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# Reduction Towards Collapsibility

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**Abstract**

Let $G$ be a graph and let $R \subseteq V(G)$ be a subset with $|R|$ even. A subgraph $\Gamma$ is called an $\underline{R\text{-subgraph}}$ of $G$ if both $G - E(\Gamma)$ is connected and the set of odd vertices of $\Gamma$ is $R$. A graph $G$ is collapsible if for every subset $R \subseteq V(G)$ with $|R|$ even, $G$ has an $R$-subgraph. The reduction of $G$ is the graph obtained from $G$ by contracting all nontrivial collapsible subgraphs of $G$. It is known that if $H$ is a collapsible subgraph of $G$, then $G$ has a spanning closed trail if and only if $G/H$, the graph obtained from $G$ by contracting $H$, has a spanning closed trail. A similar equivalence holds for graphs with nowhere-zero 4-flows. It is shown that the 3-cycle is collapsible but not the $n$-cycle with $n \geq 4$. In [Congressus Numerantium 58 (1987), 233 - 246], Catlin introduced the concept of $\pi$-collapsiblity and use it to show a way to reduce the 4-cycle. In this note, we shall show that a graph $G$ is $\pi$-collapsible if and only if the reduction of $G$ is $\pi$-collapsible. We also find a way to reduce the $n$-cycle, for $n \geq 4$.

**1. Introduction**. The graphs in this note are finite, undirected and loopless, but may have multiple edges. We shall use the notation of Bondy and Murty [1], except for contractions. Let $G$ be a graph and let $E \subseteq E(G)$, the contraction $G/E$ is the graph obtained from $G$ by identifying the ends of each edge of $E$ and then deleting the resulting loops. If $H$ is a connected subgraph of $G$, then we use $G/H$ for $G/E(H)$. For convenience, we define $G/\emptyset = G$. Let $S$ be a nonemptyset. A partition of $S$ is a collection $< S_1, S_2 >$ of two nonempty subsets of $S$ such that they are pairwise disjoint and such that their union is $S$.

Collapsible graphs are defined in [5] by Catlin. Let $G$ be a graph and let $R \subseteq V(G)$ be a subset with $|R|$ even. A subgraph $\Gamma$ is called an

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$R$-subgraph of $G$ if both $G - E(\Gamma)$ is connected and the set of odd vertices of $\Gamma$ is $R$. A graph $G$ is collapsible if for every subset $R \subseteq V(G)$ with $|R|$ even, $G$ has an $R$-subgraph. Collapsible graphs are shown to be useful in double cycle covers, spanning eulerian subgraphs and hamiltonian line graphs, (see [5], [6], [3] and [9]), among others. In [5], Catlin showed that for any graph $G$, there is a unique collection of maximal collapsible subgraphs. When all these collapsible subgraphs of $G$ are contracted, the resulting graph is called the reduction of $G$. A graph is reduced if it is the reduction of some graph.

Theorem 1.1 (Catlin [5]) Let $G$ be a graph.

(a) $G$ is reduced if and only if $G$ has no nontrivial collapsible subgraphs.

(b) $G$ is collapsible if and only if the reduction of $G$ is $K_1$.

Let $H$ be a collapsible subgraph of $G$, and let $v_H$ denote the vertex in $G/H$ to which $H$ is contracted. Let $R \subseteq V(G)$ be a subset of even cardinality. Define

$$R/H = (R \cap [V(G) - V(H)]) \cup X, \qquad (1)$$

where $X = \emptyset$ if $|R \cap V(H)|$ is even, and $X = \{v_H\}$ if $|R \cap V(H)|$ is odd. Thus $R/H \subseteq V(G/H)$.

Theorem 1.2 (Catlin [5]) Let $G$ be a graph and let $R \subseteq V(G)$ be a subset of even cardinality. If $H$ is collapsible, then $G/H$ has an $R/H$ subgraph $\Gamma'$ if and only if $G$ has an $R$-subgraph $\Gamma$ with $E(\Gamma) \subseteq E(H) \cup E(\Gamma')$.

All complete graph of order at least 3 are collapsible. All cycles of length at least 4 are not collapsible. What can be done for cycles with length at least 4?

In [6], Catlin invented a method to reduced the 4-cycle. Let $H$ be a graph and let $\pi =< V_1, V_2 >$ be a partition of $V(H)$. Then $H$ is called $\pi^-$-collapsible (resp., $\pi^+$-collapsible) is for every subset $R \subseteq V(H)$ with $|R|$ even,

(i) if $|R \cap V_1|$ is odd (resp., is even), then $H$ has an $R$-subgraph;

(ii) if $|R \cap V_1|$ is even (resp., odd), then $H + e$ has an $R$-subgraph $\Gamma_e \subseteq H$, for any added edge $e = w_1w_2$ with $w_1 \in V_1$ and $w_2 \in V_2$.

If $H$ is either $\pi^-$-collapsible or $\pi^+$-collapsible, then $H$ is $\pi$-collapsible.

All nontrivial collapsible graphs are $\pi$-collapsible, for any partition of vertex set. The 4-cycle is $\pi$-collapsible with each subset in $\pi$ being an independent set of size 2. All cycles of order least 5 are not $\pi$-collapsible, for any $\pi$.

Let $H$ be a $\pi$-collapsible subgraph of $G$, for some partition $\pi =< V_1, V_2 >$ of $V(H)$. Denote by $G/\pi$ the graph obtained from $G$ by identifuing all vertices of $V_1$ to form a single vertex $v_1$, and by identifying all vertices of $V_2$ to form a single vertex $v_2$, and by joining $v_1$ and $v_2$ with exactly one edge $e_\pi = v_1v_2$.

Theorem 1.3 (Catlin [6]) Suppose that $H$ is a $\pi$-collapsible subgraph of $G$ for some partition $\pi$ of $V(H)$. Each of the following holds:

(a) If $G/\pi$ is collapsible, then $G$ is collapsible.

(b) If $G/\pi$ has a spanning eulerian subgraph, then $G$ has a spanning eulerian subgraph.

In this note, we continue the study of reducing the cycles. In Section 2, we show that a graph $G$ is $\pi$-collapsible if and only if the reduction of $G$ is $\pi$-collapsible. In Section 3, we discuss a way to reduce the $n$-cycle, for any $n \geq 4$ such that the analogues of Theorem 1.3 hold. In the last section, we discuss the reduction in searching for nowhere-zero $k$-flows and related problems.

**2. Collapsibility and $\pi$-collapsibility.** The main result of this section is Theorem 2.1 below whose proof will be divided into 2 lemmas.

Theorem 2.1 Let $G'$ denote the reduction of $G$. Then $G$ is $\pi$-collapsible for some partition $\pi$ of $V(G)$ if and only if $G'$ is $\pi'$-collapsible for some partition $\pi'$ of $V(G')$.

Lemma 2.2 Let $H$ be a collapsible subgraph of $G$. If $G/H$ is $\pi'$-collapsible for some partition $\pi'$ of $V(G/H)$, then $G$ is $\pi$-collapsible for some partition $\pi$ of $V(G)$.

Proof: We assume first that $G/H$ is $\pi^-$-collapsible for some partition $\pi' =< X_1, X_2 >$ of $V(G/H)$. We shall use $v_H$ to denote the vertex to which $H$ is contracted.

For notational convenience, we shall regard

$$V(G/H) = [V(G) - V(H)] \cup \{v_H\}. \tag{2}$$

Without loss of generality, we may assume that $v_H \in X_1$. Define

$$V_1 = (X_1 - \{v_H\}) \cup V(H),\ V_2 = X_2. \tag{3}$$

Then by (1), $\pi =< V_1, V_2 >$ is a partition of $V(G)$.

Let $R \subseteq V(G)$ be a subset of $V(G)$ with even cardinality. Define $R/H$ as in (1). Then by (1) and (2), $R/H$ is a subset of even cardinality of $V(G/H)$. By (1) and (3), both $|R \cap V_1|$ and $|(R/H) \cap X_1|$ have the same parity.

If $|(R/H) \cap X_1|$ is odd, then since $G/H$ is $\pi'^-$-collapsible, $G/H$ has an $R/H$-subgraph. By Theorem 1.2 $G$ has an $R$-subgraph.

Suppose that $|(R/H) \cap X_1|$ is even. For any pair of vertices $v_1$ and $v_2$ with $v_1 \in V_1$ and $v_2 \in V_2$, we can form an edge $e = v_1 v_2$ not in $G$. This edge $e$ induces an added edge $e' = x_1 x_2$ in $G/H$ with $x_1 \in X_1$ and $x_2 \in X_2$ such that $e' \notin E(G/H)$ and $(G/H) + e' \cong (G+e)/H$. Since $G/H$ is $\pi'^-$-collapsible, $(G/H) + e'$ has an $R/H$-subgraph in $G/H$. By $(G/H) = e' \cong (G+e)/H$, and by Theorem 1.2, $G + e$ has an $R$-subgraph in $G$.

Thus $G$ is $\pi^-$-collapsible, by definition. The proof when $G/H$ is $\pi'^+$-collapsible is similar. □

Corollary 2.3 Let $G'$ denote the reduction of $G$. If $G'$ is $\pi'$-collapsible, for some partition $\pi'$ of $V(G')$, then there is a partition $\pi$ of $V(G)$ such that $G$ is $\pi$-collapsible.

Proof: Since $G'$ is obtained from $G$ by contracting one maximal collapsible subgraph at a time, one can repeatedly apply Lemma 2.2 to get Corollary 2.3. □

Lemma 2.4 Let $H$ be a collapsible subgraph of $G$. If $G$ is $\pi$-collapsible for some partition $\pi$ of $V(G)$, then there is a partition $\pi'$ of $V(G/H)$ such that $G/H$ is $\pi'$-collapsible.

Proof: Again we assume first that $G$ is $\pi^-$-collapsible, for some partition $\pi =< V_1, V_2 >$. Define

$$X_1 = [V_1 - V(H)] \cup X, \; X_2 = [V_2 - V(H)] \cup Y,$$

where $X = \{v_H\}$ and $Y = \emptyset$ if $V_1 \cap V(H) \neq \emptyset$, and $X = \emptyset$ and $Y = \{v_H\}$ if $V_1 \cap V(H) = \emptyset$.

Then $\pi' =< X_1, X_2 >$ is a partition of $V(G/H)$. Let $R' \subseteq V(G/H)$ be a subset of even cardinality.

Case 1 $v_H \notin R'$.

Then since $v_H \notin R'$, we can regard $R' \subseteq V(G)$. Suppose that $|R' \cap X_1|$ is odd. Since $G$ is $\pi^-$-collapsible, $G$ has an $R'$-subgraph. By Theorem 1.2, $G/H$ has an $R'$-subgraph.

Suppose that $|R' \cap X_1|$ is even. For any pair of vertices $x_1$ and $x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$, we can form an edge $e' = x_1x_2$ that is not in $G/H$. This edge $e'$ induces an added edge $e = v_1v_2$ for $G$ with $v_1 \in V_1$ and $v_2 \in V_2$ such that $e \neq E(G)$ and $(G+e)/H \cong (G/H)+e'$. In fact, it suffices to consider the case when $x_1 = v_H$ or when $x_2 = v_H$. Suppose first that $x_1 = v_H$. Then since $V_1 \cap V(H) \neq \emptyset$, we can choose any $v_1 \in V_1 \cap V(H)$ and $v_2 = x_2$. Now $|R' \cap V_1|$ is even, and so $G + e$ has an $R$-subgraph in $G$, by the assumption that $G$ is $\pi^-$-collapsible. By Theorem 1.2 and by the fact that $(G+e)/H \cong (G/H) + e'$, it follows that $(G/H) + e'$ has an $R'$-subgraph in $G/H$.

The case when $x_2 = v_H$ can be proved similarily.

Case 2 $v_H \in R'$ and $V_1 \cap V(H) \neq \emptyset$.

Without loss of generality, we may assume that $v_h \in X_1$. Let $R = (R' - \{v_H\}) \cap \{v\}$, where $v \in V_1 \cap V(H)$ is an arbitrary vertex in $H$. Then we have $R \subseteq V(G)$. Since $v_H \in X_1$, for $i \in \{1,2\}$, both $|R \cap V_i|$ and $|R \cap X_i|$ have the same parity.

If $|R' \cap X_1|$ is odd, then $|R \cap V_1|$ is also odd. Since $G$ is $\pi^-$-collapsible, $G$ has an $R$-subgraph. Note that any $R$-subgraph of $G$ is contracted onto an $R'$-subgraph of $G/H$, by the definition of $R$. Thus $G/H$ has an $R'$-subgraph.

Now suppose that $|R' \cap X_1|$ is even. Then $|R \cap V_1|$ is also even. For any pair of vertices $x_1$ and $x_2$ of $G/H$ with $x_1 \in X_1$ and $x_2 \in X_2$, we can form an edge $e = x_1x_2$ that is not in $G/H$. This edge $e' = x_1x_2$ induces an added edge $e = v_1v_2$ for $G$ with $v_1 \in V_1$ and $v_2 \in V_2$ such that $e \notin E(G)$ and $(G+e)/H \cong (G/H) + e'$. Since $|R \cap V_1|$ is even and since $G$ is $\pi^-$-collapsible, $G + e$ has an $R$-subgraph in $G$. Again, this $R$-subgraph is contracted to an $R'$-subgraph of $(G/H)+e'$ in $G/H$.

Case 3 $v_H \in R'$ and $V_1 \cap V(H) = \emptyset$.

Then $v_H \in X_2$. Let $R = (R' - \{v_H\}) \cup \{v\}$ with $v \in V(H)$. The proof is similar to that of Case 2. □

Corollary 2.5 Let $G'$ denote the reduction of $G$. If $G$ is $\pi$-collapsible, for some partition $\pi$ of $V(G)$, then there is a partition $\pi'$ of $G'$ such that $G'$ is $\pi'$-collapsible.

Proof: Since $G'$ is obtained from $G$ by contracting one maximal collapsible subgraph at a time, one can repeatedly apply Lemma 2.4 to get Corollary 2.5. □

Proof of Theorem 2.1: It follows by combining Corollary 2.3 and

Corollary 2.5. □

**3. Reducing a cycle.** Let $C = x_1x_2...x_mx_1$ be an $m$-cycle of $G$ with $m \geq 4$. Let $e$ and $e'$ be two non-adjacent edges of $C$. Without loss of generality, we may assume that $e = x_1x_2$ and $e' = x_ix_{i+1}$, where $m > i > 2$. Define

$$G'(C_n) = (G - [E(C) - \{e, e'\}])/\{e, e'\}.$$

Let $w$ and $w'$ denote the two vertices of $G'(C_n)$ to which the edges $e, e'$ are contracted, respectively. For convenience, we regard

$$V(G'(C_n)) - \{w, w'\} = V(G) - \{x_1, x_2, x_i, x_{i+1}\},$$

and

$$E(G'(C_n)) = E(G) - E(C).$$

For $R \subseteq V(G)$, define $R'' \subseteq V(G'(C_n))$ by

$$R'' = \begin{cases} ([V(G) - V(C)] \cap R) \cup \{w\} & \text{if } |R \cap \{x_1, x_2\}| = 1 \\ ([V(G) - V(C)] \cap R) \cup \{w'\} & \text{if } |R \cap \{x_i, x_{i+1}\}| = 1 \\ ([V(G) - V(C)] \cap R) & \text{if } |R \cap \{x_1, x_2\}| \text{ and} \\ & |R \cap \{x_i, x_{i+1}\}| \text{ are even} \end{cases}$$

Theorem 3.1 Let $R$ be an even set of vertices of $G$. If $G'(C_n)$ has an $R''$-subgraph, then $G$ has an $R$-subgraph.

Proof: Let $\Gamma'$ be an $R''$-subgraph of $G'(C_n)$ and let $\Gamma_1 = G[E(\Gamma')]$. Note that these are equivalent:

(a) $d_{\Gamma'}(w)$ is odd,
(b) $d_{\Gamma_1}(x_1)$ is not congruent to $d_{\Gamma_1}(x_2)$ (mod 2),
(c) $|R \cap \{x_1, x_2\}| = 1$;

and that these are equivalent:

(d) $d_{\Gamma'}(w')$ is odd,
(e) $d_{\Gamma_1}(x_i)$ is not congruent to $d_{\Gamma_1}(x_{i+1})$ (mod 2),
(f) $|R \cap \{x_i, x_{i+1}\}| = 1$.

Define $P$ and $P'$ to be two subgraphs of $C$ by:

$$P = x_2x_3...x_i, \; P' = x_{i+1}...x_mx_1.$$

Define $\Gamma$ in Table 1 below, where an occurence of a 0 in the column $x_j$ means $x_j \notin R$, whereas an occurence of a 1 means $x_j \in R$.

Since $G'(C_n) - E(\Gamma')$ is connected, At most one of the two edges $e, e'$ is used in $E(\Gamma)$, and when one of $\{e, e'\}$ is in $E(\Gamma)$, all the edges in

$E(P) \cup E(P')$ are in $E(G - E(\Gamma))$ and so $G - E(\Gamma)$ is always connected. Note that by the definition of $\Gamma$, the set of vertices of odd degree in $\Gamma$ is $R$, and so $\Gamma$ is an $R$-subgraph of $G$.

Table 1 : Definition of $\Gamma$ in the proof of Theorem 2

| $x_1$ | $x_2$ | $x_i$ | $x_{i+1}$ | $d_{\Gamma_1}(x_1)$ | $d_{\Gamma_1}(x_2)$ | $d_{\Gamma_1}(x_i)$ | $d_{\Gamma_1}(x_{i+1})$ | $\Gamma$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | even | even | even | even | $\Gamma_1$ |
| 0 | 0 | 0 | 0 | even | odd | odd | even | $\Gamma_1 + E(P)$ |
| 0 | 0 | 0 | 0 | odd | even | even | odd | $\Gamma_1 + E(P')$ |
| 0 | 0 | 0 | 0 | odd | odd | odd | odd | $\Gamma_1 + (E(P) \cup E(P'))$ |
| 1 | 0 | 0 | 0 | even | even | even | odd | $\Gamma_1 + E(P')$ |
| 1 | 0 | 0 | 0 | odd | even | even | even | $\Gamma_1$ |
| 1 | 0 | 0 | 0 | even | odd | odd | odd | $\Gamma_1 + (E(P') \cup E(P))$ |
| 1 | 0 | 0 | 0 | odd | odd | odd | even | $\Gamma_1 + E(P)$ |
| 1 | 1 | 0 | 0 | even | even | odd | odd | $\Gamma_1 + (E(P) \cup E(P'))$ |
| 1 | 1 | 0 | 0 | even | odd | even | odd | $\Gamma_1 + E(P')$ |
| 1 | 1 | 0 | 0 | odd | even | odd | even | $\Gamma_1 + E(P)$ |
| 1 | 1 | 0 | 0 | odd | odd | even | even | $\Gamma_1$ |
| 1 | 1 | 1 | 0 | even | even | even | odd | $\Gamma_1 + (E(P) \cup E(P'))$ |
| 1 | 1 | 1 | 0 | even | odd | odd | odd | $\Gamma_1 + E(P')$ |
| 1 | 1 | 1 | 0 | odd | even | even | even | $\Gamma_1 + E(P)$ |
| 1 | 1 | 1 | 0 | odd | odd | odd | even | $\Gamma_1$ |
| 1 | 1 | 1 | 1 | even | even | even | even | $\Gamma_1 + (E(P) \cup E(P'))$ |
| 1 | 1 | 1 | 1 | even | odd | odd | even | $\Gamma_1 + E(P')$ |
| 1 | 1 | 1 | 1 | odd | even | even | odd | $\Gamma_1 + E(P)$ |
| 1 | 1 | 1 | 1 | odd | odd | odd | odd | $\Gamma_1$ |

The other cases not covered by Table 1 can be obtained by symmetry, or they can be excluded by the equivalences (a) $\Longleftrightarrow$ (b) $\Longleftrightarrow$ (c) and (d) $\Longleftrightarrow$ (e) $\Longleftrightarrow$ (f). □

Corollary 3.2 If $G'(C_n)$ is collapsible, then $G$ is also collapsible.

Proof: Apply Theorem 3.1 to each even subset $R$ of $V(G)$. □

Corollary 3.3 If $G'(C_n)$ is supereulerian, then $G$ is supereulerian.

Proof: Let $O(G)$ denote the set of all odd degree vertices of $G$, and let $O(G'(C_n))$ denote all the odd degree vertices of $G'(C_n)$. If

$$O(G'(C_n)) = [O(G)]'', \tag{4}$$

then by the assumption that $G'(C_n)$ has an $O(G'(C_n))$-subgraph and by Theorem 3.1, $G$ has an $O(G)$-subgraph and so $G$ is supereulerian. Thus we shall show that (4) holds always.

Clearly (4) holds when $O(G) \cap V(C) = \emptyset$. Suppose that $O(G) \cap V(C) \neq \emptyset$. Since $G'(C_n) = (G - [E(C) - \{e, e'\}])/\{e, e'\}$, the parity of $d_G(x_1) + d_G(x_2)$ is the same as that of $d_{G'(C_n)}(w)$, and the parity of $d_G(x_i) + d_G(x_{i+1})$ is the same as that of $d_{G'(C_n)}(w')$. These, together with the definition of $[O(G)]''$, imply that (4) must hold also. □

When $n = 4$, Corollary 3.2 was used in [8], together with Theorem 1.3 and other reduction techiques, to prove two conjectures of Paulraja ([10], [11] and [12]) and one conjecture in [6].

**4. Nowhere-zero flows.** A nowhere-zero $k$-flow of $G$ is an assignment of edge directions and integer weights in $\{-k+1, \cdots, -1, 1, 2, \cdots, k-1\}$ to the edges of $G$ so that at each vertex of $G$ the flow in is the same as the flow out. The family of graphs that admit a nowhere-zero $k$-flow is denoted by $F_k$. Thus one has

$$F_k \subseteq F_{k+1}. \tag{5}$$

Tutte [7] has conjectured that every 2-edge-connected graph without a subgraph contractible to the Petersen graph is in $F_4$, and every 2-edge-connected graph is in $F_5$.

Theorem 4.1 Let $k \geq 4$ be an integer. If $G'(C_n)$ is in $F_k$, then $G$ is in $F_k$.

Proof: Suppose that $G'(C_n)$ has an assignment of edge directions and integer weights in $\{-k+1, \cdots, -1, 1, 2, \cdots, k-1\}$ to the edges of $G'(C_n)$ so that at each vertex of $G'(C_n)$ the flow in is the same as the flow out. This assignment gives rise to an assignment of edge directions and integer weights to edges of $E(G) - E(C)$ so that at each vertex of $V(G) - V(C)$, the flow in is the same as the flow out. Let $f(v)$ denote the amount of flow into $v$ minus the amount of flow out of $v$. Then by the assumption that $G'(C_n) \in F_k$,

$$f(x_1) + f(x_2) = 0, \text{ and } f(x_i) + f(x_{i+1}) = 0. \tag{6}$$

Assign edge directions from $x_j$ to $x_{j+1}$ to $E(C)$ for all $j = 1, 2, \cdots, m$ and from $x_m$ to $x_1$. Assign an unknown amount of integer $\alpha$ to each edge in $E(C)$. In addition, assign an integer $f(x_1)$ to $x_1x_2$ and an integer $f(x_i)$ to $x_ix_{i+1}$. Since $k \geq 4$ and by (6), one can choose

$\alpha \in \{-k+1, \cdots, -1, 1, \cdots, k-1\}$ so that the resulting assignment is a nowhere-zero $k$-flow of $G$. □

In [3], Catlin explicitly asked for an answer when $k = 4$ of the following problem.

Problem 1: What graph $H$ satisfies

$$G \in F_k \Longleftrightarrow G/H \in F_k, \tag{7}$$

for any supergraph $G$ of $H$?

Tutte's 5-flow conjecture, if true, implies that the $H$ in (7) can be any 2-edge-connected graph. (The converse holds also). In searching for such graphs, we are lead to a more genernal problem. Let $F(H,k)$ denote the set of functions $f : V(H) \longrightarrow \{-k+1, \cdots, 0, 1, \cdots, k-1\}$ such that

$$\sum_{v \in V(H)} f(v) = 0. \tag{8}$$

Let $\mathcal{F}(k)$ denote the family of graphs such that $H \in \mathcal{F}(k)$ if and only if for any $f \in F(H,k)$, there is an assignment of edge directions and integer weights in $\{-k+1, \cdots, -1, 1, 2, \cdots, k-1\}$ to $E(H)$ such that at each vertex $v$ of $H$, the amount of flow in minus the amount of flow out is the same as $f(v)$. As $f \equiv 0$ in $F(H,k)$ gives the original nowhere-zero $k$-flows of $H$, we naturally consider also the problem below.

Problem 2: Can we characterize $\mathcal{F}(k)$?

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