Europ. J. Combinatorics (1996) **17**, 447–450

A Property on Edge-disjoint Spanning Trees

HONG-JIAN LAI, HONGYUAN LAI AND CHARLES PAYAN

Let G be a simple graph with n vertices and let G^c denote the complement of G. Let $\omega(G)$ denote the number of components of G and $G(E)$ the spanning subgraph of G with edge set E. Suppose that $|E(G)| = k(|V(G)| - 1)$. Consider the partition $\Pi = \langle X_1, X_2, \ldots, X_k \rangle$ of $E(G)$ such that $|X_i| = n - 1$ $(1 \leq i \leq k)$. Define

$$\varepsilon(\Pi) = \sum_{i=1}^{k} \omega(G(X_i)) - k \qquad \text{and} \qquad \varepsilon(G) = \min \varepsilon(\Pi),$$

where the minimum is taken over all such partitions. In [*Europ. J. Combin.* **7** (1986), 263–270], Payan conjectures that if $\varepsilon(G) > 0$, then there exist edges $e \in E(G)$ and $e' \in E(G^c)$ such that $\varepsilon(G - e + e') < \varepsilon(G)$. This conjecture will be proved in this note.

1. INTRODUCTION

The graphs in this note are finite and undirected. We allow multiple edges, but forbid loops. We shall use the notation of Bondy and Murty [1], unless otherwise stated. Let G be a graph with $E(G) \neq \varnothing$. Let $\tau(G)$ denote the number of edge-disjoint spanning trees of G. For $X \subseteq E(G)$, the *notation* $G(X)$ denotes the spanning subgraph of G with edge set X, whereas $G[X]$ denotes the subgraph of G induced by X. The *contraction* G/X is the graph obtained from G by identifying the ends of each edge in X and then deleting the resulting loops. When H is a connected subgraph of G, we use G/H for $G/E(H)$. For convenience, we define $G/\varnothing = G$. as in [1], $\omega(G)$ denotes the number of components of G. By $H \subseteq G$ we mean that H is a subgraph of G. Throughout this note, $\mathbf{N}$ denotes the set of all positive integers. For a set S, an *m-partition* $\langle X_1, X_2, \ldots, X_m \rangle$ of S is a collection of m subsets of S such that:

$$S = \bigcup_{i=1}^{m} X_i, \qquad |X_i| > 0 \quad (1 \leq i \leq m) \qquad \text{and} \qquad X_i \cap X_j = \varnothing, \quad \text{whenever } i \neq j.$$

Let $k \geq 1$ be an integer and let G be a graph with $|V(G)| = n$, and with

$$|E(G)| = k(|V(G)| - 1). \tag{1}$$

Let $\Pi = \langle X_1, X_2, \ldots, X_k \rangle$ be a k-partition of $E(G)$. If Π further satisfies

$$|X_i| = n - 1 \qquad (1 \leq i \leq k), \tag{2}$$

then Π is called a *uniform k-partition*. Let $\mathscr{P}_k(G)$ denote the set of all uniform k-partitions of $E(G)$, and let $\mathscr{P}'_k(G)$ denote the set of all k-partitions of $E(G)$. Define

$$\varepsilon(\Pi) = \sum_{i=1}^{k} \omega(G(X_i)) - k, \qquad \text{for any } \Pi \in \mathscr{P}_k(G).$$

Note that we always have $\varepsilon(\Pi) \geq 0$. Define

$$\varepsilon(G) = \min_{\Pi \in \mathscr{P}_k(G)} \varepsilon(\Pi) \qquad \text{and} \qquad \varepsilon'(G) = \min_{\Pi \in \mathscr{P}'_k(G)} \varepsilon(\Pi). \tag{3}$$

Note that $\varepsilon(G) = 0$ iff $\tau(G) = k$. In [3], Payan posed the following conjectures.

0195-6698/96/050447 + 04 $18.00/0

CONJECTURE 1. If G is a simple graph satisfying (1) and if $\varepsilon(G)>0$, then there is always an edge $e \in E(G)$ and an edge $e' \in E(G^c)$ such that $\varepsilon(G-e+e')<\varepsilon(G)$.

CONJECTURE 2. For graph G with $|E(G)|/(|V(G)|-1)=k/t$, where $k, t \in \mathbf{N}$ are relatively prime, let G_t be the graph obtained from G by replacing each edge in G by a set of t parallel edges with identical ends. Define $\varepsilon(G)=\varepsilon(G_t)$. If $\varepsilon(G)>0$, then there are edges $e \in E(G)$ and $e' \in E(G^c)$ such that $\varepsilon(G-e+e')<\varepsilon(G)$.

It is clear that Conjecture 2 implies Conjecture 1. Conjecture 1 will be proved in this note. However, Conjecture 2 remains open.

2. PROOF OF CONJECTURE 1

We start with the following observation.

LEMMA 1. *For any graph G satisfying* (1), $\varepsilon(G)=\varepsilon'(G)$.

PROOF. By (3) and by $\mathcal{P}_k(G) \subset \mathcal{P}'_k(G)$, we have $\varepsilon'(G) \leqslant \varepsilon(G)$. What left is to show $\varepsilon(G) \leqslant \varepsilon'(G)$.

For each $\Pi = \langle X_1, \ldots, X_k \rangle \in \mathcal{P}'_k(G)$, define

$$f(\Pi)=\sum_{i=1}^{k} \max\{|X_i|-n+1, 0\}.$$

Thus $f(\Pi)=0 \Leftrightarrow \Pi \in \mathcal{P}_k(G)$. Choose a $\Pi \in \mathcal{P}'_k(G)$ such that $\varepsilon'(G)=\varepsilon(\Pi)$ and such that $f(\Pi)$ is minimized.

We claim that $\Pi \in \mathcal{P}_k(G)$. If not, then we may assume that $\Pi=\langle X_1, X_2, \ldots, X_k\rangle$ and that $|X_1|>n-1$ and $|X_2|<n-1$. Thus $G(X_1)$ must have a cycle C. Pick an edge $e \in E(C)$, and define

$$X'_i=\begin{cases} X_1-\{e\} & \text{if } i=1,\\ X_2 \cup \{e\} & \text{if } i=2,\\ X_i & \text{if } i>2. \end{cases}$$

Then $\Pi'=\langle X'_1, X'_2, \ldots, X'_k\rangle \in \mathcal{P}'_k(G)$. By the facts that $\omega(G(X_1))=\omega(G(X'_1))$ and $\omega(G(X_2)) \geqslant \omega(G(X'_2))$, we have $\varepsilon(\Pi') \leqslant \varepsilon(\Pi)=\varepsilon'(G)$. But $f(\Pi') \leqslant f(\Pi)-1$, contrary to the choice of Π. Hence $\Pi \in \mathcal{P}_k(G)$ and so $\varepsilon(G) \leqslant \varepsilon'(G)$. This proves Lemma 1. □

The following has been proved by Nash-Williams.

THEOREM 2 (*Nash-Williams* [2]). *Let G be a graph and let $k \in \mathbf{N}$ be an integer. If $|E(G)| \geqslant k(|V(G)|-1)$, then G has a subgraph H with $\tau(H) \geqslant k$.*

LEMMA 3. *Let G be a graph and let H be a subgraph of G with $\tau(H) \geqslant k>0$. If $\tau(G/H) \geqslant k$, then $\tau(G) \geqslant k$.*

PROOF. Suppose that $Y_1, Y_2, \ldots, Y_k$ are disjoint edge subsets of $E(H)$ such that each $H(Y_i)$ is a spanning tree of H, and suppose that $X_1, X_2, \ldots, X_k$ are disjoint edge subsets of $E(G/H)$ such that each $G/H(X_i)$ is a spanning tree of G/H. Then each $G(X_i \cup Y_i)$ $(1 \leq i \leq k)$, is a spanning tree of G. □

LEMMA 4. *Let G be a graph satisfying* (1). *If* $\varepsilon(G) > 0$, *then G has an induced subgraph H with*

$$|E(H)| > k(|V(H)| - 1) \quad \text{and} \quad \tau(H) \geq k. \tag{4}$$

PROOF. By (1) and by Theorem 2, G has a subgraph H with $\tau(H) \geq k$. Let $H_1, H_2, \ldots, H_c$ denote all the subgraphs of G such that for each $1 \leq i \leq c$,

$$\tau(H_i) \geq k \text{ and } H_i \text{ is maximal with respect to this property,} \tag{5}$$

and let $v_1, v_2, \ldots, v_c$ be the vertices in G' onto which the subgraphs $H_1, H_2, \ldots, H_c$ are contracted, respectively. Note that the $G(H_i)$'s are vertex-disjoint induced subgraphs of G. Since $\varepsilon(G) > 0$, and therefore $\tau(G) < k$, $G \neq \bigcup_{i=1}^{c} H_i$ and so G' is non-trivial.

We first claim that G' has no subgraph with k edge-disjoint spanning trees. By contradiction, we suppose that G' has a subgraph H' with $\tau(H') \geq k$. If $V(H') \cap \{v_1, \ldots, v_c\} = \varnothing$, then $H' \subset G$, and so H' should be one of the H_i's, a contradiction. Hence we may assume that $v_1, v_2, \ldots, v_d$ are in $V(H')$, where $d \leq c$. But then, by Lemma 3, $H = G[E(H') \cup E(H_1) \cup \cdots \cup E(H_d)]$ is a subgraph of G with $\tau(H) \geq k$, contrary to (5). Thus no subgraph of G' has k edge-disjoint spanning trees and so, by Theorem 2, and by the fact that G' is not trivial, we have

$$|E(G')| < k(|V(G')| - 1). \tag{6}$$

To complete the proof of Lemma 4, we argue by contradiction to assume that

$$|E(H_i)| = k(|V(H_i)| - 1) \qquad (1 \leq i \leq c). \tag{7}$$

Note that

$$|V(G')| = |V(G)| - \sum_{i=1}^{c} |V(H_i)| + c. \tag{8}$$

Hence, by (6), (7) and (8),

$$|E(G)| = |E(G')| + \sum_{i=1}^{c} |E(H_i)| < k\,|V(G')| - k + k\left(\sum_{i=1}^{c} |V(H_i)| - c\right) = k(|V(G)| - 1),$$

contrary to (1). Hence one of the H_i's must satisfy the conclusion of Lemma 4. By (5), each H_i is an induced subgraph of G. This proves Lemma 4. □

THEOREM 5. *Let G be a simple graph satisfying* (1). *If* $\varepsilon(G) > 0$, *then there is an edge* $e' \in E(G^c)$ *and an edge* $e \in E(G)$ *such that*

$$\varepsilon(G - e + e') < \varepsilon(G).$$

PROOF. By Lemma 4, G has a maximal induced subgraph H with $|E(H)| > k(|V(H)| - 1)$ and $\tau(H) \geq k$. Since G is simple, H is simple, and so by $\tau(H) \geq k$, $|V(H)| \geq k$.

For any $v \in V(G) - V(H)$, if v is adjacent to every vertex in H, then by $|V(H)| \geq k$, $G[V(H) \cup \{v\}]/H$ is a graph of two vertices and with at least k parallel edges, and so by Lemma 3, $\tau(G[V(H) \cup \{v\}] \geq k$, contrary to the maximality of H. Therefore,

$$\forall v \in V(G) - V(H), \text{ there is a } w \in V(H) \text{ such that } vw \notin E(G). \tag{9}$$

Since $\tau(H) \geq k$, $E(H)$ has a k-partition $\langle Y_1, Y_2, \ldots, Y_k \rangle$ such that each $H(Y_i)$ is a spanning connected subgraph of H $(1 \leq i \leq k)$. By (4), $|E(H)| > k(|V(H)| - 1)$, and so we may assume, without loss of generality, that $H(Y_1)$ has a cycle C. Pick an edge $e \in E(C)$.

Let $\Pi = \langle X_1, X_2, \ldots, X_k \rangle \in \mathscr{P}'_k(G)$ such that $\varepsilon'(G) = \varepsilon(\Pi)$. Define $X'_i = (X_i - E(H)) \cup Y_i$ $(1 \leq i \leq k)$, and $\Pi' = \langle X'_1, X'_2, \ldots, X'_k \rangle$. Since each $H(Y_i)$ is a spanning connected subgraph of H, $\omega(G(X'_i)) \leq \omega(G(X_i))$ and so $\varepsilon'(G) \leq \varepsilon(\Pi') \leq \varepsilon(\Pi) = \varepsilon'(G)$. Thus $\varepsilon(\Pi) = \varepsilon(\Pi')$. Since $\varepsilon'(G) = \varepsilon(G) > 0$ (by Lemma 1), we may assume that $\omega(G(X'_j)) > 2$, for some j. Since $G(X'_j)$ has a connected subgraph $H(Y_j)$, $G(X'_j)$ must have a vertex $v \in V(G) - V(H)$ such that v is not in the component of $G(X'_j)$ that contains $H(Y_j)$. By (9), there is some $w \in V(H)$ such that $vw \notin E(G)$. Now let

$$X''_i = \begin{cases} X'_1 - \{e\} & \text{if } i = 1 \\ X'_j \cup \{vw\} & \text{if } i = j \\ X'_i & \text{if } i \neq 1 \text{ and } i \neq j. \end{cases} \tag{10}$$

Then $\Pi'' = \langle X''_1, X''_2, \ldots, X''_k \rangle \in \mathscr{P}'_k(G - e + vw)$. Let G_1 denote $G - e + vw$. By (10) and since $e \in E(C)$, $\omega(G(X'_i)) \leq \omega(G_1(X''_i))$; by the choice of vw, $\omega(G(X'_j)) = \omega(G_1(X''_j)) + 1$, and so, by Lemma 1,

$$\varepsilon(G) = \varepsilon'(G) = \varepsilon(\Pi') > \varepsilon(\Pi'') \geq \varepsilon'(G_1) = \varepsilon(G_1).$$

This proves Theorem 5. □

Acknowledgements

The work of the first author was partially supported by NSA Grant MDA904-94-H-2012.

References

1. J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications,* American Elsevier, New York, 1976.
2. C. St. J. A. Nash-Williams, Decompositions of finite graphs into forests, *J. Lond. Math. Soc.*, **39** (1964), 12.
3. C. Payan, Graphes équilibrés et arboricité rationnelle, *Europ. J. Combin.*, **7** (1986), 263–270.

Received 28 March 1991 and accepted in revised form 12 July 1995

HONG-JIAN LAI
West Virginia University,
Morgantown, WV 26506, U.S.A.

HONGYUAN LAI
Wayne State University,
Detroit, MI 48202, U.S.A.

CHARLES PAYAN
CNRS LSD2, BP53,
38041 Grenoble Cedex 9, France