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Graphs without spanning closed trails

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Abstract

Jaeger (1979) proved that if a graph has two edge-disjoint spanning trees, then it is supereulerian, i.e., that it has a spanning closed trail. Catlin (1988) showed that if G is one edge short of having two edge-disjoint spanning trees, then G has a cut edge or G is supereulerian. Catlin conjectured that if a connected graph G is at most two edges short of having two edge-disjoint spanning trees, then either G is supereulerian or G can be contracted to a K_2 or a $K_{2,t}$ for some odd integer $t \geqslant 1$. We prove Catlin's conjecture in a more general context. Applications to spanning trails are discussed.

1. Introduction

Graphs in this note are finite and loopless. Undefined terms and notation are from [2]. As in [2], $\omega(G)$ and $\kappa'(G)$ denote the number of components and the edge-connectivity of G, respectively. We use $H \subseteq G$ ($H \subset G$) to denote the fact that H is a subgraph of G (proper subgraph of G). Let V, W be disjoint subsets of $V(G)$. Then $[V, W]_G$ denotes the set of edges in G that have one end in V and the other end in W. Let $X \subseteq E(G)$. The *contraction* G/X is obtained from G by contracting each edge of X and deleting the resulting loops. If $H \subseteq G$, we write G/H for $G/E(H)$.

For a graph G, $O(G)$ denotes the set of all vertices of odd degree in G. A connected graph G with $O(G) = \emptyset$ is called an *eulerian graph*. A graph is *supereulerian* if it has a spanning eulerian subgraph. The collection of all supereulerian graphs will be denoted by $\mathcal{SL}$. The topic of supereulerian graphs was surveyed in [7].

Let $F(G)$ denote the minimum number of extra edges that must be added to G so that the resulting graph has 2 edge-disjoint spanning trees. Thus for a connected graph G, if G has a spanning tree T such that $\omega(G - E(T)) = k + 1$, then $F(G) \leqslant k$.

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Theorem 1.1 (Jaeger [10]). *If $F(G)=0$, then G is supereulerian.*

Theorem 1.1 was later improved by Catlin. In [3], Catlin defined collapsible graphs. Let $R \subseteq V(G)$. A subgraph Γ of G is called an *R-subgraph* if both

(i) $G - E(\Gamma)$ is connected, and

(ii) $v \in R$ if and only if v has odd degree in Γ.

A graph G is *collapsible* if for any even subset R of $V(G)$, G has an R-subgraph. The collection of all collapsible graphs is denoted by $\mathscr{CL}$. Setting $R = O(G)$, one sees that $\mathscr{CL} \subset \mathscr{SL}$.

Catlin showed [3] that every vertex of G lies in a unique maximal collapsible subgraph of G. The *reduction* of G is obtained from G by contracting all maximal collapsible subgraphs, and is denoted by G'. A graph G is *reduced* if G is the reduction of some graph.

Theorem 1.2 (Catlin [3]). *Let G be a graph.*

(i) *If $F(G)=0$, then G is collapsible.*

(ii) *If $F(G)=1$, then either $G \in \mathscr{CL}$, or the reduction of G is a K_2.*

Our main results are Theorem 1.3 below, which was conjectured by Catlin in [3], and Theorem 1.4, which is a special case of Theorem 1.3.

Theorem 1.3. *If G is connected, and if $F(G) \leqslant 2$, then either G is collapsible, or the reduction of G is a K_2 or a $K_{2,t}$ for some integer $t \geqslant 1$.*

Theorem 1.4. *If G is a graph with $\kappa'(G) \geqslant 3$, $\kappa(G) \geqslant 2$ and with $F(G) \leqslant 2$, then G is collapsible.*

The next result, conjectured by Catlin in [4], follows directly from Theorem 1.3.

Theorem 1.5. *Let G be a connected graph. If $F(G) \leqslant 2$, then exactly one of the following holds*:

(i) *G is supereulerian.*

(ii) *G has a cut edge.*

(iii) *The reduction of G is $K_{2,s}$, for some odd integer $s \geqslant 3$.*

Graphs that are obtained from the Petersen graph by replacing each vertex of the Petersen graph by a complete subgraph K_m ($m \geqslant 4$) show that the bound $F(G) \leqslant 2$ is best possible in Theorems 1.3, 1.4 and 1.5.

Theorem 1.6 below extends a prior result of Jaeger [10] (weaker than Theorem 1.1) that 4-edge-connected graphs are supereulerian. The Petersen graph, being noncollapsible and having 10 vertices of degree 3, shows that Theorem 1.6 is best possible.

Theorem 1.6. *If G is a 3-edge-connected graph with at most 9 edge cuts of size 3, then G is collapsible.*

Theorem 1.7 below improves a former result of Zhan [13] that the line graph of a 4-edge-connected graph is hamiltonian-connected.

Theorem 1.7 (Catlin and Lai [8]). *If G is a graph with $F(G) = 0$, then for two edges e, e' of G, exactly one of the following holds*:

(i) *G has a spanning trail that starts with e and ends with e'.*

(ii) *The edges $\{e, e'\}$ form an edge cut of G such that each component of $G - \{e, e'\}$ has an edge.*

Proof (*Sketch*). Let G' be the graph obtained from G by subdividing each of e and e' into a path of length 2. Let $v(e)$ and $v(e')$ denote the new vertices. Note that $F(G') = 2$. Apply Theorem 1.3 to G' to find a spanning $(v(e), v(e'))$-trail in G'. □

Examples showing that the containment $\mathscr{CL} \subset \mathscr{SL}$ is strict are ubiquitous. For any graph G with $F(G) \leqslant 2$, though, we have the following equivalence characterizing collapsible graphs in terms of spanning trails.

Theorem 1.8. *Let G be a graph. If $F(G) \leqslant 2$ then the following are equivalent*:

(a) $G \in \mathscr{CL}$;

(b) *$G \in \mathscr{SL}$, and for any distinct vertices $x, y \in V(G)$, there is a spanning trail in G with origin x and terminus y.*

In Section 2, we display the mechanisms needed for the proofs. Assuming Theorem 1.4, we shall prove Theorems 1.3, 1.6 and 1.8 in Section 3. Theorem 1.4 will be proved in the last section.

2. Mechanisms

We summarize some of Catlin's conclusions on collapsible and reduced graphs in Theorems 2.1 and 2.2 below.

Theorem 2.1 (Catlin [3]). *Let G be a graph.*

(i) *G is reduced iff G has no nontrivial collapsible subgraph.*

(ii) *If G is reduced, then G is simple, K_3-free, and G cannot have a nontrivial subgraph with 2 edge-disjoint spanning trees, and for any $H \subseteq G$, either $H \in \{K_1, K_2\}$ or $|E(H)| \leqslant 2|V(H)| - 4$.*

(iii) *If H is a collapsible subgraph of G, then $G \in \mathscr{CL}$ ($G \in \mathscr{SL}$) if and only if $G/H \in \mathscr{CL}$ ($G \in \mathscr{SL}$).*

(iv) *If H_1 and H_2 are collapsible subgraphs of G, and if $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2$ is collapsible.*

(v) *A reduced graph can be covered by at most 2 edge-disjoint forests.*

Theorem 2.2 (Catlin [6]). *Let $C_4 = u_1v_1u_2v_2u_1$ be a 4-cycle of G. Denote by G/π the graph obtained from $G - E(G[V(C_4)])$ by identifying u_1 and u_2 to form a new vertex u, and by identifying v_1 and v_2 to form a new vertex v, and by joining u and v with a new edge $e_\pi = uv$. Each of the following holds:*

(i) *If G/π is collapsible, then G is collapsible.*

(ii) *If G/π is supereulerian, then G is supereulerian.*

(iii) *$K_{3,3} - e \in \mathscr{CL}$.*

Lemma 2.3. *If for any $H \subset G$ with $|V(H)| < |V(G)|$, H is reduced, and if $|V(G)| \geqslant 3$, then*

$$F(G) = 2|V(G)| - 2 - |E(G)|. \tag{1}$$

Proof. By the definition of $F(G)$, there is a set E of $F(G)$ new edges which may be added to G so that the resulting graph $G + E$ has 2 edge-disjoint spanning trees, say T_1 and T_2, and so

$$|E(G)| + F(G) = |E(G+E)| \geqslant |E(T_1)| + |E(T_2)| \geqslant 2|V(G)| - 2.$$

Conversely, we note that no proper subgraph of G has two edge-disjoint spanning trees (by (i) of Theorem 2.1 and (i) of Theorem 2.2), and so $E(G)$ can be covered by 2 edge-disjoint forests F_1 and F_2 (say) of G. As each F_i $(1 \leqslant i \leqslant 2)$ is $\omega(F_i) - 1 = |V(G)| - |E(F_i)| - 1$ edges short of being a tree,

$$F(G) \leqslant \sum_{i=1}^{2} \omega(F_i) - 1 = 2|V(G)| - |E(G)| - 2. \qquad \square$$

Lemma 2.4. *Let G be a graph with $\kappa'(G) \geqslant 3$ and $|E(G)| = 2|V(G)| - 4$. If for any $H \subset G$, H is reduced, and if G contains a $K_{2,3}$ as a subgraph, then $G \in \mathscr{CL}$.*

Proof. Let $\{u_1, u_2\}$ $(\{v_1, v_2, v_3\})$ be the vertices of degree 3 (degree 2) of this $K_{2,3}$ subgraph in G. Since $\kappa'(G) \geqslant 3$, we may assume that $\{u_1v_1, u_1v_2, u_2v_1, u_2v_2\}$ is not an edge cut of G (otherwise we consider $\{u_1v_3, u_1v_2, u_2v_3, u_2v_2\}$). Let $C_4 = u_1v_1u_2v_2u_1$ and let G/π be the graph defined in Theorem 2.2 with $e_\pi = uv \in E(G/\pi)$.

By (ii) of Theorem 2.1, and by the fact that $|E(G/\pi)| = 2|V(G/\pi)| - 3$, G/π is not reduced. Let $(G/\pi)'$ be the reduction of G/π. If $e_\pi = uv \in E(G/\pi)'$, then every nontrivial collapsible subgraph H of G/π must contain u or v, and so by (iv) of Theorem 2.1, G/π has at most two nontrivial maximal collapsible subgraphs H_1 and H_2 (say). (In the event that only H_1 is nontrivial, we set $H_2 = K_1$.) By the assumption that the preimages of H_1 and H_2 in G are reduced, and by (ii) of Theorem 2.1, we have $|E(H_i)| \leqslant 2|V(H_i)| - 2$, and so

$$|E((G/\pi)')| = |E(G)| - |E(C_4)| + 1 - \sum_{i=1}^{2} |E(H_i)|$$

$$\geqslant 2|V(G)| - 7 - \sum_{i=1}^{2}(2|V(H_i)| - 2)$$

$$\geqslant 2|V((G/\pi)')| - 3.$$

By (ii) of Theorem 2.1, and since $(G/\pi)'$ is reduced, we conclude that either $(G/\pi)' = K_1$, whence by (i) of Theorem 2.2, $G \in \mathscr{CL}$, or $(G/\pi)' = K_2$, whence $E(C_4)$ is an edge cut of G, contrary to the assumption that $E(C_4)$ is not an edge cut of G.

Thus we assume that $e_\pi = uv \notin E(G/\pi)'$. Then e_π lies in some collapsible subgraph H' of G/π, and there is a subgraph $H \subset G$ with $C_4 \subset H$ such that $H/\pi = H'$. By (i) of Theorem 2.2 with H in place of G, $H \in \mathscr{CL}$, a contradiction. Hence $H' = G/\pi$ and so $G \in \mathscr{CL}$, by (i) of Theorem 2.2. □

3. Proof of the main results

We shall assume Theorem 1.4 to prove Theorem 1.3. Let G be a connected noncollapsible graph. We argue by induction on $|V(G)|$. Let G' denote the reduction of G. Note that by the definition of $F(G)$, for any $X \subseteq E(G)$, we have

$$F(G) \geqslant F(G/X), \tag{2}$$

and so $F(G) \geqslant F(G')$. If $G \neq G'$, then $|V(G')| < |V(G)|$. By induction, G', which is also the reduction of G', is either K_2, or a $K_{2,t}$ for some $t \geqslant 1$, and so we are done. Hence we assume that $G = G'$ is a reduced noncollapsible graph.

Suppose first that G has a cut vertex v. Then G has two nontrivial subgraphs H_1 and H_2 such that $V(H_1) \cap V(H_2) = \{v\}$, $H_1 \cup H_2 = G$, and $F(H_1) \leqslant F(H_2)$. By Lemma 2.3, $2 \geqslant F(G) = F(H_1) + F(H_2)$. Therefore either $F(H_1) \leqslant 1$ and $F(H_2) \leqslant 1$, whence, by Theorem 1.2, $H_1, H_2 \in \{K_1, K_2\}$, and so $G \in \{K_2, K_{1,2}\}$; or $F(H_1) = 0$ and $F(H_2) = 2$, whence, by Theorem 1.2 and (i) of Theorem 2.1, $H_1 = K_1$, contrary to the assumption that H_1 is nontrivial. Hence we assume $\kappa(G) \geqslant 2$. Thus, $\kappa'(G) \geqslant 2$.

If $\kappa'(G) \geqslant 3$, then by Theorem 1.4, G is collapsible, contrary to the assumption that G is not collapsible. Therefore G has an edge cut X with $|X| = 2$. Let G_1 and G_2 be the two components of $G - X$, where $F(G_1) \leqslant F(G_2)$.

By Lemma 2.3, $2 \geqslant F(G) = F(G_1) + F(G_2)$. Either $F(G_1) \leqslant 1$ and $F(G_2) \leqslant 1$, whence, by Theorem 1.2, $G_1, G_2 \in \{K_1, K_2\}$, and so by $\kappa'(G) \geqslant 2$ and by (i) of Theorem 2.1, $G = K_{2,2}$; or $F(G_1) = 0$ and $F(G_2) = 2$, whence, by induction, $G_2 = K_{2,t'}$ for some $t' \geqslant 1$, and so by (iii) of Theorem 2.2 and by (i) of Theorem 2.1, $G = K_{2,t}$, where $t = t' + 1$. This proves Theorem 1.3. □

Proof of Theorem 1.6. Let G' denote the reduction of G. By contradiction, we assume that $G \notin \mathscr{CL}$ and so $G' \neq K_1$. Since $\kappa'(G) \geqslant 3$, $G' \neq K_2$, either. Note that G' has at most 9 vertices of degree 3 and all other vertices have degree at least 4. Thus

$2|E(G')| \geqslant 27 + 4(|V(G)| - 9)$, and so $|E(G')| \geqslant 2|V(G')| - 4$. By (ii) of Theorem 1.2 and by Lemma 2.3, $F(G') \leqslant 2$ and so, by Theorem 1.3, $G' \in \mathscr{CL}$, a contradiction. □

Proof of Theorem 1.8. For sets A and B, $A \oplus B = A \cup B - A \cap B$ is the symmetric difference of A and B.

Suppose (a) holds, and let $x, y \in V(G)$. Let Γ be an R-subgraph of G, where $R = \mathrm{O}(G)$ if $x = y$ and $R = \mathrm{O}(G) \oplus \{x, y\}$ if $x \neq y$. Then $G - E(\Gamma)$ has a spanning trail of G from x to y.

Instead, suppose (a) is false. If $G \notin \mathscr{SL}$, then (b) fails. Suppose $G \in \mathscr{SL}$. Then to prove (b) false, we must find $x, y \in V(G)$ such that there is no spanning (x, y)-trail. Since $F(G) \leqslant 2$ and $G \notin \mathscr{CL}$, Theorem 1.3 implies that the reduction of G is K_2 or $K_{2,t}$ $(t \geqslant 1)$. By $G \in \mathscr{SL}$, the reduction of G is neither K_2 nor $K_{2,t}$ with t odd. Hence, G is contractible to $K_{2,t}$ for some even number $t \geqslant 2$. Let $\psi : G \to K_{2,t}$ denote that contraction. Pick x and y to be vertices of G that are mapped by that reduction-contraction ψ to nonadjacent degree t vertices x' and y' (say) of the $K_{2,t}$. Since there is no spanning (x', y')-trail in $K_{2,t}$, G has no spanning (x, y)-trail. Thus, (b) of Theorem 1.8 fails. □

4. Proof of Theorem 1.4

Let G be a graph with $\kappa'(G) \geqslant 3$, $\kappa(G) \geqslant 2$ and with $F(G) \leqslant 2$, and let G' denote the reduction of G. By contradiction, we assume that G is not collapsible and among all counterexamples, $|V(G)|$ is minimized. Therefore, by the minimality of $|V(G)|$, by (ii) of Theorem 1.2, and by (2), we may assume that $G = G'$ is reduced, $\kappa'(G) = 3$ and $F(G) = 2$.

We need a few more terms and lemmas. For a vertex $v \in V(G)$, $N_G(v)$ denotes that set of vertices that are adjacent to v in G.

Definition. Let T be a tree of G and U a forest of G with U_1 being a component of U and $U_2 = U - V(U_1)$. Suppose each of the following holds:

(T1) $E(T) \cap E(U) = \emptyset$,

(T2) $V(U_1) \subset V(T)$,

(T3) $V(U_2) \supseteq V(T) - V(U_1) \neq \emptyset$.

Then the ordered triple (T, U_1, U_2) is called a 3-*forest* of G.

A vertex is called *pendant* in a subgraph H if its degree in H is 1. The subset of all pendant vertices of H is denoted by $D_1(H)$.

Lemma 4.1. *Let (T, U_1, U_2) be a 3-forest of a reduced graph G. Then for any vertex $v \in V(U_1)$, G has a 3-tree $(\bar{T}, \bar{U}_1, \bar{U}_2)$ such that $V(T) = V(\bar{T})$, $V(\bar{U}_1) = \{v\}$, $V(\bar{U}_1 \cup \bar{U}_2) = V(U_1 \cup U_2) = V(U)$ and $|E(\bar{U}_1 \cup \bar{U}_2)| = E(U_1 \cup U_2)|$.*

Proof. Suppose that the 3-forest (T, U_1, U_2) has been chosen with $v \in V(U_1)$ such that $|V(U_1)|$ is minimized.

If $|V(U_1)|=1$, then we are done. Thus we assume that $|V(U_1)|\geqslant 2$. If $T[V(U_1)]$ is connected, then $F(G[V(U_1)])=0$, contrary to (ii) of Theorem 2.1. Thus $\omega(T[V(U_1)]) \geqslant 2$. Since T is connected, there is a component T_0 in $T - V(U_1)$ such that $|[V(T_0), V(U_1)]_T| = r \geqslant 2$. Let $S_0 = V(T_0)$, and let $T_1, T_2, \ldots, T_r$ denote the components of $T - S_0$, and let $S_i = V(T_i) \cap V(U_1)$ $(1 \leqslant i \leqslant r)$. Since T is a tree, there is exactly one edge $u_i t_i$ (say) in $E(T)$, with $u_i \in S_0$ and $t_i \in S_i$ $(1 \leqslant i \leqslant r)$.

Let U_{11} be a minimal subtree of U_1 such that $\{t_1, \ldots, t_r\} \subseteq V(U_{11})$. By the minimality of U_{11}, $D_1(U_{11}) \subseteq \{t_1, \ldots, t_r\}$. Without loss of generality, we assume that $t_1, t_2 \in D_1(U_{11})$, and so U_{11} has a (t_1, t_2)-path $t_1, \ldots, x, y, \ldots, t_2$ (say) such that the vertices in the section $t_1, \ldots, x$ are all in S_1, while $y \in S_i$ $(2 \leqslant i \leqslant r)$. Let the two components of $U_1 - xy$ be U_{1x} and U_{1y} with $x \in V(U_{1x})$ and $y \in V(U_{1y})$.

If $v \in V(U_{1x})$, then let $U_1' = U_{1x}$, $U_2' = (U_2 \cup U_{1y}) + t_i u_i$ and $T' = T + xy - t_i u_i$; and if $v \in V(U_{1y})$, then let $U_1' = U_{1y}$, $U_2' = (U_2 \cup U_{1x}) + t_1 u_1$ and $T' = T + xy - t_1 u_1$. In any case, we have $v \in V(U_1')$, $V(T') = V(T), V(T') - V(U_1') \subseteq V(U_2'), V(U_1') \cup V(U_2') = V(U_1 \cup U_2)$ and $|E(U_1' \cup U_2')| = |E(U_1 \cup U_2)|$, whereas $|V(U_1')| < |V(U_1)|$, contrary to the minimality of $|V(U_1)|$. □

As a by-product of the proof of Lemma 4.1, we obtain the corollary below also.

Corollary 4.2. *Let (T, U_1, U_2) be a 3-forest of a reduced graph G such that U_1 contains the given edge e. Then G has a 3-forest $(\bar{T}, \bar{U}_1, \bar{U}_2)$ such that $V(T) = V(\bar{T}), E(\bar{U}_1) = \{e\}, V(\bar{U}_1 \cup \bar{U}_2) = V(U_1 \cup U_2) = V(U)$ and $|E(\bar{U}_1 \cup \bar{U}_2)| = |E(U_1 \cup U_2)|$.*

The proof of Corollary 4.2 is an imitation of the proof of Lemma 4.1, but with e in place of v. The hypothesis $v \in V(U_{1x})$ $(v \in V(U_{1y}))$ is replaced by $e \in E(U_{1x})$ $(e \in V(U_{1y}))$.

Lemma 4.3. *Let G be a reduced graph with $\kappa'(G) = 3$ and $F(G) = 2$. For any $v \in V(G)$, G has edge-disjoint forests T and U with the following properties:*

(i) *T has exactly two components T_1 and T_2 such that $T_2 \cong K_2$.*

(ii) *U has exactly two components U_1 and U_2 such that $V(U_1) = \{v\}$.*

(iii) *$V(T_2) \subset V(U_2)$.*

Proof. Since G is reduced, by (v) of Theorem 2.1, G has two edge-disjoint spanning forests T' and U' such that $E(G) = E(T') \cup E(U')$. Since $F(G) = 2$, we have $\omega(T') + \omega(U') = 4$. If T' has two components T_1' and T_2', then by the assumption that G is connected, G has an edge $st \in E(U')$, with $s \in V(T_1')$ and $t \in V(T_2')$, such that $T' + st$, $U' - st$ are edge-disjoint spanning forests. Therefore we assume that $\omega(T') = 1$, and U' has exactly 3 components U_1', U_2' and U_3'. Note that $(T', U_1', U_2' \cup U_3')$ is a 3-forest and so, by Lemma 4.1, G has a 3-forest $(T, U_1, U_2 \cup U_3)$ such that T and U are edge-disjoint spanning forests of G, where U has components U_1, U_2 and U_3, and such that $V(U_1) = \{v\}$.

By Corollary 4.2, it suffices to show that G has a 3-forest $(\bar{U}, \bar{T}_2, \bar{T}_1)$ such that $V(\bar{U}) = V(U_2 \cup U_3)$, $v \notin V(\bar{T}_2)$ and $|V(\bar{T}_2)| \geqslant 2$. In the following, we assume that no such 3-forest $(\bar{U}, \bar{T}_2, \bar{T}_1)$ exists, to obtain a contradiction.

Let $T_1, T_2, \ldots, T_r$ be the components of $T - v$, and for each $1 \leqslant i \leqslant r$, denote $N_T(v) \cap V(T_i) = \{t_i\}$. Note that $r \geqslant \kappa'(G) \geqslant 3$. We may assume that for some $1 \leqslant r' < r$,

$$\{t_1, t_2, \ldots, t_{r'}\} \subseteq V(U_2) \quad \text{and} \quad \{t_{r'+1}, t_{r'+2}, \ldots, t_r\} \subseteq V(U_3). \tag{3}$$

Without loss of generality and by the fact that $r \geqslant 3$, we assume that

$$r' \leqslant r - 2. \tag{4}$$

Among all the decompositions (T, U_1, U_2, U_3) of $E(G)$ such that T is a spanning tree, such that $U = U_1 \cup U_2 \cup U_3$ is a spanning forest with $V(U_1) = \{v\}$ and such that (3) and (4) are satisfied, we choose one so that

$$|V(U_2)| \text{ is maximized.} \tag{5}$$

Claim 1. *$V(T_i) \cap V(U_3) \subseteq D_1(T_i)$ $(1 \leqslant i \leqslant r')$ and $V(T_j) \cap V(U_2) \subseteq D_1(T_j)$ $(r'+1 \leqslant j \leqslant r)$.*

Let $x \in V(T_i) \cap V(U_3)$. Then we may assume that there is a vertex $x' \in V(T_i) \cap V(U_2)$ such that $xx' \in E(T_i)$. If the component T' (say) of $T - xx'$ containing x has at least 2 vertices, then $((U_2 \cup U_3) + xx', T', T - V(T'))$ is the desired 3-forest. Hence we must have $x \in D_1(T_i)$. The proof for the other half of the claim is similar.

Claim 2. *$V(T_i) \cap V(U_3) = \emptyset$ $(1 \leqslant i \leqslant r')$ and $V(T_i) \cap V(U_2) = \emptyset$ $(r' + 1 \leqslant j \leqslant r)$.*

Assume that $X = \bigcup_{i=1}^{r'} (V(T_i) \cap V(U_3)) \neq \emptyset$. Since U_3 is connected, there must be an $x \in X$ and an $x'' \in V(U_3) - X$ such that $xx'' \in E(U_3)$. We assume that $x \in V(T_i)$ and $x'' \in V(T_j)$ with $1 \leqslant i \leqslant r' < j \leqslant r$ and both $1 \leqslant j < r$ and $i \neq j$. By Claim 1, there is a vertex $x' \in V(T_i) \cap V(U_2)$ with $xx' \in E(T_i)$. Denote by U_x the component of $U_3 - xx''$ containing x. Thus $(T + xx'' - xx', \{v\}, U_2 + xx', U_3 - U_x)$ violates the choice of (T, U_1, U_2, U_3), including (5). The proof for the other half of the claim is similar. This proves Claim 2.

Define

$$H = G\left[\{v\} \cup \bigcup_{j=r'+1}^{r} V(T_i)\right].$$

By Claim 2 and the definitions of U_1, U_2, and U_3, the tree U_3 spans $H - v$. Clearly, $T[V(H)]$ is connected. Hence,

$$\begin{aligned} F(H) &= 2|V(H)| - 2 - (|E(U_3)| + |E(H)|) \\ &= 2|V(H)| - 2 - (|E(U_3)| + |E(T[V(H)])|) \\ &= 2|V(H)| - 2 - (|V(H)| - 2 + |V(H)| - 1) = 1. \end{aligned}$$

By (i) of Theorem 1.2 and by the assumption that G is reduced, H is reduced, and so by (ii) of Theorem 1.2, $H \cong K_2$, contrary to the fact that $|V(H)| \geqslant 3$. □

Lemma 4.4 (Catlin [3]). *Let $R \subseteq V(G)$. Then the graph G has a spanning tree such that each component of $G - E(T)$ has an even number of vertices in R if and only if G has an R-subgraph. (Such a spanning tree T is called an R-tree of G.)*

Theorem 4.5 (Chen [9]). *If G is a 3-edge-connected graph with at most 11 vertices, then either G is collapsible, or the reduction of G is the Petersen graph.*

In the proof of Theorem 1.4 below, we use the following notation: if a subgraph H is a tree, and if $u, v \in V(H)$, then $H(u, v)$ denotes the unique (u, v)-path in H.

Proof of Theorem 1.4 (*continued*). Since G is assumed to be noncollapsible, by Lemma 4.4, there is an $R \subseteq V(G)$ with $|R|$ even such that G does not have an R-tree.

If $R = V(G)$, then choose v so that $d_G(v) = \Delta(G)$; if $R \neq V(G)$, then choose $v \in V(G) - R$. By Lemma 4.3, G has disjoint forests T (with components T_1 and T_2) and U (with components U_1 and U_2) satisfying (i), (ii) and (iii) of Lemma 4.3. Let the components of $T_1 - v$ be T_{1i} and $N_T(v) \cap V(T_{1i}) = \{t_i\}$ $(1 \leqslant i \leqslant r)$, and let $V(T_2) = \{u_1, u_2\}$.

Claim 1. *If $|\{u_1, u_2\} \cap R|$ is even, then for all $1 \leqslant i \leqslant r$, $|R \cap V(T_{1i})|$ is odd.*

If $|\{u_1, u_2\} \cap R|$ is even, and if for some i, $|R \cap V(T_{1i})|$ is even, then $U' = (U_1 \cup U_2) + vt_i$ is a spanning tree of G such that $G - E(U')$ has 3 components, T_2, T_{1i}, and $T_1 - V(T_{1i})$. By Lemma 4.4, U' is an R-tree, a contradiction. This proves Claim 1.

An edge $xy \in E(U_2)$ is *good* if for some $1 \leqslant i < j \leqslant r$, $x \in V(T_{1i})$ and $y \in V(T_{1j})$ and if t_i and t_j are in different components of $U_2 - xy$. Assume that such an edge exists and that $|R \cap \{u_1, u_2\}|$ is even. Then by Claim 1, $U' = U_2 - xy + \{vt_i, vt_j\}$ is an R-tree. Therefore we have proved the following:

Claim 2. *If $|\{u_1, u_2\} \cap R|$ is even, then all edges in U_2 are not good.*

Let U_{21} be the smallest subtree of U_2 that contains $\{t_1, t_2, \ldots, t_r\}$. Then $D_1(U_{21}) \subseteq \{t_1, t_2, \ldots, t_r\}$.

Obtain a tree U'_{21} by contracting each edge $e = xy \in E(U_{21})$ such that both $x, y \in V(T_{1i})$, for the same i with $1 \leqslant i \leqslant r$. Note that the preimage of a vertex in $D_1(U'_{21})$ must contain a vertex in $D_1(U_{21})$. Without loss of generality, we assume that $t_1, t_2, \ldots, t_l \in D_1(U_{21})$ such that they are in distinct contraction-images of vertices in $D_1(U'_{21})$. Therefore, for each $i \in \{1, 2, \ldots, l\}$, there exists an edge $t'_i t''_i \in E(U_2)$ such that $t'_i \in V(T_i)$ and $t''_i \notin V(T_i)$, and such that $U_2 - t'_i t''_i$ has two components U'_2, U''_2 with $\{t_i, t'_i\} \subseteq V(U'_2)$ and $\{t_1, t_2, \ldots, t_r\} - \{t_i\} \subseteq V(U''_2)$.

Claim 3. *If* $|R \cap \{u_1, u_2\}|$ *is even, then* $\{t_1'', t_2''\} \subseteq \{u_1, u_2\}$.

If $t_i'' \notin \{u_1, u_2\}$, for $i = 1$ and 2, then $t_i' t_i''$ is a good edge, contrary to Claim 2. This proves Claim 3.

Let $U_2(u_1, u_2) = u_1 s_1 s_2 \ldots s_k u_2$. By Claim 3 and without loss of generality, we assume that $t_1'' = u_1$. Note that by Claim 2, if $|R \cap \{u_1, u_2\}|$ is even, then $t_1' t_1'' \notin E(U_2(u_1, u_2))$.

It remains to consider the following possible cases, depending on $|\{u_1, u_2\} \cap R|$. In the first two cases below, Claims 1, 2 and 3 apply.

Case 1: $\{u_1, u_2\} \subset R$. Assume first that $s_k \in V(T_{1j})$ with $j \neq 1$. Then $U' = U_2 + \{u_1 u_2, vt_1, vt_j\} - \{t_1' t_1'', s_k u_2\}$ is an R-tree: $G - E(U')$ has 3 components $T_{11} + t_1' t_1''$, $T_{1j} + s_k u_2$, and $T_1 - (V(T_{11}) \cup V(T_{1j}))$, a contradiction. (Similarly, a contradiction obtains if $t_2'' = u_2$ and if $s_1 \in V(T_{1j})$ with $j \neq 2$.)

Hence $s_k \in V(T_{11})$. If $t_2'' = u_1$, then $U' = U_2 + \{u_1 u_2, vt_1, vt_2\} - \{t_1' t_2'', u_2 s_k\}$ is an R-tree: $G - E(U')$ has 3 components $T_{11} + s_k u_2$, $T_{12} + t_2' t_2''$ and $T_1 - (V(T_{11}) \cup V(T_{12}))$. Therefore we may assume that $t_1'' = u_1$, $t_2'' = u_2$, $s_1 \in V(T_{12})$ and $s_k \in V(T_{11})$.

Note that in this case, $U_2(u_1, u_2)$ is a subgraph of U_{21}. If U_{21} is not a path, then $l \geqslant 3$. By symmetry, we assume that $t_3'' = u_1$. Thus $U' = U_2 + \{u_1 u_2, vt_3, vt_1\} - \{t_3' t_3'', s_k u_2\}$ is an R-tree. Therefore, U_{21} must be a path, and so for any $3 \leqslant i \leqslant r$,

$$\text{there exists an } s_{n_i} \ (1 \leqslant n_i \leqslant k) \text{ such that } V(U_2(t_i, s_{n_i})) \subset V(T_{1i}). \tag{6}$$

If $r = d_G(v) \geqslant 4$, then by (6), $E(U_2(u_1, u_2))$ contains a good edge, contrary to Claim 2. Hence $r = 3$, and so by $u_1, u_2 \in R$ and by Claim 1, $v \in R$. By the choice of v, G must be 3-regular, and so by (1), $3|V(G)| = 2|E(G)| = 4|V(G)| - 8$. Hence $|V(G)| \leqslant 8$, and so by Theorem 4.5, G is collapsible, a contradiction. This proves Case 1.

Case 2: $\{u_1, u_2\} \cap R = \emptyset$. Recall the assumption that $t_1'' = u_1$. If $s_1 \in V(T_{1i})$ for some $i \neq 1$, then $U' = U_2 + \{vt_1, vt_i, u_1 u_2\} - \{t_1' u_1, u_1 s_1\}$ is an R-tree: $G - E(U')$ has 3 components $\{u_2\}$, $(T_{11} \cup T_{1i}) + \{t_1' u_1, u_1 s_1\}$, and $T_1 - (V(T_{11}) \cup V(T_{1i}))$.

Hence we assume that $s_1 \in V(T_{11})$, and so by Claim 2, $V(U_2(u_1, u_2)) - \{u_1, u_2\} \subset V(T_{11})$ (and so $s_k \in V(T_{11})$). If $t_2'' = u_1$, then $U' = U_2 + \{vt_1, vt_2, u_1 u_2\} - \{u_1 s_1, u_1 t_2'\}$ is an R-tree: $G - E(U')$ has components $\{u_2\}$, $T_{11} \cup T_{12} + \{u_1 s_1, u_1 t_2'\}$ and $T_1 - (V(T_{11}) \cup V(T_{12}))$.

Hence we may assume that $t_1'' = u_1, s_1 \in V(T_{11})$ and $t_2'' = u_2$, and so $s_k \in V(T_{11})$. Thus $U' = U_2 + \{vt_1, vt_2, u_1 u_2\} - \{t_2' t_2'', u_2 s_k\}$ is an R-tree: $G - E(U')$ has 3 components $\{u_1\}$, $(T_{11} \cup T_{12}) + \{t_2' t_2'', u_2 s_k\}$, $T_1 - V(T_{11} \cup T_{12})$, a contradiction.

Case 3: $|\{u_1, u_2\} \cap R| = 1$. We may assume that $u_2 \in R$ and $u_1 \notin R$ and that in the path $U_2(u_1, u_2) = u_1 s_1 \ldots s_k u_2$, $s_k \in V(T_{1i'})$, for some $1 \leqslant i' \leqslant r$. If for some $i \in \{1, 2, \ldots, r\} - \{i'\}$, $|R \cap V(T_{1i})|$ is even, or if when $i = i'$, $|R \cap V(T_{1i'})|$ is odd, then $U' = U_2 + \{u_1 u_2, vt_i\} - u_2 s_k$ is an R-tree: when $i \neq i'$, $G - E(U')$ has components $\{u_1\}, T_{1i}$ and $(T_1 - V(T_{1i})) + \{u_2 s_k\}$; when $i = i'$, $G - E(U')$ has components $\{u_1\}$, $T_{1i} + u_2 s_k$ and $T_1 - V(T_{1i})$. Therefore we may assume that

$$|R \cap V(T_{1i})| \text{ is odd, } i \in \{1, 2, \ldots, r\} - \{i'\} \text{ and } |R \cap V(T_{1i'})| \text{is even.} \tag{7}$$

Suppose that U_2 has a good edge xy with $x \in V(T_{1i})$ and $y \in V(T_{1j})$ and if t_i and t_j are in different components of $U_2 - xy$, for some $1 \leqslant i < j \leqslant r$. If $i' \notin \{i,j\}$, then $U' = U_2 + \{vt_i, vt_j, u_1u_2\} - \{xy, u_2s_k\}$ is an R-tree: $G - E(U')$ has components $\{u_1\}$, $(T_{1i} \cup T_{1j}) + \{xy\}$, and $(T_1 - (V(T_{1i}) \cup V(T_{1j})) + \{u_2s_k\}$.

Thus we assume that $i \neq j = i'$. Then $U' = U_2 + \{vt_i, vt_{i'}, u_1u_2\} - \{xy, u_2s_k\}$ is an R-tree: $G - E(U')$ has components $\{u_1\}, T_{1i} \cup T_{1i'} \cup \{u_2\} + \{xy, u_2s_k\}$, and $T - V(T_{1i}) \cup V(T_{1i'})$. Hence we assume that

$$\text{all edges in } U_2 \text{ are not good.} \tag{8}$$

Since $|D_1(U_{21})| \geqslant 2$, we may assume that $i' > 1$. By (8), $t_1'' \in \{u_1, u_2\}$. Let $U' = U_2 + \{vt_1, vt_{i'}\} - t_1't_1''$. Then by (7), U' is an R-tree: $G - E(U')$ has three components $(T_{11} \cup T_2) + t_1't_1'', T_{1i'}$ and $T_1 - V(T_{11} \cup T_{1i'})$, a contradiction.

These contradictions establish Theorem 1.4. □

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