The strength and the l-edge-connectivity of a graph

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Abstract

For an integer $l \geq 2$ the *l-edge-connectivity* $\lambda_l(G)$ of a connected graph G of order $p \geq l$ is the number of edges that need to be deleted from G to produce a disconnected graph with at least l components. The strength of G, denoted by $\bar{\lambda}_2(G)$, is the maximum value of $\lambda_2(H)$, where H runs over all subgraphs of G. In this note we investigate the relationship between λ_l and $\bar{\lambda}_2$, especially the extremal cases. A structural characterization of the extremal when $\lambda_2(G) = 2$ and arbitrary $l \geq 2$ is found.

1 Introduction

Graphs in this paper are finite and loopless. Undefined terms and notation can be found in [3]. For a graph G, $k(G)$ denotes the number of components of G. We write $H \subseteq G$ to mean H is a subgraph of G.

For subsets S and S' of V, we denote by $[S, S']$ the set of edges with one end in S and the other in S'. An *edge-cut* of G is a subset of E of the

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form $[S, \bar{S}]$, where S is a nonempty proper subset of V and $\bar{S} = V - S$. A minimal edge-cut of G is called a *bond*. The *edge-connectivity* $\lambda(G)$ of a connected graph G is the minimum cardinality of an edge-cut of G. For an integer $l \geq 2$ Chartrand, Kapoor, Lesniak and Lick [2] defined the *l-edge-connectivity* $\lambda_l(G)$ of G to be the smallest number of edges whose removal leaves a graph with at least l components. For an overview on this parameter, see Oellermann's resent survey [11]. In particular if $l = 2$, then $\lambda_l(G) = \lambda(G)$. Matula [8] defined the *strength* $\bar{\lambda}(G)$ of G to be the following number:

$$\bar{\lambda}(G) = \max_{H \subset G} \lambda(H).$$

The invariant $\bar{\lambda}(G)$, has been studied in [1, 9, 4], among others. In this paper, We shall show that $(l-1)\bar{\lambda}(G) \geq \lambda_l(G)$ and characterize the graphs when equality holds for $\lambda(G) = 2$ and for all values of $l \geq 2$. As a byproduct, we characterize graphs with $\lambda(G) = \bar{\lambda}(G) = 2$.

2 Relationship between $\bar{\lambda}_2(G)$ and $\lambda_l(G)$

Let G be a graph with $|V(G)| \geq 2$. An *l-sequential cut* is a sequence $\{X_1, X_2, \cdots, X_{l-1}\}$ of edge subsets of G such that X_i is an edge-cut of a component of $G_i = G - \cup_{j=0}^{i} X_j$ such that $k(G_i) + 1 = k(G_i - X_{i+1})$, where $X_0 = \emptyset$. The sequence $\{X_1, X_2, \cdots, X_{l-1}\}$ is a *minimal l-sequential cut* if each X_i is a minimum edge-cut of G_i. The sequence $\{X_1, X_2, \cdots, X_{l-1}\}$ is a *minimum l-sequential cut* if, among all l-sequential cuts of G, $\sum_{i=1}^{l-1} |X_i|$ is minimum.

Lemma 2.1 Let G be a graph with $|V(G)| \geq 2$. G has an l-sequential cut if and only if $|V(G)| \geq l$.

Proof Let the sequence $\{X_1, X_2, \cdots, X_{l-1}\}$ be an l-sequential cut. Then $G - \{X_1, X_2, \cdots, X_{l-1}\}$ has l non empty components $C_1, C_2, \cdots, C_l$, and so $|V(G)| = \sum_{i=1}^{l} |V(C_i)| \geq l$.

Conversely, we shall prove that if $|V(G)| \geq l$ then G has an l-sequential cut by induction on l. For the case $l = 2$, $|V(G)| \geq 2$. Let $u \in V(G)$, such that u is not a cut vertex. Let X_1 be the set of edges incident to u. Then since u is not a cut vertex, X_1 is a bond of G. Thus $G - X_1$ separate G into two components u and $G - u$. Thus X_1 is a 2-sequential cut, and the result holds for $l = 2$. Assume the result hold for $l = s - 1$. That is if $|V(G)| \geq s - 1$ then G has an $(s-1)-$sequential cut. Now if $|V(G)| \geq s$, by induction G has an $(s-1)-$sequential cut $\{X_1, X_2, \cdots, X_{s-2}\}$ such that $G - \{X_1, X_2, \cdots, X_{s-2}\}$ has $(s-1)$ components $C_1, C_2, \cdots, C_{s-1}$. Since $|V(G)| \geq s$ at least one component C_i for $1 \leq i \leq s-1$ has more than one vertex. Thus if $|V(C_i)| \geq 2$ we can find a bond X_{s-1} such that C_i is separated into 2-components. Therefore $\{X_1, X_2, \cdots, X_{s-1}\}$ is an l-sequential cut. □

Lemma 2.2 If $\{X_1, \cdots, X_{l-1}\}$ is a minimum l-sequential cut of G, then there is a minimum l-sequential cut $\{X'_1, \cdots, X'_{l-1}\}$ such that for each i with $1 \leq i \leq l-1$, X'_i is a bond of a component of $G_i = G - \cup_{j=0}^{i} X_j$, where $X_0 = \emptyset$.

Proof Let $\{X_1, \cdots, X_{l-1}\}$ be a minimum l-sequential cut of G. Define $\{X'_1, \cdots, X'_{l-1}\}$ as follows.

Let $X'_1 \subseteq X_1$ be a minimal edge-cut of G, and after $X'_1, X'_2, \cdots, X'_m$ are defined, for some m with $1 \leq m < l-1$, let $X'_{m+1} \subseteq \cup_{i=1}^{m+1} X_i - \cup_{i=1}^{m} X'_i$ be a minimal edge-cut of $G - \cup_{i=1}^{m} X'_i$. By the way $\{X'_1, \cdots, X'_{l-1}\}$ is defined, it is an l-sequential cut of G.

Note that the $X'_1, \cdots X'_{l-1}$ are disjoint and that $\cup_{i=1}^{l-1} X'_i \subseteq \cup_{i=1}^{l-1} X_i$, and so $\sum_{i=1}^{l-1} |X'_i| \leq \sum_{i=1}^{l-1} |X_i|$. But equality must hold since $\{X'_1, \cdots, X'_{l-1}\}$ is an l-sequential cut of G and since $\{X_1, \cdots, X_{l-1}\}$ is minimum. □

We use the following definition for *contractions*. Let G be a graph and let X be a subset of $E(G)$. We use G/X to denote the graph obtained from G by identifying the two ends of each edge in X and then deleting all loops produced.

Observation 2.3 Let $H \subseteq G$. Then each of the following holds:

(i) $\lambda_2(G/E(H)) \geq \lambda_2(G)$.

(ii) If $\lambda_2(G/E(H)) \geq k$ and $\lambda_2(H) \geq k$ then $\lambda_2(G) \geq k$.

Proof (i). Let H be a subgraph of G. Then $E(G/E(H)) \subseteq E(G)$ and any edge-cut of $G/E(H)$ is also an edge-cut of G. Thus $\lambda_2(G/E(H))$ edge-cut separates the graph G into at least two components, which may not be the minimum cardinality of an edge-cut of G.

(ii). Let $X \subseteq E(G)$ be a minimal edge-cut of G, and let $X_1 = X \cap E(H)$ and $X_2 = X - X_1 = X \cap E(G/H)$. If $X_1 \neq \emptyset$, then since X is an edge-cut of G, X_1 must be an edge-cut of H. It follows that $|X| \geq |X_1| \geq k$, by the assumption that $\lambda_2(H) \geq k$. If $X_1 = \emptyset$, then since X is an edge-cut of G, X_2 must be an edge-cut of $G/E(H)$. Therefore $|X| = |X_2| \geq k$, by the assumption that $\lambda_2(G/E(H)) \geq k$. □

Lemma 2.4 Let H be a connected graph with l vertices. The vertices of H can be labeled as $v_1, v_2, \cdots, v_l$ so that for each i $(1 \leq i \leq l)$ the graph $H - \{v_1, v_2, \cdots, v_i\}$ is connected.

Proof We prove this by induction on $|V(H)|$. If $|V(H)| = 1$ there is nothing to prove. Suppose the result is true for $|V(H)| < l$. Let H be a connected graph with l vertices. Let H_1 be a spanning tree of H. Label a vertex of degree 1 in H_1 by v_1. Thus $H - v_1$ is a connected graph since $H_1 - v_1$ is a spanning tree of $H - v_1$. Furthermore $|V(H - v_1)| < l$. Therefore the vertices of $H - v_1$ can be labeled as $v_2, v_3, \cdots, v_l$ so that for each i $(2 \leq i \leq l)$ the graph $(H - v_1) - \{v_2, v_3, \cdots, v_i\}$ is connected. Thus for each i $(1 \leq i \leq l)$ the graph $H - \{v_1, v_2, \cdots, v_i\}$ is connected. □

Lemma 2.5 Let $l \geq 2$ be an integer and let G be a connected graph with $|V(G)| \geq l$. Then the followings are equivalent.

(i) There is an edge subset T of $E(G)$ such that $G - T$ has l components.

(ii) G has an l-sequential cut $\{X_1, X_2, \cdots, X_{l-1}\}$ such that $T = \cup_{i=1}^{l-1} X_i$.

Proof Let $T \subseteq E(G)$ such that $G - T$ has l components. Obtain a graph H from G by contracting each component of $G - T$ into a vertex. Thus H has l vertices and $|T|$ edges. Further more H is connected. By Lemma 2.4, the vertices of H can be labeled as $v_1, v_2, \cdots, v_l$ such that $H - \{v_1, v_2, \cdots, v_i\}$

is connected for all i. Let $C_1, C_2, \cdots, C_l$ be the l-components of $G-T$, and the component C_i is contracted to the vertex v_i. Let

$$X_1 = \{e \in T : e \text{ has exactly one end in } C_1\}$$

and let

$$X_i = \{e \in T - \cup_{j=1}^{i-1} X_j : e \text{ has exactly one end in } C_i\}.$$

Since $H - \{v_1, v_2, \cdots, v_i\}$ is connected, each X_i is a bond of $G - \cup_{j=0}^{i-1} X_j$, where $X_0 = \emptyset$. Thus $\{X_1, X_2, \cdots, X_{l-1}\}$ is an l-sequential cut of G and $T = \cup_{i=1}^{l-1} X_i$.

Conversely assume G has a l-sequential cut $\{X_1, X_2, \cdots, X_{l-1}\}$ such that $T = \cup_{i=1}^{l-1} X_i$. By the definition of an l-sequential cut, $G-T$ has l components. □

Lemma 2.6 Let $l \geq 2$ be an integer and let G be a graph with $|V(G)| \geq l$. Then the followings are equivalent.

(i) There is an edge subset T of $E(G)$ such that $G-T$ has at least l components and $\lambda_l(G) = |T|$.

(ii) G has a minimum l-sequential cut $\{X_1, X_2, \cdots, X_{l-1}\}$ such that $T = \cup_{i=1}^{l-1} X_i$.

Proof Let T be an edge subset of $E(G)$ such that $G-T$ has at least l components and $\lambda_l(G) = |T|$. Since $\lambda_l(G) = |T|$, $G-T$ has exactly l components. Thus by Lemma 2.5 G has an l sequential cut. Let $\{X_1, X_2, \cdots, X_{l-1}\}$ be an l-sequential cut of G. Let $\{Y_1, Y_2, \cdots Y_{l-1}\}$ be a minimum l-sequential cut of G. Then $G - \cup_{i=1}^{l-1} Y_i$ has l components, and so

$$\lambda_l(G) \leq \sum_{i=1}^{l-1} |Y_i| \leq \sum_{i=1}^{l-1} |X_i| = |T| = \lambda_l(G),$$

which implies $\{X_1, X_2, \cdots, X_{l-1}\}$ is a minimum l-sequential cut of G.

Conversely assume G has a minimum l-sequential cut $\{X_1, X_2, \cdots, X_{l-1}\}$ such that $T = \cup_{i=1}^{l-1} X_i$. By the definition of l-sequential cut, $G-T$ has l components thus $\lambda_l(G) \leq |T|$. Let T_1 be an edge subset of G such that $\lambda_l(G) = |T_1|$ and $G - T_1$ has exactly l components. By Lemma 2.5 G has an l-sequential cut $\{Y_1, Y_2, \cdots, Y_{l-1}\}$ such that $T_1 = \cup_{i=1}^{l-1} Y_i$. Therefore, $T = \cup_{i=1}^{l-1} X_i$ and $\{X_1, X_2, \ldots, X_{l-1}\}$ is minimum, $\lambda_l(G) = |T_1| \geq |T| = \sum_{i=1}^{l-1} |X_i|$. Thus $\lambda_l(G) = |T|$. □

<u>**Lemma 2.7**</u> For any connected graph G with $|V(G)| \geq l$, $(l-1)\bar{\lambda}_2(G) \geq \lambda_l(G)$.

<u>**Proof**</u> Let $\{X_1, X_2, \cdots, X_{l-1}\}$ be a minimal l-sequential cut. Then

$$\lambda_l(G) \leq \sum_{i=1}^{l-1} |X_i| \leq \sum_{i=1}^{l-1} \bar{\lambda}_2(G) = (l-1)\bar{\lambda}_2(G). \square$$

<u>**Theorem 2.8**</u> Let G be a simple connected graph. Then $(l-1)\bar{\lambda}_2(G) = \lambda_l(G)$ if and only if each of the following holds:

(i) any minimal l-sequential cut is a minimum l-sequential cut.

(ii) for some integer k, $\bar{\lambda}_2(G) = \lambda_2(G) = k$.

(iii) for any minimal l-sequential cut $\{X_1, X_2, \cdots, X_{l-1}\}$; $|X_i| = k$.

<u>**Proof**</u> Assume $(l-1)\bar{\lambda}_2(G) = \lambda_l(G)$. Let $\{X_1, X_2, \cdots, X_{l-1}\}$ be a minimal l-sequential cut. Then since each X_i is a minimum edge cut of a subgraph of G, $|X_i| \leq \bar{\lambda}_2(G)$. It follows that

$$(l-1)\bar{\lambda}_2(G) = \lambda_l(G) \leq \sum_{i=1}^{l-1} |X_i| \leq (l-1)\bar{\lambda}_2(G)$$

and so equality must hold. This gives (i), and $|X_1| = |X_2| = \cdots = |X_{l-1}| = \bar{\lambda}_2(G)$. By the definition of a minimal l-sequential cut, $|X_1| = \lambda_2(G)$. Hence $\lambda_2(G) = \bar{\lambda}_2(G)$. Denote this common value by k. Then both (ii) and (iii) hold.

Conversely, by (iii), if $\{X_1, X_2, \cdots, X_{l-1}\}$ is a minimal l-sequential cut, then $\sum_{i=1}^{l-1} |X_i| = (l-1)k$. By (ii) $\sum_{i=1}^{l-1} |X_i| = (l-1)\bar{\lambda}_2(G)$. By (i) $\{X_1, X_2, \cdots, X_{l-1}\}$ is a minimum l-sequential cut. Hence by Lemma 2.6, $\lambda_l(G) = \sum_{i=1}^{l-1} |X_i|$. Thus $(l-1)\bar{\lambda}_2(G) = \lambda_l(G)$. □

The next theorem indicates that graphs reaching the upper bound $(l-1)\bar{\lambda}_2(G) = \lambda_l(G)$ possesses an interesting property: nontrivial components of such graphs when a minimum edge-cut is removed inherit the same equality. Such a property shall play an important role in the study of extremal graphs.

Theorem 2.9 Let G be a simple graph with $|V(G)| \geq l \geq 3$ and assume $\lambda_2(G) > 1$ and $(l-1)\bar{\lambda}_2(G) = \lambda_l(G)$. Let $X \subseteq E(G)$ be a minimum edge-cut of G such that $G - X$ has two components H_1 and H_2. Then either $H_1 = K_1$ or H_1 satisfies

(i) $\lambda_2(H_1) = \lambda_2(G)$; and

(ii) $(l-2)\bar{\lambda}_2(H_1) = \lambda_{l-1}(H_1)$.

Proof Assume $H_1 \neq K_1$. If $1 < |V(H_1)| = l_1 < l-1$, then by Lemma 2.1, H_1 has a minimal l_1-sequential cut $\{W_2, W_3, \cdots, W_{l_1}\}$. Note that $|W_{l_1}| = 1$, since H_1 is a simple graph. Since $|V(G)| \geq l$, the minimal l_1-sequential cut $\{W_2, W_3, \cdots, W_{l_1}\}$ of H_1 together with X becomes an (l_1+1)-sequential cut $\{X, W_2, \cdots, W_{l_1}\}$ in G. Since X is a minimum edge-cut of G, $\{X, W_2, \cdots, W_{l_1}\}$ is a minimal (l_1+1)-sequential cut of G. Since $|V(H_2)| = |V(G)| - |V(H_1)| \geq l - l_1$ and since H_2 is connected, it follows from Lemma 2.1 that the minimal (l_1+1)-sequential cut $\{X, W_2, W_3, \cdots, W_{l_1}\}$ may be extended to a minimal l-sequential cut $\{X, W_2, W_3, \cdots, W_{l_1}, \cdots, W_{l-1}\}$. But then, by Theorem 2.8(ii) and Theorem 2.8(iii), one must have, for each i, $|W_i| = \lambda_2(G) > 1$, contrary to $|W_{l_1}| = 1$.

Thus $|V(H_1)| \geq l-1$. By Lemma 2.1, H_1 has a minimal $(l-1)$-sequential cut $\{Y_2, Y_3, \cdots, Y_{l-1}\}$. Then $\{X, Y_2, \cdots, Y_{l-1}\}$ is a minimal l-sequential cut of G. By definition of minimal sequential cut and by Theorem 2.8(iii), $\lambda_2(H_1) = |Y_2| = |X|$. Since X is a minimum cut and by Theorem 2.8(ii) and Theorem 2.8(iii) $|X| = \lambda_2(G) = \bar{\lambda}_2(G)$. Thus $\lambda_2(H_1) = \lambda_2(G)$, and so (i) holds.

Let $\{Z_2, \cdots, Z_{l-1}\}$ be a minimum $(l-1)$-sequential cut of H_1. By Lemma 2.6

$$\lambda_{l-1}(H_1) = \sum_{i=2}^{l-1} |Z_i|.$$

Since $\{X, Z_2, \cdots, Z_{l-1}\}$ is an l-sequential cut of G, and since $\{X, Y_2, \cdots, Y_{l-1}\}$

is a minimum l-sequential cut of G,

$$|X| + \lambda_{l-1}(H_1) = |X| + \sum_{i=2}^{l-1} |Z_i| \geq |X| + \sum_{i=2}^{l-1} |Y_i| \geq |X| + \lambda_{l-1}(H_1).$$

Therefore, equalities must hold everywhere, and so $\lambda_{l-1}(H_1) = \sum_{i=2}^{l-1} |Y_i|$. By Theorem 2.8, $|Y_i| = \bar{\lambda}_2(G)$. Therefore

$$\lambda_{l-1}(H_1) = \sum_{i=2}^{l-1} |Y_i| = \bar{\lambda}_2(G)(l-2) = \lambda_2(H_1)(l-2).$$

Note that $\lambda_2(H_1) \leq \bar{\lambda}_2(H_1) \leq \bar{\lambda}_2(G)$, and so $\bar{\lambda}_2(H_1)(l-2) = \lambda_{l-1}(H_1)$ follows. □

3 Characterization of Extremal Graphs

An *elementary subdivision* of a nonempty graph H is a graph obtained from H by removing some edge $e = uv$ and by adding a new vertex w and new edges uw and wv. A *subdivision* of H is a graph obtained from H by a succession of elementary subdivisions (including the possibility of none). A subdivision of a graph H is denoted by $S(H)$.

Let H be a graph with a specified distinguished vertex v. Let $v_1, v_2, \cdots, v_d$ be the vertices on H adjacent to v. Let L be a graph disjoint from H. Then $H(L)$ denotes the collection of simple graphs obtained from the disjoint union of L and $H - v$ by adding d new edges $e_1, e_2, \cdots, e_d$ such that e_i joins v_i to a vertex in L, for $1 \leq i \leq d$. If $G \in H(L)$, then we say that G is obtained by *replacing* a vertex v in H by the graph L. If G_1 is isomorphic to G_2, then we say G_1 and G_2 are *isomorphic* or *equal* and write $G_1 = G_2$.

Observation 3.1 If $G \in H(L)$, then L is a subgraph of G ($L \subseteq G$) and $G/E(L) = H$. Thus $E(G) = E(H) \cup E(L)$.

Proof Let $G \in H(L)$. From the definition of $H(L)$, L is a subgraph of of G. We get $G/E(L) = H$, by the definition of contraction and by the definition of $H(L)$. Thus $E(G) = E(H) \cup E(L)$. □

Let $M(2)$ be a collection of simple graphs having K_1 (a single vertex graph without edges), and C_3 (a cycle of length 3), such that $G \in M(2) - \{K_1,\ C_3\}$ if and only if either

(M1) $G = S(H)$ for some graph $H \in M(2)$, or

(M2) G can be obtained from a graph $H \in M(2)$ by replacing a vertex of H by a graph in M(2); *i. e.*, $G \in H(L)$, for some $L \in M(2)$.

Theorem 3.2 Let G be a simple connected graph. Then $\lambda_2(G) = \bar{\lambda}_2(G) = 2$ if and only if $G \in M(2) - \{K_1\}$.

Proof Suppose $G \in M(2) - \{K_1\}$. Let C_n denote the n-cycle. Note that $\lambda_2(C_n) = 2$ and $\bar{\lambda}_2(C_n) = 2$. Thus for $G = C_n$, the result is trivial. For any other graph $G \in M(2) - \{K_1\}$ we shall prove $\lambda_2(G) = \bar{\lambda}(G) = 2$ by induction on $|V(G)|$.

Note first that if $G = S(H)$ for some $H \in M(2)$, then $\lambda_2(G) = \lambda_2(H) = \bar{\lambda}_2(G) = \bar{\lambda}_2(H) = 2$. Hence we only need to consider the case when $G \in H(L)$, for some $H,\ L \in M(2)$. We shall show $\lambda_2(G) = \bar{\lambda}_2(G) = 2$. If $\lambda_2(G) = 1$, then G has an edge-cut e. By Observation 3.1, either $e \in E(H) = E(G/L)$ or $e \in E(L)$. By induction, we have $\lambda_2(H) = 2$, $\lambda_2(L) = 2$, and so e cannot be an edge-cut of G. Thus $\lambda_2(G) \geq 2$.

If G has a subgraph Γ with $\lambda(\Gamma) \geq 3$, then by Observation 2.3, either $\Gamma/\Gamma \cap L$ is a subgraph of $H = G/L$, whence $\lambda_2(\Gamma/\Gamma \cap L) \geq \lambda_2(\Gamma)$, contrary to $\bar{\lambda}_2(H) = 2$; or $\Gamma \subseteq L$, contrary to $\bar{\lambda}_2(L) = 2$. Therefore $\bar{\lambda}_2(G) = 2$.

Conversely, assume that $\lambda_2(G) = \bar{\lambda}_2(G) = 2$. We argue by induction on $|V(G)|$ to show that $G \in M(2)$. Since $\lambda_2(G) = \bar{\lambda}_2(G) = 2$ and $G \neq K_1$, G has an edge-cut X with $|X| = 2$. Define a relation $\sim$ on $E(G)$ as follows: $e \sim e'$ if and only if $\{e, e'\}$ is an edge-cut or $e = e'$. It is well known that this defines an equivalence relation. Let $W = \{e_1, e_2, \cdots, e_m\}$ be an equivalence class and let $C_1, C_2, \cdots, C_m$ be the components of $G - W$.

Note that for each i with $1 \leq i \leq m$, either $C_i = K_1$ or $\lambda_2(C_i) = 2$, for if C_i has an edge-cut e', then e' would have been in W. Also, $G/(C_1 \cup C_2 \cup \cdots \cup C_m) =$ an m-cycle, which is in $M(2)$. Note that for those $C_i \neq K_1$, $2 \leq \lambda_2(C_i) \leq \bar{\lambda}_2(C_i) \leq \bar{\lambda}_2(G) = 2$. Thus $\bar{\lambda}_2(C_i) = 2$. By induction, $C_i \in M(2)$. Hence G is obtained by replacing some vertices of an m-cycle by graphs in $M(2)$. □

Denote $L_{2,l} = K_2 + (l-1)K_1$. Let $[G_1, G_2]_2$ denotes the collection of all simple graphs obtained from the disjoint union of G_1 and G_2, by joining vertices in G_1 to those in G_2 with exactly 2 edges. Let $L(2,l)$ be the collection of simple graphs such that K_1, $L_{2,l} \in L(2,l)$, and such that a graph $G \in L(2,l) - \{K_1, L_{2,l}\}$ if and only if for some G_1, $G_2 \in L(2, l-1)$, $G \in [G_1, G_2]_2$.

Theorem 3.3 Let G be a simple graph and $|V(G)| \geq l \geq 3$ and suppose $\lambda_2(G) = 2$. Then $(l-1)\bar{\lambda}_2(G) = \lambda_l(G)$ if and only if $G \in L(2,l)$.

Proof Assume $(l-1)\bar{\lambda}_2(G) = \lambda_l(G)$. Let X be a minimum edge-cut of G. Denote the two components by H_1 and H_2. By Theorem 2.9, either $H_i = K_1$ or H_i satisfies Theorem 2.9(i) and Theorem 2.9(ii), for $1 \leq i \leq 2$, and so $G \in L(2,l)$.

Conversely, assume $G \in L(2,l)$. Then for some G_1, $G_2 \in L(2, l-1)$, $G \in [G_1, G_2]_2$. Note that $[G_1, G_2]_2 \subseteq M(2)$, and by induction on $|V(G)|$, $G_i \in M(2)$. Thus $G \in M(2)$ and so $\bar{\lambda}_2(G) = \lambda_2(G) = 2$. Thus G satisfies Theorem2.8(ii). Hence, for each value $l \geq 2$,

$$L(2,l) \subseteq M(2). \tag{1}$$

Let $\{X_1, X_2, \cdots, X_{l-1}\}$ be an arbitrarily selected minimal l-sequential cut of G. By (the above equation), $|X_i| \geq 2$. Since $\bar{\lambda}_2(G) = 2$, $|X_i| \leq 2$. Thus $|X_i| = 2$ for all $1 \leq i < l$. So G satisfies Theorem2.8(iii). By Lemma 2.7, $\lambda_l(G) \leq (l-1)\bar{\lambda}_2(G) = (l-1)2 = \sum_{i=1}^{l-1} |X_i|$. Let $\{Y_1, Y_2, \cdots, Y_{l-1}\}$ be a minimum l-sequential cut of G. By Lemma 2.2, we may assume that for each i with $1 \leq i \leq l-1$,

$$Y_i \text{ is a bond of } G_i = G - \cup_{j=0}^{i} Y_j, \text{ where } Y_0 = \emptyset. \tag{2}$$

If $\sum_{i=1}^{l-1} |Y_i| = 2(l-1)$, then every minimal l-sequential cut is also a minimum l-sequential cut; and then follows that G satisfies Theorem 2.8(i)-(iii), and so $(l-1)\bar{\lambda}_2(G) = \lambda_l(G)$.

Therefore, we assume that

$$\sum_{i=1}^{l-1} |Y_i| < 2(l-1). \tag{3}$$

Note that by (2) $|Y_1| \geq \lambda_2(G) = 2$.

Suppose first that $|Y_1| = 2$. Let H_1 and H_2 be the two components of $G - Y_1$. By Theorem 2.9, we may assume that H_1 satisfies $(l-2)\bar{\lambda}_2(H_1) = \lambda_{l-1}(H_1)$. Let $\{Y_2', \cdots, Y_{l-1}'\}$ be a minimal $(l-1)$-sequential cut of H_1. Then $\{Y_1, Y_2', \cdots, Y_{l-1}'\}$ is a minimal l-sequential cut of G. By applying Theorem 2.8 to H_1 $\{Y_1, Y_2', \cdots, Y_{l-1}'\}$ is a minimum l-sequential cut of G with $|Y_i'| = |Y_1| = 2$, for each i with $2 \leq i \leq l-1$. It follows by Theorem 2.8 that $\{X_1, X_2, \cdots, X_{l-1}\}$ would be a minimum l-sequential cut, contrary to the assumption (3).

Therefore, we must have $|Y_1| \geq 3$. Since $G \in [G_1, G_2]_2$, we may assume that $Y_1' = Y_1 \cap E(G_1) \neq \emptyset$. For if both $Y_1 \cap E(G_1) = \emptyset$ and $Y_1 \cap E(G_2) = \emptyset$, then one must have $|Y_1| = |E(G) - (E(G_1) \cup E(G_2))| = 2$, a contradiction.

Next, we claim that $l - 1 \geq 3$. For if $l = 3$, then by $|Y_1| \geq 3$, and by $\sum_{i=1}^{2} |Y_i| < 2(3-1)$, one must have $Y_2 = \emptyset$, a contradiction.

Now we can exclude the possibility that $G_2 = K_1$. For if $G_2 = K_1$, then by (3), $\{Y_1, Y_2, \cdots, Y_{l-2}\}$ would be an $(l-1)$-sequential cut of G_1 with $\sum_{i=1}^{l-2} |Y_i| \leq 2l - 3 - |Y_{l-1}| \leq 2(l-2)$. Recall that $G_1 \in L(2, l-1)$. Since $l - 1 \geq 3$ and by induction, G_1 satisfies $(l-2)\bar{\lambda}_2(G) = \lambda_{l-1}(G)$. Thus by $\sum_{i=1}^{l-2} |Y_i| \leq 2l - 3 - |Y_{l-1}| \leq 2(l-2)$, $\{Y_1, Y_2, \cdots, Y_{l-2}\}$ is a minimum $(l-1)$-sequential cut of G_1. By Theorem 2.8, $|Y_1| = 2$, contrary to the assumption that $|Y_1| \geq 3$.

Thus both G_1 and G_2 are not K_1. Now define $Y_i' = Y_i \cap E(G_1)$ and $Y_i'' = Y_i \cap E(G_2)$ for each i with $1 \leq i \leq l-1$. Since $G_1 \in L(2, l-1) \subset M(2)$ and by (2), either $|Y_i'| = 0$ or $|Y_i'| \geq 2$. Similarly, we conclude that either $|Y_i''| = 0$ or $|Y_i''| \geq 2$.

Hence we may assume that there are p_1 nonempty sets in $Y_1', Y_2', \cdots Y_{l-1}'$ and p_2 nonempty sets in $Y_1'', Y_2'', \cdots Y_{l-1}''$. Since $\{Y_1, \cdots, Y_{l-1}\}$ is an l-sequential cut of G, $p_1 + p_2 \geq l - 1$.

If $p_1 = l-1$, then $\{Y_1', Y_2', \cdots Y_{l-2}'\}$ is an $(l-1)$-sequential cut of G_1, and so $\sum_{i=1}^{l-2} |Y_i'| \geq 2(l-2)$, by induction. Since $|Y_1| \geq 3$, either $\sum_{i=1}^{l-2} |Y_i'| >$

$2(l-2)$ or $|Y_1''| \geq 1$. This together with $|Y_{l-1}| \geq 1$, yields $\sum_{i=1}^{l-1} |Y_i| \geq 2(l-1)$, contrary to (3).

Therefore we may assume that $p_1 \leq l-2$ and $p_2 \leq l-2$. Since $G_j \in L(2, l-1) \subseteq L(2, p_j+1)$, by induction, $\sum_{i=1}^{p_j} |Y_i'| \geq 2p_j$. It follows that $\sum_{i=1}^{l-2} |Y_i| = \sum_{j=1}^{2} \sum_{i=1}^{p_j} |Y_i'| \geq 2(p_1+p_2) \geq 2(l-1)$, contrary to (3).

These contradictions indicate that the arbitrarily selected minimal l-sequential cut $\{X_1, \cdots X_{l-1}\}$ must be minimum, and so G satisfies $(l-1)\bar{\lambda}_2(G) = \lambda_l(G)$. □

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