N) \geq n$. Note that if $\{X, E-X\}$ is a Tutte n-separation, then $|E - X_N| \geq |E-X| \geq n$; if $\{X, E-X\}$ is a vertical n-separation, then $r(E-X_N) \geq r(E-X) \geq n$; if $\{X, E-X\}$ is a cyclical n-separation, then $M|(E-X_N)$, containing $M|(E-X)$ as a restriction, has a circuit. Thus (i) and (ii) follow by Lemma 3.4.

(iii) and (iv). By (i) and (ii), for Tutte or cyclical connection, it suffices to show that $r(X_N) \geq n$. If not, we assume that there is an X_i with $r(X_N) < |X_N|$ and $r(X_N) < n$. Note that $r(X_N) \geq k(X_N)$. By Lemma 3.4, $k(X_N) \leq k(X) \leq n-1$. Since $|E-X_N| \geq |E-X| \geq n > r(X_N)$ (since $r(X_N) < |X_N|$ implies that X_N contains a circuit, resp.), $\{X_N, E-X_N\}$ would be a Tutte (cyclical, resp.) $r(X_N)$-separation, contrary to the assumption that M is Tutte (cyclical, resp.) n-connected. Thus $r(X_N) \geq n$, and so (ii) follows from (i). For vertical connection, if there is some X_i with $k(X_i) < r(X_i)$, then by (i), $r(X_i) < n$, and so $r(E - X_i) \geq r(E-X) \geq n > r(X_i)$. Therefore $\{X_i, E-X_i\}$ is a vertical $r(X_i)$-separation of M, contrary to the assumption that M is vertically n-connected.

(v) and (vi). Assume $\omega(G[E-X]) \leq \omega(G[X])$ and $c \geq 2$. Among all Tutte (vertical, resp.) n-separations $\{X, E-X\}$ such that $V' = V(G) - V(G[E-X]) \neq \emptyset$ for (v), or such that both X and $E-X]$ are independent for (vi), choose one $\{X, E-X\}$ such that $o(X)$ is minimized.

By (i) - (iv), we may assume that $r(X_N) = |X_N| < n$ (in the Tutte connection case) and $k(X_N) = r(X_N)$ (in the vertical connection case), $\forall \emptyset \neq N \subset \{1, 2, \cdots, c\}$. Therefore we observe:

(A) $\forall N$ with $\emptyset \neq N \subset \{1, 2, \cdots, c\}$, $k(X_N) = r(X_N)$ and $r(E - X_N) = r(E)$.

If $k(X_N) < |X_N| < n$, then since $|E - X_N| > |E-X| \geq n > |X_N|$, $\{X_N, E-X_N\}$ is a Tutte $|X_N|$-separation, contrary to the assumption that M is n-connected. Therefore $k(X_N) = |X_N|$, and so (A) follows from Lemma 3.1. This proves (A).

Assume (v). Pick $v \in V'$. By (A) with $N = \{1, \cdots, c\} - \{i\}$, $v \in V(G[X_i])$, for each i. Thus there exist distinct i and j such that $v \in$

$V(G[X_i]) \cap V(G[X_j])$, contrary to the assumption that $G[X_i]$'s are the components of $G[X]$.

Now assume (vi). Label the components of $G[X]$ so that $|X_1| \geq |X_2| \geq \cdots \geq |X_c|$. We further choose $\{X, E-X\}$, subject to minimizing $o(X)$, such that $|X_1|$ is maximized. We shall show that $c=1$ and so (vi) follows from (v) (with X in (vi) replacing $E-X$ in (v)). Suppose $c \geq 2$. By (A), there is an edge $e' \in X - X_1$ such that $r((E-X) \cup e') = r(E-X)+1$. Since G is connected, there is an edge $e'' \in E-X$ incident with exactly one vertex in $G[X_1]$. Let $X' = (X-e') \cup e''$. Then $r(X) = r(X') = |X'|$ and $r(E-X) = r(E-X') = |E-X'|$, and so $\{X', E-X'\}$ is also an n-separation with both X' and $E-X'$ independent in M, contrary to the choice of $\{X, E-X\}$. Hence $c=1$. □

Lemma 3.7. *Let $M = M(G)$ or $M = B(G)$, and let $\{X, E-X\}$ be a Tutte (vertical, resp.) n-separation of M. Suppose that some component of $G[X]$ has an end edge e and that some component of $G[E-X]$ has a cut edge e' such that in G, e' is not incident with the isolate vertex in $G[X-e]$. Let $X' = (X-e) \cup e'$. Then $\{X', E-X'\}$ is a Tutte (vertical, resp.) n'-separation of M such that $n' \leq n$ and $V(G) - V(G[X']) \neq \emptyset$.*

Proof: Note that $r(X') \leq r(X)$ and $r(E-X') \leq r(E-X)$, and so $k(X') \leq k(X)$. Note also that the vertex of degree one incident with e in X becomes an isolated vertex in $G[X']$. Thus $\{X', E-X'\}$ is a Tutte (vertical, resp.) n'-separation with $n' = k(X')+1 \leq n$ and $V(G) - V(G[X']) \neq \emptyset$. □

Note that such a pair (e, e') in Lemma 3.7 can always be found if both $G[X]$ and $G[E-X]$ have a component which is a tree of at least 3 vertices.

Theorem 3.8. *Let $n \geq 1$ be an integer, let G be a connected graph with $|E(G)| \geq 2$, and let M be either the cycle matroid or the bicircular matroid of G. Let $E = E(G)$. If one of the following holds:*

(i) *M is Tutte n-connected and $\{X, E-X\}$ is a Tutte n-separation of M such that*

$$o(X) = \min\{o(X') : \{X', E-X'\} \text{ is a Tutte } n\text{-separation of } M\}, \text{ or} \tag{7}$$

(ii) *M is vertically n-connected and $\{X, E-X\}$ is a vertical n-separation of M such that*

$$o(X) = \min\{o(X') : \{X', E-X'\} \text{ is a vertical } n\text{-separation of } M\}, \text{ or} \tag{8}$$

(iii) *M is cyclical n-connected and $\{X, E-X\}$ is a cyclical n-separation of M such that*

$$o(X) = \min\{o(X') : \{X', E-X'\} \text{ is a cyclical } n\text{-separation of } M\}, \tag{9}$$

then both $G[X]$ and $G[E-X]$ are connected.

Proof:: We argue by contradiction. Assume that G is a counterexample. We may assume that $|E(G)| \geq 3$ since otherwise Theorem 3.8 holds trivially.

We may also assume that $n \geq 2$. For if $n = 1$, then $k(X) = 0$ for some $X \subset E$ implies that $G[X]$ is a union of blocks of G, and so Theorem 3.8 follows trivially.

We assume that there is an n-separation $\{X, E-X\}$ of $M(G)$ satisfying one of the conditions of Theorem 3.8 but $o(X) \geq 3$. We may assume that $\omega(G[E-X]) \leq \omega(G[X])$ and $c = \omega(G[X]) \geq 2$. Let $X_1, X_2, \cdots, X_c$ are nonempty subsets of X such that $X = \cup_{i=1}^{c} X_i$ and such that each $G[X_i]$ is a component of $G[X]$. Let $d = \omega(G[E-X])$ and $Y_1, Y_2, \cdots, Y_d$ are nonempty subsets of $E-X$ such that $E-X = \cup_{i=1}^{d} Y_i$ and such that each $G[Y_i]$ is a component of $G[E-X]$.

If $\{X, E-X\}$ is an exact cyclical n-separation, then there must be an X_i with $r(X_i) < |X_i|$, and so by Lemma 3.6(iii), $c = 1$, a contradiction. This proves Theorem 3.8(iii).

Assume then that $\{X, E-X\}$ is an exact Tutte n-separation. Apply Lemma 3.6(iii) to both X and $E-X$, we may assume either $\omega(G[E-X]) = 1$, or both X and $E-X$ are independent in M. Note that $k(X) = n-1$ implies $r(E) > r(E-X)$. Hence by Lemma 3.6 (v) and (vi), and by $\omega(G[E-X]) = 1$, we conclude that $M = B(G)$ and $G[E-X]$ is a spanning tree of G. If $G[X]$ has an acyclic component, then by Lemma 3.7, there is an n-separation $\{X', E-X'\}$ with $V(G) - V(G[X']) \neq \emptyset$, and so Theorem 3.8(i) follows from Lemma 3.6(v). Hence every component of $G[X]$ is cyclic. But then $r(X) = \sum_{i=1}^{c} |V(G[X_i])| = |V(G)| = r(E)$, and so $n-1 = k(X) = r(X) + r(E-X) - r(E) = r(E-X) > n$, a contradiction. This proves Theorem 3.8(i).

Hence we assume that $\{X, E-X\}$ is an exact vertical n-separation. By Lemma 3.6, we observe:

(A) $r(X_N) < n$, $\forall N$ with $\emptyset \neq N \subset \{1, 2, \cdots, c\}$; and $r(Y_N) < n$, $\forall N$ with $\emptyset \neq N \subset \{1, 2, \cdots, d\}$.

(B) $k(X_N) = r(X_N)$ and $r(E - X_N) = r(E)$, $\forall N$ with $\emptyset \neq N \subset \{1, 2, \cdots, c\}$; and $k(Y_N) = r(Y_N)$ and $r(E - Y_N) = r(E)$, $\forall N$ with $\emptyset \neq N \subset \{1, 2, \cdots, d\}$.

(C) $V(G) = V(G[E-X]) = V(G[X])$.

Suppose $M = B(G)$. By (A), (B), and Lemma 3.7, either G has a pair of parallel edges $\{e_1, e_2\}$ such that $Y_{i'} = \{e_1\}$ and $X_{i''} = \{e_2\}$ for some i', i'', whence M has a vertical $(n-1)$-separation $\{X - e_2, E - (X - e_2)\}$, contrary to the assumption that M is vertically n-conncted; or we may assume that every $G[Y_i]$ is cyclic, whence by (C), $r(E-X) = \sum_{i=1}^{d} |V(G[Y_i])| =$

$|V(G)| = r(E)$, and so $n-1 = k(X) = r(X)+r(E-X)-r(E) = r(X) > n$, a contradiction. This proves Theorem 3.8(ii) when $M = B(G)$.

Suppose $M = B(G)$. For vertices $u, v \in V(G)$, define

$$\kappa(u,v) = \text{maximum number of internly disjoint } (u,v)\text{-paths in } G,$$

and let $k = \min\{\kappa(u,v) : u, v \in V(G)\}$. Since M has a vertical n-separation, G is not spanned by a complete subgraph. Hence one can find nonadjacent $u, v \in V(G)$ such that $\kappa(u,v) = k$. By Menger's Theorem, there is a subset $V' \subset V(G) - \{u,v\}$ such that G has connected subgraphs G_1 and G_2 with $G = G_1 \cup G_2$, with $u \in V(G_1)$ and $v \in V(G_2)$ and with $|V(G_1) \cap V(G_2)| = k$. Therefore, $\{E(G_1), E(G_2)\}$ is a vertically k-separation of M with $o(E(G_1)) = 2$. Clearly $k \geq n$.

To complete the proof of Theorem 3.8(ii) when $M = M(G)$, it suffices to show that $k = n$. However, this is proved by Cunningham in [2]. □

4 Connectivity in cycle matroids and bicircular matroids

Throughout this section, G denotes a nontrivial connected graph. If H is a subgraph of G, then $\bar{H} = G[E(G) - E(H)]$.

Proposition 4.1. (Cunningham, Proposition 1 of [2]) *If $X \subset E(G)$, then in $M(G)$, $k(X) \leq |A_G(G[X])| - 1$, where equality holds if and only if both $G[X]$ and $G[E(G) - X]$ are connected.*

Theorem 4.2. (Tutte, Theorem 3.5 of [9]) *Let G be a nontrivial connected graph. Then $M(G)$ is Tutte n-connected if and only if G is Tutte n-connected.*

Theorem 4.3. (Cunningham, Theorem 2 of [2]) *Let G be a nontrivial connected graph. Then $M(G)$ is cyclically n-connected if and only if G is cyclically n-connected.*

Theorem 4.4. (Cunningham, Theorem 1 of [2], Inukai and Weinberg, Theorems 1 and 2 of [3], and Oxley, Theorem 2 of [6]) *Let G be a nontrivial connected graph. Then $M(G)$ is vertically n-connected if and only if G is vertically n-connected.*

Proofs of Theorems 4.2 and 4.3: Let $E = E(G)$. Assume that $M(G)$ is Tutte n-connected but G is not Tutte n-connected. Then G has a Tutte l-separation $\{X, E-X\}$, for some $1 \leq l < n$. By Proposition 4.1, $\{X, E-X\}$ is a Tutte l-separation of $M(G)$, contrary to the assumption that $M(G)$ is Tutte n-connected. Hence G must be Tutte n-connected.

Assume that G is Tutte n-connected but $M(G)$ is not Tutte l-connected. Let l be an integer with $1 \leq l < n$ such that $M(G)$ is Tutte l-connected but not Tutte $(l+1)$-connected. Then by Theorem 3.8(i), there is a Tutte

l-separation $\{X, E-X\}$ of $M(G)$ such that both $G[X]$ and $G[E(G)-X]$ are connected. It follows by Proposition 4.1 that $l-1 \geq k(X) = |A_G(G[X])|-1$. Thus $l \geq |A_G(G[X])|$, and so by (4), $\{X, E-X\}$ is a Tutte l-separation of G. Therefore G is not Tutte n-connected. This proves Theorem 4.2.

The proof for Theorems 4.3 is similar, using Theorem 3.8(iii) in place of Theorem 3.8(i) in the argument.

Theorem 4.4 can also be proved by a similar argument using Theorem 3.8(ii). However, as our proof for Theorem 3.8(ii) when $M = M(G)$ uses the same idea in Cunningham's proof for Theorem 4.4, this should not be regarded as a different proof.

Let $l \geq 1$ be an integer. Define $\mathcal{F}(l; G)$ to be the collection of partitions $\{E(H), E(G) - E(H)\}$, where H is a subgraph of G such that both H and $\bar{H}$ are connected, and such that

$$|A_G(H)| = \begin{cases} l-1 & \text{if } \omega_a(H) = \omega_a(\bar{H}) = 0 \\ l & \text{if } \omega_a(H) + \omega_a(\bar{H}) = 1 \text{ or } \omega_a(G) = 1 \\ l+1 & \text{if } \omega_a(H) = \omega_a(\bar{H}) = 1 \text{ and } \omega_a(G) = 0 \end{cases}$$

Let G be a connected graph and let $E = E(G)$. Let $X \subset E$ with $\{X, E-X\} \in \mathcal{F}(l; G)$. The partition $\{X, E-X\}$ is a ***l-biseparation*** if

$$\min\{|X|, |E-X|\} \geq l,$$

a ***vertical l-biseparation*** if

$$\min\{|V(G[X])| - \omega_a(G[X]), |V(G[E-X])| - \omega_a(G[E-X])\} \geq l,$$

and a ***cyclical l-biseparation*** if

$$\text{both } G[X] \text{ and } G[E-X] \text{ have a bicycle.}$$

The graph G is ***n-biconnected*** if G has no l-biseparation for any $1 \leq l < n$. We define a graph G to be ***vertically n-biconnected*** and ***cyclically n-biconnected*** similarly.

Examples: Fix $i \in \{1,2\}$. Let H_i be the graph with $V(H_i) = \{v_j^i : 1 \leq j \leq 4\}$ and $E(H_i) = \{v_j^i v_{j'}^i : 1 \leq j < j' \leq 4\} - \{v_1^i v_2^i\}$. (That is, H_i is isomorphic to K_4 minus an edge.) Let G be obtained from the disjoint union from H_1 and H_2 by adding four more edges $\{v_j^1 v_j^2 : 1 \leq j \leq 4\}$. Let $X_1 \subset E(G)$ be the three edges incident with a vertex of degree 3 in G; let $X_2 = E(H_1)$. Then it can be seen that $\{X_1, E(G) - X_1\}$ is a vertical 3-biseparation, and that $\{X_2, E(G) - X_2\}$ is both a 5-biseparation and a cyclical 5-biseparation of G. It can be verified that G is 5-biconnected, vertically 3-biconnected, and cyclically 5-biconnected.

Proposition 4.5. *Let G be a nontrivial connected graph. If $X \subset E(G)$, and if $H = G[X]$, then in $B(G)$,*

$$k(X) \leq |A_G([X])| - \begin{cases} 0 & \text{if } \omega_a(H) = \omega_a(\bar{H}) = 0 \\ 1 & \text{if } \min\{\omega_a(H), \omega_a(\bar{H})\} = 0 \text{ and } \{\omega_a(H), \omega_a(\bar{H})\} \geq 1, \\ & \text{or if } \omega_a(G) = 1 \\ 2 & \text{if } \min\{\omega_a(H), \omega_a(\bar{H})\} \geq 1 \text{ and } \omega_a(G) = 0 \end{cases}$$

where equality holds when both $G[X]$ and $G[E(G) - X]$ are connected.

Proof: Let $E = E(G)$, and let r denote the rank function of $B(G)$. By the definition of bicircular matroids, for any $X \subseteq E$, $r(X) = |V(G[X])| - \omega_a(G[X])$, and so

$$\begin{aligned} k(X) &= r(X) + r(E - X) - r(E) \\ &= |V(H)| + |V(\bar{H})| - |V(G)| - \omega_a(H) - \omega_a(\bar{H}) + \omega_a(G) \\ &= |A_G(H)| - \omega_a(H) - \omega_a(\bar{H}) + \omega_a(G). \end{aligned}$$

This completes the proof. □

Theorem 4.6. (Wagner, [10]) *Let G be a connected graph. Then $B(G)$ is Tutte n-connected if and only if G is n-biconnected.*

The proof for Theorem 4.6 will be similar to that for Theorem 4.7, and so it will be omitted.

Theorem 4.7. *Let G be a connected graph. Each of the following holds.*

(i) *$B(G)$ is vertically n-connected if and only if G is vertically n-biconnected.*

(ii) *$B(G)$ is cyclically n-connected if and only if G is cyclically n-biconnected.*

Proof: Let $E = E(G)$. Suppose that $B(G)$ is vertically (cyclically, resp.) n-connected but G is not vertically (cyclically, resp.) n-connected. Then G has a vertical (cyclical, resp.) l-biseparation $\{X, E - X\}$ with $1 \leq l < n$. Then by Proposition 4.5, $\{X, E - X\}$ is a vertical (cyclical, resp.) l-separation of $B(G)$, contrary to the assumption that $B(G)$ is not vertical (cyclical, resp.) n-connected.

Suppose that G is vertically (cyclically, resp.) n-connected but $B(G)$ is not vertically (cyclically, resp.) n-connected. Then there is an integer l, $1 \leq l < n$, such that $B(G)$ is vertically (cyclically, resp.) l-connected, but not vertically (cyclically, resp.) $(l+1)$-connected. Then by Theorem 3.8(ii) and (iii), there is a vertical (cyclical, resp.) l-separation $\{X, E - X\}$ of $B(G)$ such that both $G[X]$ and $G[E - X]$ are connected. By Proposition 4.5, $\{X, E - X\} \in \mathcal{F}(l; G)$, and so $\{X, E - X\}$ is a vertical (cyclical, resp.) l-biseparation, contrary to the assumption that G is vertically (cyclically, resp.) n-biconnected. This proves Theorem 4.7. □

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Hamiltonian Connected Line Graphs

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Abstract. Let G be a simple graph with n vertices. Let $L(G)$ denote the line graph of G. We show that if $\kappa'(G) \geq 2$ and if for every pair of nonadjacent vertices $v, u \in V(G)$, $d(v) + d(u) > 2n/3 - 2$, then for any pair of vertices $e, e' \in V(L(G))$, either $L(G)$ has a hamilton (e, e')-path, or $\{e, e'\}$ is a vertex-cut of $L(G)$. When G is a triangle-free graph, this bound can be reduced to $n/3$. These bounds are all best possible and they partially improve prior results in [J. Graph Theory, 10 (1986), 411–425] and [Discrete Math. 76 (1989) 95–116].

1. Introduction.

We shall follow the notation of Bondy and Murty [2], unless otherwise stated. Let G be a graph and e, e' be two edges of G. A trail in G whose first edge is e and whose last edge is e' is called an (e, e')-*trail*. An (e, e')-trail T is called a *spanning* (e, e')-trail of G if $V(T) = V(G)$ and if every edge of G is incident with an internal vertex of T. A trial T of G is *dominating* if $G - V(T)$ is edgeless. For convenience, the graph K_1 is regarded as having a closed trail.

The *line graph* of G, denoted by $L(G)$, has $E(G)$ as its vertex set in which two vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G.

Theorem A. *(Harary and Nash-Williams [12]) Let G be a graph with at least three edges. Then G has a dominating closed trial if and only if $L(G)$ is hamiltonian.*

Theorem B. *(Lesniak-Foster and Williamson [13], Zhan [14]) Let G be a graph and let e, e' be in $E(G)$. If G has a spanning (e, e')-trail, then $L(G)$ has a spanning (e, e')-path.*

The definition of spanning (e, e')-trails was used in [9]. We shall say a few words about this definition. Let G be the 4-cycle and e, e' be two nonadjacent edges in G. Then G has an (e, e')-trail that is spanning in G but $L(G)$ does not have a hamilton (e, e')-path. This is why we define a spanning (e, e')-trail in the way above.

If for every pair of vertices u, v of G, G has a spanning (u, v)-path, then G is said to be *hamiltonian connected*. With the help of Theorem B, Zhan showed the following:

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Theorem C. *(Zhan [14]) If $\kappa'(G) \geq 4$, then for every pair of edges $e, e' \in E(G)$, G has a spanning (e, e')-trail and so $L(G)$ is hamiltonian connected.*

If $X \subseteq E(G)$ is an edge-cut such that at least two components of $G - X$ have edges, then X is called an *essential edge-cut*. It is easy to see that if $\{e, e'\}$ is an essential edge-cut of G, then G cannot have any spanning (e, e')-trails. It has been noted by Catlin [7], (and by Zhan [14], for the case when $k = 2$), that G is $2k$-edge-connected if and only if $|E(G)| \geq k$ and for any k-subset $X \subseteq E(G)$, $G - X$ has k edge-disjoint spanning trees. In particular, 4-edge-connected graphs always have 2 edge-disjoint spanning trees. Thus, the following improves Theorem C:

Theorem D. *(Catlin and Lai [9]) Let G be a graph with 2 edge-disjoint spanning trees. For two edges $e, e' \in E(G)$, one of the following holds:*

(i) *G has a spanning (e, e')-trail;*
(ii) *$\{e, e'\}$ is an essential edge-cut of G.*

2. Main results.

The proofs of the following theorems appear in the last section.

Theorem 1. *Let G be a simple graph with $|V(G)| = n \geq 27$ and with $\kappa'(G) \geq 2$. If for every pair of nonadjacent vertices $u, v \in V(G)$,*

$$d(u) + d(v) > \frac{2n}{3} - 2, \tag{1}$$

then for every pair of edges $e, e' \in E(G)$, exactly one of the following holds:

(i) *G has a spanning (e, e')-trail;*
(ii) *$\{e, e'\}$ is an essential edge-cut of G.*

Corollary 1A. *Let G satisfy the hypothesis of Theorem 1. Then either $L(G)$ has a 2-vertex-cut or $L(G)$ is hamiltonian connected.*

Corollary 1B. *(Catlin [4] and Benhocine, Clark, Köler, and Veldman [1]) Let G be a 2-edge-connected simple graph with $n = |V(G)| \geq 27$. If for every pair of nonadjacent vertices $u, v \in V(G)$,*

$$d(u) + d(v) > \frac{2n}{3} - 2, \tag{2}$$

then $L(G)$ is hamiltonian.

Proof: The truth of Corollary 1A follows immediately from Theorem B and Theorem 1. Call an *end-block* of G a maximal 2-connected subgraph H such that either $H = G$ or H contains exactly one cut-vertex of G. If G satisfies the conclusion of Theorem 1 but $L(G)$ is not hamiltonian, then it would follow that every pair of adjacent edges are incident with a cut-vertex of G, which leads to an obvious contradiction, since in an end block of G, one can always find two adjacent edges that are not both incident with a cut-vertex of G. Thus, Corollary 1B follows from Theorem 1 and Theorem A. ∎

In fact, it was proved in [1] and in [4] that G has a spanning closed trail with $|V(G)| \geq 4$ and with a weaker lower bound $(2n+1)/3$, and in [8], Catlin showed that when $n = |V(G)| \geq 20$, then bound in (2) can be lowered $(2n-9)/5$.

Theorem 2. *Let G be a 2-edge-connected triangle-free simple graph with $n \geq 33$ vertices. If for every pair of distinct nonadjacent vertices $u, v \in V(G)$,*

$$d(u) + d(v) > \frac{n}{3}, \tag{3}$$

then for every pair of edges $e, e' \in E(G)$, exactly one of the following holds:

(i) *G has a spanning (e, e')-trail;*

(ii) *$\{e, e'\}$ is an essential edge-cut of G.*

Corollary 2. *Let G be a graph satisfying the hypothesis of Theorem 2, then $L(G)$ is either hamiltonian connected or has a vertex-cut of size 2.*

Theorem 1, Theorem 2, and Corollary 2, are best possible in some sense. Let $s \geq 10$ be an integer, and let $G(s)$ and $G(s, s)$ be defined as follows.

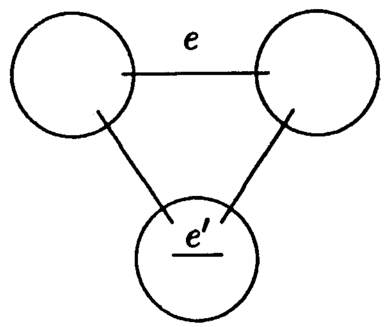

Figure 1: $G(s)$ or $G(s, s)$ with edges e and e'.

For $G(s)$, each circle in Figure 1 denotes a complete subgraph K_s, and a line joining two circles denotes a single edge joining two vertices in two distinct K_s's. Let $n = |V(G(s))| = 3s$. Apparently for every pair of nonadjacent u, $v \in V(G(s))$, $d(v)+d(u) \geq (2n)/3-2$. But for the given edges e, e', neither (i) nor (ii) of Theorem 1 holds. We then obtain $G(s, s)$ by replacing each circle in Figure 1 by a complete bipartite subgraph $K_{s,s}$ and by arranging the 3 edges between the 3 $K_{s,s}$'s so that the resulting graph is a bipartite one. Let $n = |V(G(s, s))| = 6s$ this time. Then for every pair of nonadjacent u, $v \in V(G(s, s))$, $d(v) + d(u) \geq n/3$. But for the given edges e, e', neither (i) nor (ii) of Theorem 2, nor the conclusion of Corollary 2, holds.

We shall also consider the computational complexity of the following decision problem: given a graph G and a pair of edges e, e', does G have a spanning (e, e')-trial?

Theorem 3. *The problem to determine if G has a spanning (e, e')-trail is NP-complete.*

3. Collapsible and reduced graphs.

Let G be a graph and let $X \subseteq E(G)$. The *contraction* G/X is the graph obtained from G by identifying the ends of each edge of X and deleting the resulting loops. If H is a subgraph of G, then we use G/H for $G/E(H)$.

Let $O(G)$ denote the set of vertices of odd degree in G. A graph G is *eulerian* if G is connected and $O(G) = \emptyset$. A graph is *supereulerian* if it has a spanning eulerian subgraph. Let $R \subseteq V(G)$ be a subset with even cardinality. An *R-subgraph* of G is a subgraph Γ of G such that $G - E(\Gamma)$ is connected and such that $O(\Gamma) = R$. A graph G is *collapsible* if for every $R \subseteq V(G)$ with $|R|$ even, G has an R-subgraph. Note that by definition, K_1 is both collapsible and supereulerian. In [5], Catlin proved that every graph G has a unique collection of maximal collapsible subgraphs, say $H_1, H_2, \ldots, H_c$. Thus, the graph $G' = G/(\cup_{i=1}^{c} E(H_i))$ is unique, and is called the *reduction* of G. A vertex v in the reduction of G is *trivial* if its preimage in G under the contraction is a K_1 in G. A graph is *reduced* if it is the reduction of some graph.

Theorem E. *(Catlin [5]) Let G be a graph, and let $F(G)$ denote the minimum number of extra edges that must be added to G so that the resulting graph has 2 edge-disjoint spanning trees.*

(i) *Let H be a collapsible subgraph of G. Then G is supereulerian if and only if G/H is supereulerian; and G is collapsible if and only if G/H is collapsible.*

(ii) *G is reduced if and only if G has no nontrivial collapsible subgraphs; if and only if the reduction of G is G itself. In particular, a reduced graph does not contain 2-cycles and 3-cycles.*

(iii) *G is collapsible if and only if the reduction of G is K_1.*

(iv) *If G has 2 edge-disjoint spanning trees, that is $F(G) = 0$, then G is collapsible.*

(v) *If G is reduced, then $\delta(G) \leq 3$.*

(vi) *If G is reduced and if $F(G) = 1$, then $G = K_2$.*

In [9], it is noted that if G is reduced, then

$$F(G) = 2|V(G)| - |E(G)| - 2. \tag{4}$$

The following result, conjectured by Catlin in [3] and recently proved by Catlin, Han, and Lai, will be applied in this paper.

Theorem F. *(Catlin, Han, and Lai [10]) If G is a connected reduced graph with $F(G) \leq 2$, then either $G = K_1$, or $G = K_2$, or there is an integer $t \geq 1$ such that $G = K_{2,t}$.*

4. The proofs.

The following notation and terminology will be used in this section. For a graph G and an integer $i \geq 1$, $D_i(G)$ denotes the number of vertices of degree i in G.

We say that an edge $e \in E(G)$ is *subdivided* when it is replaced by a path of length 2 whose internal vertex, denoted by $v(e)$, has degree 2 in the resulting graph. This process is called *subdividing* e. For a graph G and distinct edges e, $e' \in E(G)$, let $G(e, e')$ denote the graph obtained from G by subdividing both e and e'. Thus,

$$V(G(e,e')) - V(G) = \{v(e), v(e')\}.$$

The reason for introducing $G(e, e')$ can be found in Lemma 1 below.

Lemma 1. *For a graph G and $e, e' \in E(G)$, G has a spanning (e, e')-trail if and only if either $G(e, e')$ has a spanning $(v(e), v(e'))$-trail, or both e and e' are incident with the same vertex v in G such that $G(e, e') - v$ has a spanning $(v(e), v(e'))$-trail.*

Proof: The proof is straightforward and so is omitted. ∎

Lemma 2. *Let G be a reduced graph with n vertices. Then*

$$2F(G) + 4 \leq \sum_{i=1}^{3} (4 - i)|D_i(G)|. \tag{5}$$

Proof: This follows by counting the incidences of G and by (4). ∎

Lemma 3. *Let G be a graph and let G' be the reduction of G. For vertices u, $v \in V(G)$, define u', v' to be vertices in G' whose preimages contain u and v, respectively. (Note even $v \neq v$, it may still happen that $u' = v'$). Then G has a spanning (u, v)-trail if and only if G' has a spanning (u', v')-trail.*

Proof: Let u, v, u', v', and G satisfy the hypothesis of Lemma 3. Let x be a vertex not in $V(G)$. Define a new graph H from G with $V(H) = V(G) \cup \{x\}$ and $E(H) = E(G) \cup \{ux, xv\}$. Then G has a spanning (u, v)-trail if and only if H is supereulerian, if and only if the reduction of H is supereulerian (by (i) of Theorem E), if and only if G' has a spanning (u', v')-trail. ∎

In the proof below we often need to go back and forth from subgraphs L' of $G(e, e')$ and subgraphs L of G. For any subgraph L' of $G(e, e')$ such that $d_{L'}(v(e)) = 2$ whenever $v(e) \in V(L')$ ($d_{L'}(v(e') = 2$ whenever $v(e') \in V(L')$), let L denote the corresponding subgraph of G such that $L = L'$ if $V(L') \cap \{v(e), v(e')\} = \emptyset$, and L is the graph obtained from L' by contracting exactly one edge incident with $v(e)$ if $v(e) \in V(L')$ (with $v(e')$ if $v(e') \in V(L')$). We say that L is obtained from L' by ***undoing the subdivision***. For any $v \in V(G)$, the ***neighborhood*** of v in G, denoted by $N(v)$, consists of the vertices in G that are adjacent to v.

Proof of Theorem 1: Suppose that G, e, e' satisfy the hypothesis of Theorem 1. Let G'' denote the reduction of $G(e,e')$. If $G(e,e')$ is collapsible, that is $G'' = K_1$, then by Lemma 3, $G(e,e')$ has a spanning $(v(e), v(e'))$-trail and so (i) of Theorem 1 follows from Lemma 1. Hence, we assume that $G'' \neq K_1$. Let w, w' denote the vertices in G'' whose preimages contain $v(e)$, $v(e')$, respectively. Thus, when w and w' are trivial vertices, $w = v(e)$ and $w' = v(e')$.

By $\kappa'(G) \geq 2$, $D_1(G'') = \emptyset$. By $\kappa'(G) \geq 2$ and by (vi) of Theorem E, $F(G'') \geq 2$, and so by Lemma 2, $|D_2(G'') \cup D_3(G'')| \geq 4$, where equality holds only if $D_3(G'') = \emptyset$.

Claim 1: Let $v_{H'} \in D_2(G'') \cup D_3(G'')$ be a nontrivial vertex with preimage H' in $G(e,e')$, and let H be the subgraph of G obtained from H' by undoing the subdivision. If $D_2(G'') \cup D_3(G'')$ has a trivial vertex $v' \notin \{w, w'\}$, then $|V(H)| > 2n/3 - 5$. Moreover, if $v_{H'} \notin \{w, w'\}$, then $|V(H)| \geq 2n/3 - 3$.

Note that H is a simple collapsible subgraph of G and so $|V(H)| \geq 3$. Choose a vertex $v \in V(H)$ such that $vv' \notin E(G)$ and v is incident with at most one edge in $E(G'')$. By $vv' \notin E(G)$, $|V(H)| \geq d(v)$ and so Claim 1 follows from (1).

Claim 2: If $D_2(G'') \cup D_3(G'')$ has a trivial vertex not in $\{v(e), v(e')\}$, then $D_2(G'') \cup D_3(G'')$ has at most one nontrivial vertex.

Claim 2 follows from Claim 1 and the hypothesis of $n \geq 27$.

Claim 3: $D_2(G'') \cup D_3(G'') - \{w, w'\}$ cannot have 3 trivial vertices.

By (ii) of Theorem E, G'' is reduced and so has no cycles of length less than 4. Thus, if $D_2(G'') \cup D_3(G'')$ has 3 trivial vertices other than $v(e)$, $v(e')$, then two of them must be nonadjacent and so by (1), $n < 12$, contrary to $n \geq 27$. This proves Claim 3.

Claim 4: $D_2(G'')$ has at most 2 nontrivial vertices.

Suppose that $D_2(G'')$ has three nontrivial vertices v_1', v_2', v_3' whose preimages in $G(e,e')$ are H_1', H_2', and H_3', respectively. Let H_i denote the subgraph of G obtained from H_i' by undoing the subdivision and let $n_i = |V(H_i)|$, $(1 \leq i \leq 3)$. Since $v_i' \in D_2(G'')$, each H_i has a vertex v_i that not incident with edges in G'' and so by (1),

$$n_i + n_j \geq 2 + d(v_i) + d(v_j) > \frac{2n}{3}. \tag{6}$$

Thus, $2n \geq 2\sum_{i=1}^{3} n_i > 2n$, a contradiction. This proves Claim 4.

If $F(G'') \leq 2$, then by Theorem F and by $\kappa'(G') \geq 2$, there is some integer $t \geq 2$ such that $G'' = K_{2,t}$. Assume first that $t = 2$. By Lemma 1 and Lemma 3, G has a spanning (e, e')-trail unless $v(e)$ and $v(e')$ are contained in the preimages of two distinct nonadjacent vertices of w, w' G''. In the latter case, at least one of the two vertices x and y in $V(G'') - \{w, w'\}$ is nontrivial by (1). If both x and y are nontrivial, then by Claim 4, w and w' must be trivial, whence (ii) of Theorem 1 holds. If x or y is trivial, then by Claim 2, both w and w' are trivial, whence G has a spanning (e, e')-trail.

Thus, we assume that $t \geq 3$. By Claim 4 and by $t \geq 3$, $D_2(G'')$ has at least $t-2$ trivial vertices. If $D_2(G'') - \{w, w'\}$ has a trivial vertex, then by Claim 2, $D_2(G'') \cup D_3(G'')$ has at most one nontrivial vertex. Thus, if $t \geq 5$, then $D_2(G'')$ must have at least two trivial vertices u and v (say), and so by (1), $4 = d(u) + d(v) > (2n+1)/3$, contrary to the assumption that $n \geq 27$. Similarly, if $t = 4$, then w, w' must be two trivial vertices in $D_2(G'')$, and so G'' has a spanning (w, w')-trail, which implies (i) of Theorem 1 by Lemma 1 and Lemma 3.

Therefore, we assume that $t = 3$ and that G'' does not have a spanning (w, w')-trail, whence w and w' cannot be both in $D_2(G'')$. Hence, we assume that $w \in D_3(G'')$, and so w is nontrivial, and that either $w' = w$ or $w' \in D_2(G'')$. If $D_2(G'') - \{w'\}$ has a trivial vertex, then by Claim 2 and $w \in D_3(G'')$ being nontrivial, $D_2(G'') - \{w'\}$ must have 2 trivial vertices, contrary to the assumption that $n \geq 27$, by (1). Note that by Claim 4, if $w = w'$, then $D_2(G'')$ must have a trivial vertex, which would lead to the same contradiction. It follows that $w' \in D_2$ and there are two nontrivial vertices $v_1, v_2 \in D_2(G'')$. Let v_3 denote the vertex in $D_3(G'') - \{w\}$. Let H_i $(1 \leq i \leq 3)$ denote the preimages of v_i in G, and let H_0 denote the subgraph of G obtained from the preimage of w in $G(e, e')$ by undoing the subdivision. If v_3 is trivial, then by Claim 1, we have

$$n - 1 \geq |V(H_0)| + |V(H_1)| + |V(H_2)| \geq 2n/3 - 5 + 4n/3 - 6 = 2n - 11,$$

and so $n \leq 10$, a contradiction. Thus, v_3 is also nontrivial. By choosing $u_i \in V(H_i)$ such that u_i is incident with as few edges in $E(G'')$ as possible, we have by (1)

$$|V(H_1)| + |V(H_2)| \geq d(u_1) + d(u_2) > \frac{2n}{3} - 2 \text{ and}$$
$$|V(H_0)| + |V(H_3)| \geq d(u_0) + d(u_3) > \frac{2n}{3} - 2 - 3,$$

and so $n \geq \sum_{i=0}^{3} |V(H_i)| \geq 4n/3 - 7$. It follows that $n \leq 21$, a contradiction.

Hence, we may assume that $F(G'') \geq 3$ and so by Lemma 2, $|D_2(G'') \cup D_3(G'')| \geq 5$ where equality holds only if $D_3(G'') = \emptyset$.

Case 1: $D_2(G'') \cup D_3(G'')$ has at least 4 nontrivial vertices.

Let H'_i, $(1 \leq i \leq 4)$ denote the preimages in $G(e, e')$ of the 4 nontrivial vertives in $D_2(G'') \cup D_3(G'')$. Let H_i denote the subgraph of G obtained from H'_i by undoing the subdivision. Simce G is simple, $|V(H_i)| \geq 3$, and so for H_i,

H_j, there are vertices $v_i \in V(H_i)$ and $v_j \in V(H_j)$ such that $v_iv_j \notin E(G)$ and each of v_i and v_j is incident with at most one edge in $E(G'')$. It follows by (1) that

$$2n \geq 2\sum_{i=1}^{4}|V(H_i)| > 4\left(\frac{2n}{3}-2\right) = \frac{8n}{3} - 8,$$

and so $n < 12$, a contradiction.

Case 2: $D_2(G'') \cup D_3(G'')$ has exactly 3 nontrivial vertices.

By Claim 4, we must have $D_3(G'') \neq \emptyset$. Thus, $|D_2(G'') \cup D_3(G'')| \geq 6$ and so there is a trivial vertex $v \in D_2(G'') \cup D_3(G'') - \{w, w'\}$. Now the conclusion of Claim 2 contradicts the hypothesis of Case 2.

Case 3: $D_2(G'') \cup D_3(G'')$ has at most two nontrivial vertices.

By Lemma 2 and by $F(G'') \geq 3$, $|D_2(G'') \cup D_3(G'')| \geq 5$. By Claim 2 and Claim 3, we must have $D_3(G'') = \emptyset$ and $v(e), v(e') \in D_2(G'')$, and $D_2(G'')$ must have exactly one nontrivial vertex and two trivial vertices other than $v(e)$, $v(e')$. Let v, u be the two trivial vertices in $D_2(G'') - \{v(e), v(e')\}$. It follows by (1) and $n \geq 27$ that

$$uv \in E(G). \tag{7}$$

If $V(G'') = D_2(G'')$, then G'' is a 5-cycle and so (ii) of Theorem 1 must hold. Otherwise, let H be the preimage in G of the unique nontrivial vertex in $D_2(G'')$. By Claim 1, $|V(H)| \geq 2n/3 - 3$. Pick a vertex y that is in the preimage of some vertex in $V(G'') - D_2(G'')$. We may assume that $yu \notin E(G)$ (or $yv \notin E(G)$), since that yu and yv are both in $E(G'')$ implies that G'' has a 3-cycle by (7), contrary to (ii) of Theorem E. By (1) and by $yu \notin E(G)$, we have $d(y) \geq 2n/3 - 4$. On the other hand, since $|N(y) \cap V(H)| \leq 1$, one has $d(y) \leq n - (|V(H)| - 1) - 2 < n/3 + 2$. This, together with $d(y) \geq 2n/3 - 4$, implies that $n < 18$, a contradiction.

This completes the proof of Theorem 1. ∎

Proof of Theorem 2: The proof of Theorem 2 is analogous to that of Theorem 1 and so it is omitted. ∎

Proof of Theorem 3: Consider a special case of the problem when G is a cubic graph. Let e, e' be given. Note that when e, e' are not adjacent in G, G has a spanning (e, e')-trail if and only if $G(e, e')$ has a spanning $(v(e), v(e'))$-trail by Lemma 1. If e, e' are not adjacent, then define G^* to be the graph obtained from $G(e, e')$ as indicated in Figure 2. If e, e' are adjacent in G, then define $G^* = G$. Thus, for any given e, $e' \in E(G)$, G has a spanning (e, e')-trail if and only if G^* is hamiltonian. In Theorem 2.2 of [11], Garey *et al*. show that the problem of determining if an undirected 3-regular graph is hamiltonian is NP-complete. Thus, this NP-complete problem reduces to a special case of the problem of determining if a graph has a spanning (e, e')-trail. ∎

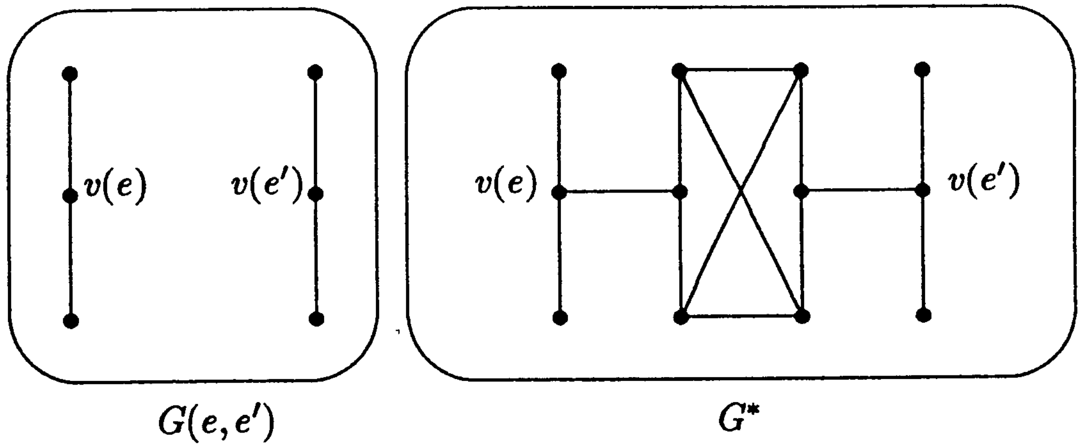

Figure 2: The graphs $G(e,e')$ and G^* with e, e' nonadjacent.

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