On Edge-integrity Maximal Graphs

Hong-Jian Lai and Xiankun Zhang
Department of Mathematics
West Virginia University
Morgantown, WV 26505

ABSTRACT. The edge-integrity of a graph G is given by $\min_{S \subseteq E(G)} \{|S| + m(G - S)\}$, where $m(G - S)$ denotes the maximum order of a component of $G - S$. Let $I'(G)$ denote the edge-integrity of a graph G. We define a graph G to be I'-maximal if for every edge e in $\overline{G}$, the complement of graph G, $I'(G + e) > I'(G)$. In this paper, some basic results of I'-maximal graphs are established, the girth of a connected I'-maximal graph is given and lower and upper bounds on the size of I'-maximal connected graphs with given order and edge-integrity are investigated. Also, the I'-maximal trees and unicyclic graphs are completely characterized.

1 Introduction

In this paper we consider finite undirected simple graphs. The edge-integrity of a graph attempts to measure the disruption caused by the removal of edges from the graph. The order of a component or graph is the number of its vertices, and we let $m(H)$ denote the maximum order of a component of graph H. The edge-integrity is defined as

$$I'(G) = \min_{S \subseteq E} \{|S| + m(G - S)\}.$$

Let G be a graph and $\overline{G}$ be the complement of G. G is I'-maximal iff $I'(G + e) > I'(G)$, for every edge e of $\overline{G}$. Let $M(k)$ denote the collection of all I'-maximal graphs with edge-integrity k and $M_n(k)$ denote the collection of all I'-maximal graphs with order n and edge-integrity k.

2 Some Basic Properties

Let $G=(V,E)$ be a simple graph with vertex set V and edge set E. For $S \subseteq E$, let $I'(S)$ denote $|S|+m(G-S)$. A set $S \subseteq E$ for which $I'(S)$ is minimized is called an I'-set of G. If $H_1, H_2, \cdots, H_r$ are the components of $G-S$ with $|V(H_i)|=n_i$ such that $n_1 \leq n_2 \leq \cdots \leq n_r$, we say that $G-S$ has type $(n_1, n_2, \cdots, n_r)$. For $V_1, V_2 \subseteq V(G)$, let $[V_1, V_2]_G = \{e = x_1x_2 \in E(G) | x_i \in V_i, i=1,2\}$. When G can be understood from the context, we write $[V_1, V_2]$ for $[V_1, V_2]_G$.

Proposition 2.1. *Assume $G \in M(k)$. Let $S \subseteq E(G)$ with $I'(G) = I'(S) = k$. Let $H_1, H_2, \cdots, H_r$ be the components of $G-S$ with type $(n_1, n_2, \cdots, n_r)$. Then $H_i \cong K_{n_i} (i = 1, \cdots, r)$.*

Proof: If there exists an edge $e \in E(\overline{H}_i)$, then in $G+e$, $I'(S)=k$, and so $I'(G+e) \leq k$, a contradiction. □

Proposition 2.2. *Assume $G \in M(k)$ and $S \subseteq E(G)$ is an I'-set. Let $H_1, H_2, \cdots, H_r$ be the components of $G-S$ with type $(n_1, n_2, \cdots, n_r)$. Assume that $r > 2$. For any i, j, we have*

$$n_r + |[V(H_i), V(H_j)]| < n_i + n_j.$$

Proof: We consider two cases.

Case 1: Suppose that there exist i, j such that $i, j < r$ and $n_r + |V(H_i), V(H_j)]| \geq n_i + n_j$. Then $|S| - |[V(H_i), V(H_j)]| + n_i + n_j \leq |S| + n_r$.

If $[V(H_i), V(H_j)] \neq \emptyset$, let $S' = S - [V(H_i), V(H_j)]$. Then

$$I'(S') = |S| - |[V(H_i), V(H_j)]| + max\{n_i + n_j, n_r\} \leq |S| + n_r = I'(S).$$

However, $H_i \cup H_j \cup [V(H_i), V(H_j)]$ can not be a complete subgraph, contrary to Proposition 2.1.

If $[V(H_i), V(H_j)] = \emptyset$, then $|S| + n_i + n_j \leq |S| + n_r$. Hence, for any edge $e \in [V(H_i), V(H_j)]_{\overline{G}}$, we have that $I'(G+e) \leq I'(G)$, contrary to the definition of I'-maximal graph.

Case 2: If $i = r$ or $j = r$, then we only need to prove that, for any $1 \leq i \leq r-1$, we have

$$|[V(H_r), V(H_i)]| < n_i.$$

Otherwise, suppose that there exists an integer i with $1 \leq i \leq r-1$ such that $|[V(H_r), V(H_i)]| \geq n_i$. Let $S' = S - |[V(H_r), V(H_i)]|$. We have

$$I'(S') = |S| - |[V(H_r), V(H_i)]| + n_r + n_i \leq |S| + n_r = I'(S).$$

Since $G[V(H_r), V(H_i)]$ is not a complete subgraph of G, we get a contradiction with Proposition 2.1. □

Corollary 2.3. *Let $G \in M(k)$ and $S \subseteq E(G)$ be an I'-set of G. Assume that $G - S$ has type $(n_1, n_2, \cdots, n_r)$ $(r > 2)$. Then $n_2 > n_r/2$.*

Proof: By Proposition 2.2, we obviously have $n_1 + n_2 > n_r$. If $n_2 \le n_r/2$, we have $n_1 + n_2 \le 2n_2 \le n_r$. Therefore $n_2 > n_r/2$. □

Since the removal of an edge which leaves an isolated vertex reduces the order of some component (not necessarily the largest) by one, we have the following proposition.

Proposition 2.4. *Let $G \in M(k)$ and $S \subseteq E(G)$ be an I'-set. And let $H_1, H_2, \cdots, H_r$ be the components of $G - S$ with type $(n_1, n_2, \cdots, n_r)$. If G is connected, then we have $n_1 \ge 2$.*

3 The girth of I'-maximal graphs

First, we construct two classes of graphs and prove three lemmas which we will use.

Let $H_0, H_1, \cdots, H_r$ be $r + 1$ complete graphs with $H_i \cong K_{p_i}$ $(0 \le i \le r)$ such that $p_0 \ge p_1 \ge p_2 \ge \cdots \ge p_r$. For each $i, 0 \le i \le r$, we choose one vertex v_i from $V(H_i)$ and construct a new graph $G(p_0; p_1, \cdots, p_r)$ as follows:

$V(G(p_0; p_1, \cdots, p_r)) = V(H_0) \cup V(H_1) \cup \cdots \cup V(H_r)$,

$E(G(p_0; p_1, \cdots, p_r)) = E(H_0) \cup E(H_1) \cup \cdots \cup E(H_r) \cup \{v_0v_1, v_0v_2, \cdots, v_0v_r\}$.

If $p_1 = p_2 = \cdots = p_j = p$, we simply denote this graph by $G(p_0; (p)_j, p_{j+1}, \cdots, p_r)$.

Lemma 3.1. *Suppose that G is a graph with unique I'-set $S \subseteq E(G)$ and $H_1, H_2, \cdots, H_r$ are the components of $G - S$ with $H_i \cong K_{n_i}$ $(1 \le i \le r)$. If for any i, j such that $1 \le i, j \le r$, $n_i + n_j > max\{n_1, n_2, \cdots, n_r\}$, then G is an I'-maximal graph.*

Proof: For any $e \in E(\overline{G})$, we consider graph $G + e$. Assume that S' is an I'-set of $G + e$.

Case 1: $e \in S'$. Then $I'(G + e) = |S'| + m(G + e - S') = |S' - e| + 1 + m(G - (S' - e)) \ge I'(G) + 1$.

Case 2: $e \notin S'$. If $S' = S$, then $I'(G + e) = |S| + m(G + e - S) > |S| + m(G - S) = I'(G)$; if $S' \ne S$, then $I'(G + e) = |S'| + m(G + e - S') \ge |S'| + m(G - S') > I'(G)$. □

Lemma 3.2. *Suppose that $p_r \ge 2$ and for any i, j such that $1 \le i, j \le r$, $p_i + p_j > p_0$. Then*

(i) $I'(G(p_0; p_1, \cdots, p_r)) = p_0 + r$.

(ii) The I'-set of $G(p_0; p_1, \cdots, p_r)$ is unique.

(iii) $G(p_0; p_1, \cdots, p_r)$ *is* I'*-maximal.*

Proof: (i) Let $\Delta(G)$ be the maximum degree of a vertex of graph G. We have, for any graph G, $I'(G) \geq \Delta(G) + 1$ (see [3]). Thus we get that $I'(G(p_0; p_1, \cdots, p_r)) \geq p_0 + r$. Let $S = \{v_0v_1, v_0v_2, \cdots, v_0v_r\}$. Then $I'(S) = p_0 + r$ and so $I'(G(p_0; p_1, \cdots, p_r)) = p_0 + r$.

(ii) Suppose that S' is an I'-set of G such that $S' \neq S$. Let c be the number of H_i's that S' intersects. We might as well assume $S' \cap E(H_{i_t}) \neq \emptyset$, for some $1 \leq t \leq c \leq r+1$. Note that since $H_{i_t} \cong K_{p_{i_t}}$ and $p_{i_t} \geq 2$, $|S' \cap E(H_{i_t})| \geq p_{i_t} - 1 \geq 1$, $(1 \leq t \leq c)$. If $S' \cap E(H_0) = \emptyset$, then the component of $G - S'$ containing H_0 also contains at least one vertex from each H_{i_t}, $1 \leq t \leq c$, and so this component has at least $p_0 + c$ vertices. Hence $I'(S) = I'(S') = |S'| + m(G - S') \geq |S| - c + \Sigma_{t=1}^{c}(p_{i_t} - 1) + p_0 + c \geq I'(S) + c > I'(S)$, a contradiction. Thus we assume $H_{i_c} = H_0$ and $X = S' \cap E(H_0) \neq \emptyset$.

Let H_0' and H_0'' be the two parts of $H_0 - X$ such that $v_0 \in V(H_0')$. Then since $H_0 \cong K_{p_0}$, $|X| = |V(H_0')|(p_0 - |V(H_0')|)$. Since $v_0 \in V(H_0')$, the component of $G - S'$ containing $V(H_0')$ also contains at least one vertex from each H_{i_t}, $1 \leq t \leq c-1$, and so $m(G - S') \geq |V(H_0')| + c$. Hence $I'(S) = I'(S') = |S'| + m(G - S') \geq |S| - c + \Sigma_{t=1}^{c-1}(p_{i_t} - 1) + |V(H_0')|(p_0 - |V(H_0')|) + |V(H_0')| + c \geq I'(S) + c - 1 \geq I'(S)$. This implies $c = 1$ and $V(H_0') = \{v_0\}$.

Let $E_{H_0}(v_0) \subseteq E(H_0)$ be the set of edges in H_0 incident with v_0. Since $H_0 \cong K_{p_0}$, $|E_{H_0}(v_0)| = p_0 - 1$. Since $c = 1$ and $V(H_0') = \{v_0\}$, $S' = S \cup E_{H_0}(v_0)$ and $G - S'$ has components $H_1, H_2, \cdots, H_r$ and $H_0 - v_0$. Hence $I'(S) = I'(S') \geq |S| + (p_0 - 1) + |V(H_0 - v_0)| = r + 2p_0 - 2 = I'(S) + (p_0 - 2)$, and so $p_0 = 2$. Since $p_r \geq 2$, we conclude that $p_0 = p_1 = \cdots = p_r = 2$. Therefore $I'(S) = I'(S') = |S'| + m(G - S') = (r+1) + 2 > r + 2 = I'(S)$, a contradiction. This proves (ii).

(iii) This result follows from (ii) and Lemma 3.1. □

Let $T(d_1, d_2)$ be a tree with vertex set $\{u_1, u_2, v_i, w_i, v_j', w_j' | 1 \leq i \leq d_1, 1 \leq j \leq d_2\}$ and edge set $\{u_1u_2, u_1w_i, w_iw_i', u_2v_j, v_jv_j' | 1 \leq i \leq d_1, 1 \leq j \leq d_2\}$; let Δ_1 be a graph obtained from K_3 with vertex set $V(K_3) = \{v_1, v_2, v_3\}$ by adding three new vertices: u_1, u_2, u_3, and three new edges: u_1v_1, u_2v_2, u_3v_3; and let $\Delta(d_1, d_2, d_3)$ be a graph obtained from K_3 with vertex set $V(K_3) = \{v_1, v_2, v_3\}$ by adding a new vertex set $\{x_i, x_i', y_j, y_j', z_k, z_k' | 1 \leq i \leq d_1, 1 \leq j \leq d_2 \text{ and } 1 \leq k \leq d_3\}$ and a new edge set $\{v_1x_i, x_ix_i', v_2y_j, y_jy_j', v_3z_k, z_kz_k' | 1 \leq i \leq d_1, 1 \leq j \leq d_2 \text{ and } 1 \leq k \leq d_3\}$.

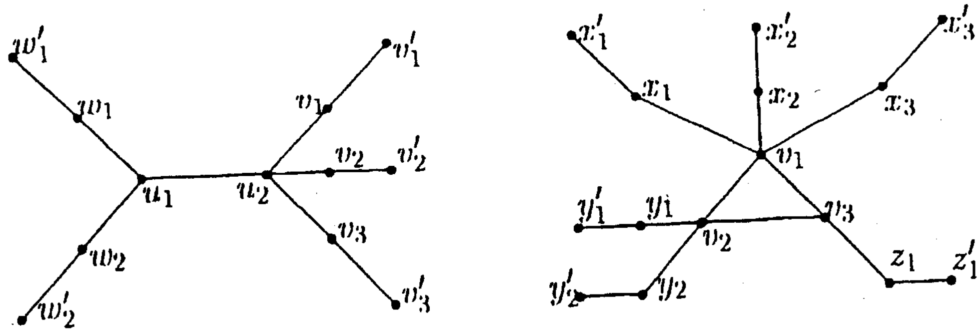

Figure 1. The graphs $T(2,3)$ and $\Delta(3,2,1)$

Lemma 3.3. *If one of d_1, d_2 and d_3 is equal to zero and at least one of them is not equal to zero, then $\Delta(d_1,d_2,d_3)$ is I'-maximal.*

Proof: Without loss of generality, we suppose that $d_3 = 0$. Let $G = \Delta(d_1,d_2,0)$. Assume that $S \subseteq E(G)$ is an I'-set of G. Let $S_1 = \{v_1x_i, v_2y_j | 1 \leq i \leq d_1, 1 \leq j \leq d_2\}$. We consider three cases.

Case 1: Only one edge e of v_1v_2, v_2v_3 and v_3v_1 belongs to S. We easily check that $S_1 \subseteq S$ and so $I'(S_1) < I'(S)$, a contradiction.

Case 2: Only two edges e_1 and e_2 of v_1v_2, v_2v_3 and v_3v_1 belong to S. Obviously, $\{e_1,e_2\} \neq \{v_2v_3, v_3v_1\}$. We might as well assume that $\{e_1,e_2\} = \{v_1v_2, v_3v_1\}$ and $S = \{v_1x_i, v_2y_j | 1 \leq i \leq r_1, 1 \leq j \leq r_2\} \cup \{e_1,e_2\}$, where $r_1 \leq d_1, r_2 < d_2$. Then $I'(S) = 2+r_1+r_2+\max\{2(d_1-r_1)+2, 2(d_2-r_2)+1\}$ $=\max\{4+2d_1-r_1+r_2, 3+2d_2+r_1-r_2\} > I'(S_1)$, a contradiction.

Case 3: $\{v_1v_2, v_2v_3, v_3v_1\} \subseteq S$. We might as well assume that $S = \{v_1x_i, v_2y_j | 1 \leq i \leq r_1, 1 \leq j \leq r_2\} \cup \{v_1v_2, v_2v_3, v_3v_1\}$, where $r_1 < d_1, r_2 < d_2$. Then $I'(S) > I'(S_1)$, a contradiction.

Hence, S does not contain v_1v_2, v_2v_3 or v_3v_1. Obviously, S can be only S_1. The result follows from Lemma 3.1. □

Proposition 3.4. *Let G be a connected I'-maximal graph. Suppose that there exists an I'-set $S \subseteq E(G)$ such that $G - S$ has components: $H_1, H_2, \cdots, H_c$ with $c = \lfloor \frac{n}{2} \rfloor$.*

(i) *If n is even, then $G \cong G(2;(2)_{c-1})$ or Δ_1.*

(ii) *If n is odd, then $G \cong \Delta(d_1,d_2,0)$, where $d_1 + d_2 = c - 1$.*

Proof: (i) If n is even, then $c = n/2$. Hence, each component $H_1, H_2, \cdots, H_c$ of $G - S$ is a K_2.

Since G is connected, we might as well assume $|[V(H_1), V(H_2)]| \neq 0$.

Claim 1: For any $i, j \geq 3$ $(i \neq j)$, $[V(H_i), V(H_j)] = \emptyset$.

If not, let $e \in [V(H_1), V(H_2)]$ and $e_1 \in [V(H_i), V(H_j)]$ for some $i, j \geq 3$ $(i \neq j)$. Let $S' = S - \{e, e_1\}$. Then $I'(S') = |S| - 2 + (2+2) = |S| + 2 = I'(S)$, contrary to Proposition 2.1.

Claim 2: If $|[V(H_1), V(H_i)]| \neq 0$ $(3 \leq i \leq c)$, then $|[V(H_2), V(H_j)]| = 0$, for any j $(3 \leq j \leq c$ and $j \neq i)$; or if $|[V(H_2), V(H_i)]| \neq 0$ $(3 \leq i \leq c)$, then $|[V(H_1), V(H_j)]| = 0$, for any j $(3 \leq j \leq c$ and $j \neq i)$.

Otherwise, we assume $e_1 \in [V(H_1), V(H_i)], e_2 \in [V(H_2), V(H_j)]$, where $3 \leq i, j \leq c$, and $i \neq j$. Let $S' = S - \{e_1, e_2\}$. Then $I'(S') = |S| - 2 + 4 = |S| + 2 = I'(S)$, contrary to Proposition 2.1.

Claim 3: For any $1 \leq i, j \leq c$, and $i \neq j$, $|[V(H_i), V(H_j)]| \leq 1$.

Otherwise, we can choose two edges $e_1, e_2 \in [V(H_i), V(H_j)]$. Let $S' = S - \{e_1, e_2\}$. Similarly, we get a contradiction.

Now we consider three cases.

Case 1: $c \geq 4$.

Without loss of generality, we assume $|[V(H_2), V(H_i)]| = 0$, for any i $(3 \leq i \leq c)$, and $V(H_1) = \{u_1, u_2\}$. Let $d(u_1) = d_1 + 1, d(u_2) = d_2 + 1$, where $d_1 + d_2 = c - 1$. Then this I'-maximal graph is isomorphic to tree $T(d_1, d_2)$. We claim that $d_1 = 0$ or $d_2 = 0$. Otherwise, assume that $[u_1, V(H_i)] \neq \emptyset$ and $[u_2, V(H_j)] \neq \emptyset$, where $i \neq j$. Let $e_1 \in [u_1, V(H_i)]$ and $e_2 \in [u_2, V(H_j)]$. Let $S' = S \cup \{u_1u_2\} - \{e_1, e_2\}$. Then $I'(S') = I'(S)$, and so S' is also an I'-set of $T(d_1, d_2)$, which yields a contradiction to Proposition 2.1. Thus G is isomorphic to $G(2; (2)_{c-1})$.

Case 2: $c = 3$.

If $|[V(H_1), V(H_3)]| = 0$ or $|[V(H_2), V(H_3)]| = 0$, then this case returns to case 1 and G is isomorphic to $G(2; (2)_2)$. If $|[V(H_1), V(H_3)]| \neq 0$ and $|[V(H_2), V(H_3)]| \neq 0$, by claim 3, $|[V(H_1), V(H_3)]| = 1$ and $|[V(H_2), V(H_3)]| = 1$. Hence, we need only consider four graphs: G_1, G_2, C_6 and Δ_1, where G_1 is a graph with vertex set $\{v_1, v_2, v_3, v_4, u_1, u_2\}$ and edge set $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, u_1v_1, u_2v_2\}$, G_2 is a graph obtained from a 5-cycle C_5 with vertex set $\{v_i | 1 \leq i \leq 5\}$ by adding a new vertex v_0 and a new edge v_0v_1 and C_6 is a 6-cycle. We easily check that G_1, G_2 and C_6 are not I'-maximal and Δ_1 is I'-maximal.

Case 3: $c = 2$.

If $r = 2$, this I'-maximal graph is a path with order 4, namely graph $G(2; 2)$.

Hence, the result follows from Lemma 3.2.

(ii) If n is odd, then $c = (n - 1)/2$. Hence, one of $H_1, H_2, ..., H_c$ is isomorphic to K_3 and the others are isomorphic to K_2.

Without loss of generality, we suppose $H_c \cong K_3$ and $H_i \cong K_2$, where $1 \leq i \leq c - 1$. We claim that $[V(H_i), V(H_j)] = \emptyset, 1 \leq i, j \leq c - 1$ and $i \neq j$. Otherwise, let $e \in [V(H_i), V(H_j)]$ and $S' = S - e$. We have $I'(S') = I'(S)$, contrary to Proposition 2.1. Now we prove that, for any i $(1 \leq i \leq c - 1)$, $|[V(H_i), V(H_c)]| = 1$. If not, let $e_1, e_2 \in [V(H_i), V(H_c)]$ and $S' = S - \{e_1, e_2\}$. Then $I'(S') = I'(S)$, contrary to Proposition 2.1.

Hence, G is isomorphic to one of three kinds of graphs: $\Delta(d_1,0,0)$, $\Delta(d_1,d_2,0)$ and $\Delta(d_1,d_2,d_3)$, where $d_1,d_2,d_3 \neq 0$, and so we need only to consider graphs: $\Delta(d_1,0,0)$, $\Delta(d_1,d_2,0)$ and $\Delta(d_1,d_2,d_3)$.

Since $d_1,d_2 \neq 0$, by Lemmas 3.2 and 3.3, $\Delta(d_1,0,0)$ and $\Delta(d_1,d_2,0)$ are I'-maximal. For graph $\Delta(d_1,d_2,d_3)$ $(d_1,d_2,d_3 \neq 0)$, let $S' = S \cup \{v_1v_2, v_2v_3, v_3v_1\}$-$\{v_1x_1, v_2y_1, v_3z_1\}$. Then $I'(S') = I'(S)$, contrary to Proposition 2.1. □

Corollary 3.5. *Let T be a tree. Then T is I'-maximal iff $T \cong G(2;(2)_d)$, $(d \geq 1)$.*

Proof: Assume tree $T \in M(k)$. Let $S \subseteq E(T)$ be an I'-set of T. Let $H_1, H_2, \cdots, H_c$ be the components of $T - S$. By Propositions 2.1 and 2.4, we have $H_i \cong K_2$ $(i = 1,2,\cdots,c)$. Hence, we know that any tree with an odd number of vertices is not I'-maximal. By Proposition 3.4, this I'-maximal tree must be isomorphic to tree $G(2;(2)_d)$, where $d = c-1$. By Lemma 3.2, for each integer $d \geq 1$, $G(2;(2)_d)$ is an I'-maximal tree, and so the proof is complete. □

Corollary 3.6. *Any connected I'-maximal graph, except $G(2;(2)_d)$ $(d \geq 1)$, has girth 3.*

Proof: Let $G = (V,E)$ be an I'-maximal graph with girth larger than 3 and $S \subseteq E(G)$ be an I'-set of G. Let $H_1, H_2, \cdots, H_c$ be the components of $G - S$.

By the assumption that G has no 3-cycle and by Propositions 2.1 and 2.4, we know that $H_i \cong K_2$ $(i = 1,2,\cdots,c)$, where $c = |V(G)|/2$. By Proposition 3.4, G is isomorphic to the tree $G(2;(2)_d)$, where $d = c-1$. □

Corollary 3.7. *Let G be a unicyclic connected graph. G is I'-maximal if and only if $G \cong \Delta_1$ or $\Delta(d_1,d_2,0)$, where d_1 and d_2 are not both zero.*

Proof: Let $S \subseteq E(G)$ be an I'-set of G and $H_1, H_2, \cdots, H_c$ be the components of $G - S$. Since G is a unicyclic connected graph, $c = \lfloor \frac{n}{2} \rfloor$. This corollary follows from Proposition 3.4. □

4 The minimum size of an I'-maximal graph

Let $m(n,k) = min\{|E(G)| : G \in M_n(k) \text{ and is connected}\}$.

Lemma 4.1. *Assume $G \in M_n(k)$. Let $S \subseteq E(G)$ be an I'-set of G. Let $H_1, H_2, \cdots, H_c$ be the components of $G - S$ with type $(n_1, n_2, \cdots, n_c)$. Then*

$$m(n,k) \geq \left\lceil \frac{(n-1)^2}{2c} - \frac{(n+1)}{2} + k \right\rceil .$$

Proof: By Proposition 2.1, we have

$$|E(G)| = \Sigma_{i=1}^{c}\binom{n_i}{2} + |S| = \Sigma_{i=1}^{c}\binom{n_i}{2} + k - n_c.$$

Combining the constraint $n_1 + n_2 + \cdots + n_c = n$, the minimum can be determined by using calculus. When $n_1 = n_2 = \cdots = n_{c-1} = (n-1)/c, n_c = (n-1+c)/c$, it is attained, namely, $|E(G)| \geq \frac{(n-1)^2}{2c} - \frac{(n+1)}{2} + k$. Since $m(n,k)$ is an integer, $m(n,k) \geq \left\lceil \frac{(n-1)^2}{2c} - \frac{(n+1)}{2} + k \right\rceil$. □

Note that the maximum value of c in Lemma 4.1 is $\lfloor \frac{n}{2} \rfloor$. When $c = \lfloor \frac{n}{2} \rfloor$, the I'-maximal graphs are what Proposition 3.4 described. Hence, by Proposition 3.4, we have

Corollary 4.2.

$$m(n,k) = \begin{cases} 2r-1, & \text{if } (n,k) = (2r, r+1), r \geq 2 \\ 2r+1, & \text{if } (n,k) = (2r+1, r+2), r \geq 2 \\ 6, & \text{if } (n,k) = (6,5). \end{cases} \quad (1)$$

Lemma 4.3. *Let G be an I'-maximal graph with order n and $S \subseteq E(G)$ be an I'-set of G. Suppose that $G - S$ has type $(n_1, n_2, \cdots, n_c)$ and $c = \lfloor \frac{n}{2} \rfloor - 1$.*

(i) *If n is even and $n \geq 6$, then $G - S$ has type $(2, 2, \cdots, 2, 3, 3)$ or $(2, 4)$.*

(ii) *If n is odd and $n \geq 7$, then $G - S$ has type $(2, 2, \cdots, 2, 3, 3, 3)$ $(n \geq 9)$, or $(2, 3, 4)$, or $(2, 5)$ or $(3, 4)$.*

Proof: (i) Suppose n is even and $n \geq 6$. Since G is I'-maximal, by Proposition 2.4 , $n_1 \geq 2$.

Claim: If $c = n/2 - 1$, then $2 < n_c < 5$.

If $n_c = 2$, then $c = n/2$; if $n_c \geq 5$, then $c - 1 \leq (n - n_c)/2 \leq (n-5)/2$ and so $c \leq (n-3)/2 < n/2 - 1$, contrary to the assumption that $c = n/2 - 1$. This proves the Claim by contradiction.

Hence $n_c = 4$ or 3. If $n_c = 4$, since $n - 4 = 2(c-1)$, then $n_{c-1} = 2$. By Corollary 2.3, we know that its type must be $(2, 4)$. If $n_c = 3$, since n is even, then $n_{c-1} = 3$. Since $n - 6 = 2(c-2)$, $n_{c-2} = 2$. Hence its type is $(2, 2, \cdots, 2, 3, 3)$.

(ii) Suppose n is odd and $c = \frac{n-1}{2} - 1$, where $n \geq 7$. By similar argument, we know that $n_c = 5$, 4 or 3. By Corollary 2.3, if $n_c = 5$, then its type is $(2, 5)$; if $n_c = 4$, then its type is $(2, 3, 4)$ or $(3, 4)$; if $n_c = 3$ and $(n \geq 9)$, then its type is $(2, 2, \cdots, 2, 3, 3, 3)$. □

Theorem 4.4. *Suppose that $(n,k) \neq (2c, c+1), (2c+1, c+2)$ and $(6,5)$, where $c \geq 2$.*

(i)

$$m(n,k) \geq \begin{cases} \left\lceil \frac{(n-1)^2}{2\lfloor \frac{n}{2} \rfloor - 2} - \frac{(n-1)}{2} + \lfloor \frac{n}{2} \rfloor \right\rceil, & \text{if } \lfloor \frac{n}{2} \rfloor \leq k-1, \\ \left\lceil \frac{(n-1)^2}{2(k-2)} - \frac{(n+1)}{2} + k \right\rceil, & \text{if } \lfloor \frac{n}{2} \rfloor > k-1. \end{cases} \quad (2)$$

(ii) *If $\lfloor \frac{n}{2} \rfloor \leq k-1$, then equality in (2) holds if and only if $(n,k) = (2c+3, c+2)$ $(c \geq 3)$.*

(iii) *If $\lfloor \frac{n}{2} \rfloor > k-1$, then equality in (2) holds if and only if $(n,k) = (3c, c+2)$ $(c \geq 4)$, $(3c-1, c+2)$ $(c \geq 5)$, $(16,8)$, $(18,9)$, $(19,9)$, $(20,10)$ or $(22,10)$.*

Proof: (i) Assume $G \in M_n(k)$. Let $S \subseteq E(G)$ be an I'-set of G. Let $H_1, H_2, \cdots, H_c$ be the components of $G - S$ with type $(n_1, n_2, \cdots, n_c)$. By Proposition 3.4 and Corollary 4.2, we know $c \leq \lfloor \frac{n}{2} \rfloor - 1$ and so $n_c \geq 3$. Since G is connected, $k \geq (c-1)+3$, so $c \leq k-2$. Hence, we have

$$c \leq min\{\lfloor \frac{n}{2} \rfloor - 1, k-2\}. \quad (3)$$

If $\lfloor \frac{n}{2} \rfloor \leq k-1$, then $c \leq \lfloor \frac{n}{2} \rfloor - 1$ and $k \geq \lfloor \frac{n}{2} \rfloor + 1$; if $\lfloor \frac{n}{2} \rfloor > k-1$, then $c \leq k-2$. Hence the inequality (2) follows from Lemma 4.1.

Next, we shall determine all values of n and k for which $m(n,k)$ reaches its minimum.

(ii) Suppose that $\lfloor \frac{n}{2} \rfloor \leq k-1$.

Claim 1: If equality holds in (2), then $(n,k) = (2c+3, c+2)$ $(c \geq 3)$.

Let $G \in M_n(k)$ and $S \subseteq E(G)$ be an I'-set of G such that $G - S$ has type $(n_1, n_2, \cdots, n_c)$. We shall show that if $|E(G)|$ reaches the lower bound in (2), then $c = \lfloor \frac{n}{2} \rfloor - 1$, $|S| = \lfloor \frac{n}{2} \rfloor - 2$ and $n_c = 3$.

If $\lfloor \frac{n}{2} \rfloor \leq k-1$, then $c \leq \lfloor \frac{n}{2} \rfloor - 1$ and $k \geq \lfloor \frac{n}{2} \rfloor + 1$. Note that a decrease of c by 1 or an increase of k by 1 must cause an increase of the lower bound in Lemma 4.1 by at least 1. Hence, by Lemma 4.1 and (2), if $G \in M_n(k)$ and $|E(G)|$ reaches the lower bound, then $c = \lfloor \frac{n}{2} \rfloor - 1$ and $k = \lfloor \frac{n}{2} \rfloor + 1$. Since G is connected, $|S| \geq c - 1 = \lfloor \frac{n}{2} \rfloor - 2$. On the other hand, $|S| = k - n_c \leq \lfloor \frac{n}{2} \rfloor + 1 - 3 = \lfloor \frac{n}{2} \rfloor - 2$. So $|S| = \lfloor \frac{n}{2} \rfloor - 2$ and $n_c = 3$.

Thus $k = |S| + n_c = c + 2$. By Lemma 4.3, since $n_c = 3$, $n = 2c+2$ or $2c+3$. Therefore $(n,k) = (2c+2, c+2)$ $(c \geq 2)$, $or (2c+3, c+2)$ $(c \geq 3)$. Note that $(n,k) \neq (2r, r+1)$, where $r \geq 2$. Hence $(n,k) = (2c+3, c+2)$ $(c \geq 3)$.

Claim 2: If $(n,k) = (2c+3, c+2)$ $(c \geq 3)$, equality in (2) holds.

By Lemma 3.2, $G(3;3,3,2,\cdots,2)$ are I'-maximal connected graphs with $(n,k)=(2c+3,c+2)$ $(c\geq 3)$. The size of $G(3;3,3,2,\cdots,2)$ is $2c+5$ $(c\geq 3)$. On the other hand, when $(n,k)=(2c+3,c+2)$ $(c\geq 3)$, $\left\lceil \frac{(n-1)^2}{2\lfloor\frac{n}{2}\rfloor-2}-\frac{(n-1)}{2}+\lfloor\frac{n}{2}\rfloor\right\rceil=2c+5$. Hence, when $(n,k)=(2c+3,c+2)$ $(c\geq 3)$, $m(n,k)$ reaches the lower bound.

(iii) Suppose that $\lfloor\frac{n}{2}\rfloor>k-1$.

Claim 3: If equality holds in (2), then $(n,k)=(3c,c+2)$ $(c\geq 4),(3c-1,c+2)$ $(c\geq 5)$, (16,8), (18,9), (19,9), (20,10) or (22,10).

Let $G\in M_n(k)$ and $S\subseteq E(G)$ be an I'-set of G such that $G-S$ has type $(n_1,n_2,\cdots,n_c)$. We shall first prove that if equality holds in (2), then $n_c=3$ and $|S|=c-1$.

If $\lfloor\frac{n}{2}\rfloor>k-1$, then $c\leq k-2$. Note that a decrease of c by 1 must cause an increase of the lower bound in Lemma 4.1 by at least 1. Hence, if equality holds in (2), then $c=k-2$, and so $n_c=k-|S|=c+2-|S|\leq(c+2)-(c-1)$, that is $n_c\leq 3$. Since $n_c\geq 3$, $n_c=3$ and so $|S|=c-1$.

Suppose that $n_1=n_2=\cdots=n_{c_1}=2, n_{c_1+1}=n_{c_1+2}=\cdots=n_c=3$. Let $c-c_1=c_2$. Then $n=3c_2+2c_1$, $c=c_1+c_2$, $k=c_1+c_2+2$ and $|S|=c_1+c_2-1$, so $\lfloor\frac{n}{2}\rfloor=\lfloor\frac{3c_2+2c_1}{2}\rfloor=c_1+c_2+\lfloor\frac{c_2}{2}\rfloor$. Since $\lfloor\frac{n}{2}\rfloor>k-1$, we have $c_1+c_2+\lfloor\frac{c_2}{2}\rfloor>c_1+c_2+1$, which implies $c_2\geq 4$.

Since equality holds in (2), we have

$$\left\lceil\frac{(n-1)^2}{2(c_1+c_2)}-\frac{(n+1)}{2}+k\right\rceil=3c_2+c_1+c-1. \tag{4}$$

Substituting $n=3c_2+2c_1$ in (4), we get

$$\left\lceil\frac{(3c_2+2c_1-1)^2}{2(c_1+c_2)}-\frac{(3c_2+2c_1+1)}{2}+c_1+c_2+2\right\rceil=4c_2+2c_1-1, \tag{5}$$

or

$$\left\lceil\frac{1+c_1-c_1c_2-c_2}{2(c_1+c_2)}+4c_2+2c_1-1\right\rceil=4c_2+2c_1-1.$$

Hence $-1<\frac{1+c_1-c_1c_2-c_2}{2(c_1+c_2)}\leq 0$, and so $c_2\geq 1$ and $c_2<(1+3c_1)/(c_1-1)$, where $c_1>1$.

Therefore, if $c_1>1$, then $4\leq c_2<(1+3c_1)/(c_1-1)$, and so $c_1<5$.

We conclude that, if $c_1=0$ or 1, then $c_2\geq 4$; if $c_1=2$, then $4\leq c_2\leq 6$; if $c_1=3$ or 4, then $c_2=4$. Hence, (n,k) must be $(3c,c+2)$ $(c\geq 4)$, $(3c-1,c+2)$ $(c\geq 5)$, (16,8), (18,9), (19,9), (20,10) or (22,10).

Claim 4: If $(n,k)=(3c,c+2)$ $(c\geq 4),(3c-1,c+2)$ $(c\geq 5)$, (16,8), (18,9), (19,9), (20,10) or (22,10), equality in (2) holds.

For each (n,k), if we can find a I'-maximal graph $G\in M_n(k)$ whose edge number reaches the lower bound, then we are done.

By Lemma 3.2, we know that the following graphs are I'-maximal.
$G(3;(3)_i)$ $(i \geq 3)$ with $c_1 = 0$,
$G(3;(3)_i, 2)$ $(i \geq 3)$ with $c_1 = 1$,
$G(3;(3)_i, 2, 2)$ $(3 \leq i \leq 5)$ with $c_1 = 2$,
$G(3;(3)_3, 2, 2, 2)$ with $c_1 = 3$,
$G(3;(3)_3, 2, 2, 2, 2)$ with $c_1 = 4$.

For these graphs, $(n,k) = (3c, c+2)$, where $c \geq 4$, $(3c-1, c+2)$, where $c \geq 5$, (16,8), (18,9), (19,9), (20,10) or (22,10), respectively. We can verify that their edge numbers reach the lower bound. □

5 The maximum size of an I'-maximal graph

Let $M(n,k) = \max\{|E(G)| : G \in M_n(k) \text{ and is connected}\}$. In order to find $M(n,k)$, first we introduce four Lemmas. The proofs of Lemmas 5.1 and 5.2 follow the routine arguments and are omitted.

Lemma 5.1. *For any positive integer n, we have*

$$\left\lceil \frac{n}{\lfloor \sqrt{n} \rfloor} \right\rceil + \lfloor \sqrt{n} \rfloor = \lceil 2\sqrt{n} \rceil. \tag{6}$$

Lemma 5.2. *Assume that $G \in M_n(k)$ and there exists an I'-set $S \subseteq E(G)$ such that $|S| = r$. Let $n = m(k-r) + b$, where m is a positive integer, $0 \leq b < k - r$.*

(i) *If $b \neq 1$, then*

$$|E(G)| \leq m\binom{k-r}{2} + \binom{b}{2} + r. \tag{7}$$

(ii) *If $b = 1$, then*

$$|E(G)| \leq (m-1)\binom{k-r}{2} + \binom{k-r-1}{2} + 1 + r. \tag{8}$$

Lemma 5.3. *For any given n and k, suppose that $n = m(k-r)+b$, where m is a positive integer, $0 \leq b < k - r$. let*

$$f(r) = \begin{cases} m\binom{k-r}{2} + \binom{b}{2} + r, & \text{if } b \neq 1, \\ (m-1)\binom{k-r}{2} + \binom{k-r-1}{2} + 1 + r, & \text{otherwise.} \end{cases} \tag{9}$$

Then when $2 \leq r \leq k - r$, $f(r) \leq f(r-1)$.

Proof: Let $g(r) = f(r) - r$. Note that, for any two positive integers n_1 and n_2 with $n_1 \leq n_2$, $\binom{n_1}{2} + \binom{n_2}{2} \leq \binom{n_1-1}{2} + \binom{n_2+1}{2} - 1$, and the equality holds if and only if $n_1 = n_2$. Suppose $n = m_1(k-r+1)+b_1, 0 \le b_1 < k-r+1$. Now we consider four cases.

Case 1: $b \ne 1$ and $b_1 \ne 1$. Then

$$\begin{aligned}
g(r) &= m\binom{k-r}{2} + \binom{b}{2} \\
&\le \binom{k-r+1}{2} + (m-1)\binom{k-r}{2} + \binom{b-1}{2} - 1 \\
&\le \binom{k-r+1}{2} + \binom{k-r+1}{2} + (m-2)\binom{k-r}{2} + \binom{b-2}{2} - 2 \\
&\le \cdots \\
&\le m_1\binom{k-r+1}{2} + \binom{b_1}{2} - m_1 \\
&= g(r-1) - m_1.
\end{aligned}$$

Similarly, we can show the results of other cases.

Case 2: $b \ne 1$ and $b_1 = 1$. Then $g(r) = m\binom{k-r}{2} + \binom{b}{2} \le (m_1-1)\binom{k-r+1}{2} + \binom{k-r}{2} + 1 - (m_1-1) = g(r-1) - (m_1-1)$.

Case 3: $b = 1$ and $b_1 \ne 1$. Then $g(r) = (m-1)\binom{k-r}{2} + \binom{k-r-1}{2} + 1 \le m_1\binom{k-r+1}{2} + \binom{b_1}{2} - m_1 = g(r-1) - m_1$.

Case 4: $b = 1$ and $b_1 = 1$. Then $g(r) = (m-1)\binom{k-r}{2} + \binom{k-r-1}{2} + 1 \le (m_1-1)\binom{k-r+1}{2} + \binom{k-r}{2} + 1 - (m_1-1) = g(r-1) - (m_1-1)$.

Note that $n > (k-r)+2$. If not, $n+r-2 \le k \le n-1$ and so $r \le 1$, contrary to the assumption of the Lemma. Hence, $n > (k-r+1)+1$, and so $m_1 \ge 1$ and if $m_1 = 1$, then $b_1 \ne 1$. For Cases 1 and 3, $f(r) = g(r)+r \le g(r-1)+r-m_1 = f(r-1)+1-m_1$ and so $f(r-1)-f(r) \ge m_1 - 1 \ge 0$. For Cases 2 and 4, since $b_1 = 1$, $m_1 \ge 2$. Therefore, $f(r) = g(r)+r \le g(r-1)+r-(m_1-1) = f(r-1)+2-m_1$ and so $f(r-1)-f(r) \ge m_1 - 2 \ge 0$. □

Lemma 5.4. *Let n, k be two given positive integers such that $\lceil 2\sqrt{n}\rceil - 1 \le k \le n-1$.*

(i) *There is a unique integer r such that $1 \le r \le \lfloor\sqrt{n}\rfloor - 1$ and $\lceil \frac{n}{r+1}\rceil + r \le k \le \lceil\frac{n}{r}\rceil + r - 2$.*

(ii) *If r satisfies (i), then $n = r(k-r)+b, 2 \le b \le k-r$.*

Proof: (i) Let $T_r = \{k \mid \lceil\frac{n}{r+1}\rceil + r \le k \le \lceil\frac{n}{r}\rceil + r - 2\}$. Then, for any i, j such that $i \ne j$ and $1 \le i, j \le \lfloor\sqrt{n}\rfloor - 1$, $T_i \cap T_j = \phi$ and $T_1 \cup T_2 \cup \cdots \cup T_{\lfloor\sqrt{n}\rfloor - 1} = \{\left\lceil\frac{n}{\lfloor\sqrt{n}\rfloor}\right\rceil + \lfloor\sqrt{n}\rfloor - 1, \left\lceil\frac{n}{\lfloor\sqrt{n}\rfloor}\right\rceil + \lfloor\sqrt{n}\rfloor, \cdots, n-1\} = \{\lceil 2\sqrt{n}\rceil - 1, \lceil 2\sqrt{n}\rceil, \cdots, n-1\}$.

(ii) Since $\lceil \frac{n}{r+1} \rceil + r \le k \le \lceil \frac{n}{r} \rceil + r - 2$ $(1 \le r \le \lfloor \sqrt{n} \rfloor - 1)$, $r(k-r)+r < n \le (k-r)(r+1)$ and so $r(k-r)+2 \le n \le (k-r)(r+1)$. □

Theorem 5.5. *Let n and k are two positive integers such that there exists a connected graph $G \in M_n(k)$. Suppose $\lceil \frac{n}{r+1} \rceil + r \le k \le \lceil \frac{n}{r} \rceil + r - 2$, where $1 \le r \le \lfloor \sqrt{n} \rfloor - 1$. Then*

$$M(n,k) = r\binom{k-r}{2} + \binom{n-r(k-r)}{2} + r.$$

Proof: Let $G \in M_n(k)$ and S be an I'-set of G. Note that, since G is connected, $k \ge \lceil 2\sqrt{n} \rceil - 1$ (see [3]). Hence, by Lemma 5.3, $k \ge \lceil 2\sqrt{n} \rceil - 1 = \left\lceil \frac{n}{\lfloor \sqrt{n} \rfloor} \right\rceil + \lfloor \sqrt{n} \rfloor - 1$. By Lemmas 5.2 and 5.3, the smaller $|S|$ is, the larger the upper bound of $|E(G)|$ becomes. When $|S| = r$, $G - S$ has at most $r+1$ components and so $k \ge \lceil \frac{n}{r+1} \rceil + r$, and when $|S| = r-1$, $k \ge \lceil \frac{n}{r} \rceil + r - 1$. Hence, for given n and k, if $\lceil \frac{n}{r+1} \rceil + r \le k \le \lceil \frac{n}{r} \rceil + r - 2$, then $|S| \ge r$ and so $|E(G)| \le f(|S|) \le f(r)$. By Lemma 5.4(ii), $n = r(k-r) + b$, where $2 \le b \le k - r$. Hence, by Lemma 5.2,

$$|E(G)| \le r\binom{k-r}{2} + \binom{b}{2} + r = r\binom{k-r}{2} + \binom{n-r(k-r)}{2} + r.$$

We shall show that this upper bound can be attained.

For given n and k such that $\lceil \frac{n}{r+1} \rceil + r \le k \le \lceil \frac{n}{r} \rceil + r - 2$ $(1 \le r \le \lfloor \sqrt{n} \rfloor - 1)$, we construct graph $G(n,k)$ as follows:

$$G(n,k) = G(k-r; (k-r)_{r-1}, n - r(k-r)).$$

By Lemma 5.4(ii), $n - r(k-r) \ne 1$. So by Lemma 3.2, each $G(n,k)$ defined above is an I'-maximal graph. It is straightforward to check that $|E(G(n,k))| = r\binom{k-r}{2} + \binom{n-r(k-r)}{2} + r$, and so this concludes the proof of Theorem 5.5. □

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