Generalized matroid packings and coverings

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Abstract

Let S be a set and let $M^1, M^2, \cdots, M^k$ be k matroids on S with rank functions $\rho^1, \rho^2, \cdots, \rho^k$, respectively. Let $\mathcal{B}^i$ be the collection of bases of M^i, $(1 \le i \le k)$. In this note we show that there is a k-tuple $(A_1, A_2, \cdots, A_k)$ with $A_i \in \mathcal{B}^i$ such that each $s \in S$ lies in at least t of the A_i's if and only if for every $X \subseteq S$,

$$t|X| \le \sum_{i=1}^{k} \rho^i(X);$$

and that there is a k-tuple $(A_1, A_2, \cdots, A_k)$ with $A_i \in \mathcal{B}^i$ such that each $s \in S$ lies in at most t of the A_i's if and only if for every $X \subseteq S$,

$$t|S - X| \ge \sum_{i=1}^{k} [\rho^i(S) - \rho^i(X)].$$

1. Introduction

We consider loopless matroids on finite nonempty sets. See [7] for undefined terms. The set of all positive integers will be denoted by $\mathbf{N}$. Let M be a loopless matroid on S with rank function ρ. The family of bases of M is denoted by $\mathcal{B}(M)$, or just $\mathcal{B}$. The family of independent sets of M is denoted by $\mathcal{I}(M)$. For $T \subseteq S$, the closure of T in M is denoted by $\sigma(T)$. A subset $T \subseteq S$ is spanning in M if $\sigma(T) = S$. The family of all spanning subsets of M is denoted by $\mathcal{S}(M)$.

Let S be a set and let $\mathbf{F}_{\mathcal{F}} = \langle \mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k \rangle$ be a k-tuple each of whose components is a family of subsets of S. For each $t \in \mathbf{N}$, an $(\mathbf{F}_{\mathcal{F}}, t)$-covering (respectively, an $(\mathbf{F}_{\mathcal{F}}, t)$-packing) of S is a k-tuple $\mathbf{A} = \langle A_1, A_2, \cdots, A_k \rangle$, with $A_i \in \mathcal{F}_i$ for all $i \in \{1, 2, \cdots, k\}$, such that every $s \in S$ is in at least (respectively, at most) t of the A_i's.

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Edmonds has the following theorem whose graphical versions are proved by Nash-Williams ([4], [5]) and by Tutte [6].

Theorem 1.1 (Edmonds [3]) Let M be a matroid on S with rank function ρ. Let $\mathcal{B}$ denote the family of bases of M and let $\mathbf{F}_{\mathcal{B}} = \langle \mathcal{B}, \mathcal{B}, \cdots, \mathcal{B} \rangle$ be a k-tuple. Each of the following holds:

(i) M has a $(\mathbf{F}_{\mathcal{B}}, 1)$-covering if and only if for each subset $X \subseteq S$,

$$|X| \leq k\rho(X).$$

(ii) M has a $(\mathbf{F}_{\mathcal{B}}, 1)$-packing if and only if for each subset $X \subseteq S$

$$|S - X| \geq k[\rho(S) - \rho(X)].$$

In this note, we shall extend this theorem.

2. Main results

Let M be a matroid on S with rank function ρ. Two elements $e_1, e_2 \in S$ are parallel if $\rho(\{e_1, e_2\}) = \rho(e_1) = \rho(e_2)$. Fix an element $e \in S$ and let e' be an element not in S. Define

$$\mathcal{I}' = \mathcal{I}(M) \cup \{I \cup \{e'\} \mid e \notin I, \text{ and } I \cup \{e\} \in \mathcal{I}(M)\}.$$

Then it is routine to check that $\mathcal{I}'$ is the family of independent sets of a matroid M' on $S \cup \{e'\}$. Note that $M'|S = M$ and e and e' are parallel elements in M'. We say that e is replaced by a set of parallel elements $\{e, e'\}$. Let $t \in \mathbf{N}$. For each $e \in S$, we replace e by a set of parallel elements $E(e) = \{e_1, e_2, \cdots, e_t\}$ such that

$$E(e) \cap E(e') = \emptyset, \text{ whenever } e \neq e'. \tag{1}$$

Denote the resulting matroid by M_t and call it the t-parallel extension of M. Let

$$S_t = \bigcup_{e \in S} E(e).$$

Then M_t is a matroid on S_t. The family of bases of M_t is denoted by $\mathcal{B}_t$ and the rank function of M_t is denoted by ρ_t. For every subset $Y \subseteq S_t$, there is a minimal subset $X \in S$ such that

$$Y \subseteq \bigcup_{e \in X} E(e). \tag{2}$$

Thus by (2) and by the minimality of X, we have

$$\rho_t(Y) = \rho(X). \tag{3}$$

In particular, we have

$$\rho_t(S_t) = \rho(S). \tag{4}$$

Theorem 2.1 Let S be a finite nonempty set and let $M^1, M^2, \cdots, M^k$ be k matroids on the same set S with rank functions $\rho^1, \rho^2, \cdots, \rho^k$. Let the family of bases of M^i be $\mathcal{B}^i$, $(1 \leq i \leq k)$. Let $\mathbf{F}_{\mathcal{B}} = \langle \mathcal{B}^1, \mathcal{B}^2, \cdots, \mathcal{B}^k \rangle$ be a k-tuple. Then for any $t \in \mathbf{N}$, each of the following holds:

(i) There is a $(\mathbf{F}_{\mathcal{B}}, t)$-covering of S if and only if for every subset $X \subseteq S$,

$$t|X| \leq \sum_{i=1}^{k} \rho^i(X). \tag{5}$$

(ii) There is a $(\mathbf{F}_{\mathcal{B}}, t)$-packing of S if and only if for every subset $X \subseteq S$,

$$t|S - X| \geq \sum_{i=1}^{k} [\rho^i(S) - \rho^i(X)]. \tag{6}$$

Proof: For each $i \in \{1, 2, \cdots, k\}$, let M_t^i denote the t-parallel extension of M^i, and let ρ_t^i denote the rank function of M_t^i. Let

$$\mathcal{I}'' = \{I \mid I = I^1 \cup I^2 \cup \cdots \cup I^k,\ I^i \in \mathcal{I}(M_t^i),\ 1 \leq i \leq k\}. \tag{7}$$

It is known ([7], page 121) that $\mathcal{I}''$ is the set of independent sets of a matroid on S_t, called the union of $M^1, M^2, \cdots, M^k$ and denoted by

$$M^o = \bigvee_{i=1}^{k} M_t^i.$$

The rank function of M^o will be given by ([7], page 121)

$$\rho^o(Y) = \min_{T \subseteq Y} \left\{ \sum_{i=1}^{k} \rho_t^i(T) + |Y - T| \right\},\ \text{for all } Y \subseteq S_t. \tag{8}$$

We shall show that the following are equivalent:

(a) There is a $(\mathbf{F}_{\mathcal{B}}, t)$-covering of S.

(b) Every subset of S_t is independent in M^o.

(c) For any $Y \subseteq S_t$, $|Y| \leq \sum_{i=1}^{k} \rho_t^i(Y)$.

(d) For any $X \subseteq S$, $t|X| \leq \sum_{i=1}^{k} \rho^i(X)$.

(a) $\Longrightarrow$ (b). Let $\mathbf{B} = \langle B^1, B^2, \cdots, B^k \rangle$ be a $(\mathbf{F}_{\mathcal{B}}, t)$-covering of S. Thus, each B^i is a base of M^i $(1 \leq i \leq k)$, and each $e \in S$ is an element in at least t components of $\mathbf{B}$. Therefore, there is a k-tuple $\mathbf{A} = \langle A^1, A^2, \cdots, A^k \rangle$ whose components satisfy $A^i \subseteq B^i$, $(1 \leq i \leq k)$, such that each $e \in S$ lies in exactly t components of $\mathbf{A}$. Define $\mathbf{B}_t = \langle B_t^1, B_t^2, \cdots, B_t^k \rangle$ to be a k-tuple obtained from $\mathbf{A}$ by replacing, for each $e \in S$, each of the t occurrences of e in the components of $\mathbf{A}$ by t distinct elements of $E(e)$ in the corresponding components of $\mathbf{B}_t$, so that each $e_i \in E(e)$ $(1 \leq i \leq k)$ occurs in just one components of $\mathbf{B}_t$. Since each $B^i \in \mathcal{I}(M^i)$, we have $A^i \in \mathcal{I}(M^i)$ also, and so $B_t^i \in \mathcal{I}(M_t^i)$. Hence by (8),

$$\bigcup_{i=1}^{k} B_t^i \in \mathcal{I}''.$$

But since each $e \in S$ lies in exactly t components of $\mathbf{A}$, it follows from the definition of $\mathbf{B}_t$ that $S_t = \cup_{i=1}^{k} B_t^i$, and so $S_t \in \mathcal{I}''$. Thus (b) follows.

(b) $\Longrightarrow$ (a). Assume that $M^o = \vee_{i=1}^{k} M_t^i$ is a matroid on S_t such that every subset of S_t is independent in M^o, and so $S_t \in \mathcal{I}''$. Thus by (8), S_t can be written by

$$S_t = \bigcup_{i=1}^{k} B_t^i, \tag{9}$$

where $B_t^i \in \mathcal{I}(M_t^i)$. For each $i \in \{1, 2, \cdots, k\}$, define a subset $A^i \subseteq S$ by

$$e \in A^i \iff E(e) \cap B_t^i \neq \emptyset.$$

Since $B_t^i \in \mathcal{I}(M_t^i)$, A^i is independent in M^i. Since $|E(e)| = t$ for all $e \in S$, we have $e \in A^i$ for at least t values of i and so by (9) and by the definition of S_t, each $e \in S$ lies in at least t components of the k-tuple $\mathbf{A} = \langle A^1, A^2, \cdots, A^k \rangle$. Since each component A^i of $\mathbf{A}$ is independent in M^i, A^i is contained in a base B^i of M^i and so $\mathbf{B} = \langle B^1, B^2, \cdots, B^k \rangle$ is a $(\mathbf{F}_{\mathcal{B}}, t)$-covering of S in $\mathcal{B}$. This proves (a).

(b) $\iff$ (c). Since (b) holds if and only if $\rho^o(S_t) = |S_t|$, it follows by (8) and (9) that (b) holds if and only if (c) holds.

(c) $\iff$ (d). For each $X \subseteq S$, define

$$Y(X) = \bigcup_{e \in X} E(e).$$

Thus $|Y(X)| = t|X|$ and so by (3), (c) implies (d). Suppose that (d) holds. For each $Y \subseteq S_t$, one can find a minimal subset $X \subseteq S$ such that

$$Y \subseteq \bigcup_{e \in X} E(e).$$

Then by $|Y| \le t|X|$, by (d) and by (3), we have

$$|Y| \le t|X| \le \sum_{i=1}^{k} \rho^i(X) = \sum_{i=1}^{k} \rho_t^i(Y),$$

which proves (c).

By the equivalence of (a) $\iff$ (b) $\iff$ (c) $\iff$ (d), we established (i) of Theorem 2.1.

To prove (ii) of Theorem 2.1, we shall show that the following are equivalent:

(a') There is a $(\mathbf{F}_{\mathcal{B}}, t)$-packing of S.

(b') The matroid M^o has rank $\sum_{i=1}^{k} \rho^i(S)$.

(c') For any $Y \subseteq S_t$, $|S_t - Y| \ge \sum_{i=1}^{k} [\rho_t^i(S) - \rho_t^i(Y)]$.

(d') For any $X \subseteq S$, $t|S - X| \ge \sum_{i=1}^{k} [\rho^i(S) - \rho^i(X)]$.

(a') $\Longrightarrow$ (b'). Suppose that $\mathbf{B} = \langle B^1, B^2, \cdots, B^k \rangle$ is a $(\mathbf{F}_{\mathcal{B}}, t)$-packing of S. Then each $e \in S$ is in at most t of the B^i's. Recall that

$$S_t = \bigcup_{e \in S} E(e),$$

where $E(e) = \{e_1, e_2, \cdots, e_t\}$. For each $i \in \{1, 2, \cdots, k\}$, define a subset B_t^i as follows:

$$B_t^i = \bigcup \{e_i \in E(e) \mid e \in B^i\}.$$

Thus $B_t^i \cap B_t^j = \emptyset$, whenever $i \neq j$. By (3), that B_t^i is base in M_t^i follows from that B^i is a base in M^i, and so

$$\rho^i(S_t) = |B_t^i| = |B^i| = \rho(S). \tag{10}$$

By (8) and (11), $\cup_{i=1}^{k} B_t^i$ is a base of M^o and so

$$\rho^o(S_t) = \left| \bigcup_{i=1}^{k} B_t^i \right| = \sum_{i=1}^{k} \rho^i(S).$$

Thus (b') must hold.

(b') $\Longrightarrow$ (a'). By (b') and by (8), we can find disjoint subsets B_t^i of S_t, $(1 \le i \le k)$, such that B_t^i is a base in M_t^i. For each $i \in \{1, 2, \cdots, k\}$, define

$$B^i = \{e \in S \mid E(e) \cap B_t^i \neq \emptyset\}.$$

Since B_t^i is independent in M_t^i, for every $e \in S$, and for every $i \in \{1, 2, \cdots, k\}$, $|E(e) \cap B_t^i| \le 1$, and so by the fact that B_t^i is a base in M_t^i again, and by (3), B^i is base in M^i. Since the B_t^i's are disjoint, every $e \in S$ is in at most t components of $\mathbf{B} = \langle B^1, B^2, \cdots, B^k \rangle$, and so S has a $(\mathbf{F}_{\mathcal{B}}, t)$-packing.

(b') $\Longleftrightarrow$ (c'). This equivalence follows from (8) and (10).

(c') $\Longleftrightarrow$ (d'). The proof of this equivalence is similar to that of (c) $\Longleftrightarrow$ (d) above.

By the equivalence of (a') $\Longleftrightarrow$ (b') $\Longleftrightarrow$ (c') $\Longleftrightarrow$ (d'), we established (ii) of Theorem 2.1. □

3. Corollaries

Let M be a matroid on S and let $\mathcal{F}$ denote a family of subsets of S, and let $\mathbf{F}_{\mathcal{F}} = \langle \mathcal{F}, \mathcal{F}, \cdots, \mathcal{F} \rangle$. In [1], a $(\mathbf{F}_{\mathcal{F}}, t)$-covering is called a t-covering of S in $\mathcal{F}$, and an $(\mathbf{F}_{\mathcal{F}}, t)$-packing is called a t-packing of S in $\mathcal{F}$. The following is an extension of Theorem 1.1.

Corollary 3.1 ([1] and [2]) Let M be a matroid on S with rank function ρ, and let $\mathcal{B}$ denote the family of bases of M. Let $\mathbf{F}_{\mathcal{B}} = \langle \mathcal{B}, \mathcal{B}, \cdots, \mathcal{B} \rangle$ be a k-tuple. Each of the following holds:

(i) M has a $(\mathbf{F}_{\mathcal{B}}, t)$-covering if and only if for every subset $X \in S$,

$$t|X| \le k\rho(X).$$

(ii) M has a $(\mathbf{F}_{\mathcal{B}}, t)$-packing if and only if for every $X \subseteq S$,

$$t|S - X| \ge k[\rho(S) - \rho(X)].$$

Proof: Apply Theorem 2.1 to the case when $M^1 = M^2 = \cdots = M^k = M$. □

Corollary 3.2 Let M be a matroid on S. Let $n_1, n_2, \cdots, n_k$ be natural numbers such that $n_i \le \rho(S)$, $(1 \le i \le k)$. Then each of the following holds:

(i) There is a t-covering $\mathbf{A} = \langle A_1, A_2, \cdots, A_k \rangle$ of S in $\mathcal{I}(M)$ such that for each $i \in \{1, 2, \cdots, k\}$, $|A_i| \le n_i$ if and only if for every subset $X \subseteq S$

$$t|X| \le \sum_{i=1}^{k} \min\{n_i, \rho(X)\}.$$

(ii) There is a t-packing $\mathbf{A} = \langle A_1, A_2, \cdots, A_k \rangle$ of S in $\mathcal{I}(M)$ such that for each $i \in \{1, 2, \cdots, k\}$, $|A_i| = n_i$ if and only if for every $X \subseteq S$,

$$t|S - X| \ge \sum_{i=1}^{k} [n_i - \min\{n_i, \rho(X)\}].$$

Proof: Apply Theorem 2.1 to the case when each M^i is the truncation of M to n_i. □

Let M be a matroid on S and let k be an integer with $\rho(S) \le k \le |S|$. Then the family of all subsets $A \in \mathcal{S}(M)$ with $|A| = k$ is a the family of bases of a matroid $M^{(k)}$ on S, called the elongation of M to k. (see [7], page 60). Let ρ and $\rho^{(k)}$ denote the rank funk functions of M and $M^{(k)}$, respectively. Then for any $X \subseteq S$, we have

$$\rho^{(k)}(X) = \rho(X) + \min\{|X| - \rho(X), k - \rho(S)\}. \tag{11}$$

In fact, let $X' \subseteq X$ be a independent subset in M with $\rho(X) = |X'|$, and let $X'' \subseteq S - X$ be an independent set in M such that $X' \cup X''$ is a base in M. Hence $\rho(S) = |X'| + |X''|$. If $|X| - \rho(X) \geq k - \rho(S)$, then one can choose $k - \rho(X)$ elements from $X - X'$ to form a subset Y. Since $X' \cup X''$ is a base of M, $X' \cup X'' \cup Y \in \mathcal{S}(M)$, and so by the definition of $M^{(k)}$, $X' \cup X'' \cup Y$ is a base of $M^{(k)}$. Therefore, $\rho^{(k)}(X) = \rho(X) + k - \rho(S)$. If $k - \rho(S) \geq |X| - \rho(X)$, then one can choose $|X| - \rho(X)$ elements from $X - X'$ to form a subset Y', and so $X' \cup Y'$ is independent in $M^{(k)}$. Thus (11) holds also. This proves (11).

Corollary 3.3 Let M be a matroid on S. Let $n_1, n_2, \cdots, n_k$ be natural numbers such that $|S| \geq n_i \geq \rho(S)$, $(1 \leq i \leq k)$. Then each of the following holds:

(i) There is a t-covering $\mathbf{A} = \langle A_1, A_2, \cdots, A_k \rangle$ of S in $\mathcal{S}(M)$ such that for each $i \in \{1, 2, \cdots, k\}$, $|A_i| \geq n_i$ if and only if for every subset $X \subseteq S$

$$t|X| \leq \sum_{i=1}^{k} [\rho(X) - \min\{n_i - \rho(S), |X| - \rho(X)\}].$$

(ii) There is a t-packing $\mathbf{A} = \langle A_1, A_2, \cdots, A_k \rangle$ of S in $\mathcal{S}(M)$ such that for each $i \in \{1, 2, \cdots, k\}$, $|A_i| = n_i$ if and only if for every $X \subseteq S$,

$$t|S - X| \geq \sum_{i=1}^{k} [n_i - \rho(X) - \min\{n_i - \rho(S), |X| - \rho(X)\}].$$

Proof: Apply Theorem 2.1 to the case when each M^i is the elongation of M to n_i. □

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