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# Reduction Techniques for Supereulerian Graphs and Related Topics — A Survey

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## Abstract

In [J. Graph Theory 12(1988) 29-44], Catlin developed a powerful reduction technique to study the existence of spanning and dominating eulerian subgraphs. This paper is intended as a survey article to cover recent developments of this reduction technique and its applications. It also covers some related topics such as supereulerian graphs, integer flows, cycle covers and hamiltonian line graphs.

- I. Notation and Terminology
- II. Reduction Methods
- III. Reductions of 4-cycles
- IV. Reduced Graphs
- V. Supereulerian Graphs
- VI. Integer Flows and Cycle Covers
- VII. Hamiltonian Line Graphs
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## I. Introduction

We use [8] for terminology and notation not defined here and consider only loopless finite graphs. A trail is a finite sequence  $T = u_0e_1u_1e_2u_2 \cdots e_ru_r$ , whose terms are alternately vertices and edges, with  $e_i = u_{i-1}u_i$  ( $1 \leq i \leq r$ ), where the edges are distinct. A trail  $T$  is a closed trail if  $u_0 = u_r$  and is called a  $(u, v)$ -trail if  $u_0 = u$  and  $u_r = v$ . A trail  $T$  is called a spanning trail if  $V(T) = V(G)$  and is called a dominating trail if  $E(G - V(T)) = \emptyset$ . A graph  $G$  is supereulerian if  $G$  has a spanning closed trail. A cycle of order  $n$  is denoted by  $C_n$ .

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A nowhere zero  $k$ -flow of  $G$  is an assignment of edge directions and integer weights in  $\{1, 2, \dots, k-1\}$  to edges of  $E(G)$  such that at every vertex  $v$ , the total number of weights of the edges directed into  $v$  is equal to the total number of weights of edges directed out from  $v$ . The collection of all graphs admitting a nowhere-zero  $k$ -flow is denoted by  $F_k$ . By definition,  $F_2$  consists of even graphs (graphs each of whose vertices has even degree). The graph  $K_1$  is regarded as having a nowhere-zero  $k$ -flow for any  $k \geq 1$ . It has been noted that supereulerian graphs are all in  $F_4$  [17].

Let  $G$  be a graph. The line graph of  $G$ , written  $L(G)$ , has  $E(G)$  as its vertex set, where two vertices are adjacent in  $L(G)$  if and only if the corresponding edges are adjacent in  $G$ . For a connected graph  $G$ , the  $i$ -th iterated line graph  $L^i(G)$  is defined recursively by  $L^0(G) = G$ ,  $L^i(G) = L(L^{i-1}(G))$ . The minimum  $i$  such that  $L^i(G)$  is hamiltonian is called the hamiltonian index of  $G$  ([25], [26]). Chartrand ([25]) showed that if a connected graph  $G$  is not a path, then the hamiltonian index of  $G$  exists. As shown by Harary and Nash-Williams [42], there is a close relationship between dominating eulerian subgraphs and Hamilton cycles in  $L(G)$ .

**Theorem 1.1** ([42]). The line graph  $L(G)$  of a connected graph  $G$  is hamiltonian if and only if  $G$  has a dominating eulerian subgraph and  $G \notin \{K_1, K_2, K_{1,2}\}$ .  $\square$

In [18], Catlin gave a survey of the research on supereulerian graphs, the reduction method, and its applications. In this paper, we shall focus on the technical aspects of the reduction method and cover some new development since then.

## II. Reduction Methods

We start with Catlin's reduction method.

### 2.1 Catlin's reduction method

Let  $H$  be a connected subgraph of  $G$ . The contraction  $G/H$  is the graph obtained from  $G$  by contracting all edges of  $H$  and deleting any resulting loops, i.e., replacing  $H$  by a new vertex  $v_H$  such that the number of edges in  $G/H$  joining any  $v \in G - V(H)$  to  $v_H$  in  $G/H$  equals the number of edges joining  $v$  in  $G$  to  $H$ . A graph  $G$  is contractible to a graph  $G'$  if  $G$  contains pairwise vertex-disjoint connected subgraph  $H_1, \dots, H_c$  with  $\bigcup_{i=1}^c V(H_i) = V(G)$  such that  $G'$  is obtained from  $G$  by successively contracting each  $H_i$  ( $1 \leq i \leq c$ ). Each subgraph  $H \in \{H_1, \dots, H_c\}$  is called the preimage of the vertex  $v_H$  of  $G'$ . A vertex  $v_H$  in  $G'$  is nontrivial if  $v_H$  is the contraction image of a nontrivial connected subgraph  $H$  of  $G$ .

For a graph  $G$ , let  $O(G)$  denote the set of all odd-degree vertices of  $G$ . A graph  $G$  is collapsible ([14]) if for every subset  $X \subseteq V(G)$  with  $|X|$  even, there is a spanning connected subgraph  $H_X$  of  $G$ , such that  $O(H_X) = X$ . We use  $\mathcal{CC}$  and  $\mathcal{SC}$  to denote the families of collapsible graphs and supereulerian graphs, respectively. The trivial graph  $K_1$ , and cycle  $C_2$ , and  $C_3$  are both supereulerian and collapsible, but  $C_4 \in \mathcal{SC} - \mathcal{CC}$ . Note that being collapsible is stronger than being supereulerian. In fact, for a collapsible graph  $G$  with  $u, v \in V(G)$ , let  $R = \{u, v\}$  if  $u \neq v$ , and  $R = \emptyset$  if  $u = v$ . Then  $|R|$  is even. By the definition of collapsible graphs,  $G$  has a spanning subgraph  $H_R$  such that  $O(H_R) = R$ . Thus  $H_R$  is a spanning  $(u, v)$ -trail in  $G$ . When  $u = v$ ,  $H_R$

is a spanning eulerian subgraph of  $G$ , and so  $\mathcal{CL} \subset \mathcal{SL}$ .

In [14], Catlin showed that every graph  $G$  has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs  $H_1, \dots, H_c$  such that  $\bigcup_{i=1}^c V(H_i) = V(G)$ . The reduction of  $G$ , denoted by  $G'$ , is the graph obtained from  $G$  by contracting each maximal collapsible subgraph  $H_i$ , ( $1 \leq i \leq c$ ), into a single vertex  $v_i$ . A graph  $G$  is reduced if  $G = G'$ .

What makes the reduction method and the collapsible graphs so useful is the following theorem: **Theorem 2.1** ([14]). Let  $G$  be a connected graph and  $G'$  the reduction of  $G$ . Then each of the following holds.

(a)  $G \in \mathcal{CL} \iff G' = K_1$ .

(b)  $G \in \mathcal{SL} \iff G' \in \mathcal{SL}$ .

(c)  $G$  has a dominating eulerian subgraph if and only if  $G'$  has a dominating eulerian subgraph containing all nontrivial vertices of  $G'$ .  $\square$

Jaeger [44] showed that a graph with two edge-disjoint spanning trees is supereulerian. Catlin [14] proved that

**Theorem 2.2** ([14]). If  $G$  is at most one edge short of having two edge-disjoint spanning trees, then either  $G \in \mathcal{CL}$  or  $G$  has a single cut-edge.  $\square$

## 2.2 A refinement of Catlin's reduction technique

In [76], Veldman gave a refinement of Catlin's reduction method that was used to solve a conjecture in [4].

Let  $G$  be a simple graph and define  $D(G) = \{v \in V(G) \mid d(v) \in \{1, 2\}\}$ . For an independent subset  $X$  of  $D(G)$ , define  $I_X(G)$  as the graph obtained from  $G$  by deleting the vertices in  $X$  of degree 1 and by replacing each path of length 2 whose internal vertex is a vertex in  $X$  of degree 2 by an edge. Note that  $I_X(G)$  may not be simple. A graph  $G$  is called  $X$ -collapsible if  $I_X(G)$  is collapsible. A subgraph  $H$  of  $G$  is an  $X$ -subgraph of  $G$  if  $d_H(x) = d_G(x)$  for all  $x \in X \cap V(H)$ . An  $X$ -subgraph  $H$  of  $G$  is called  $X$ -collapsible if  $H$  is  $(X \cap V(H))$ -collapsible. Let  $R(X)$  be the set of vertices in  $X$  that are not contained in an  $X$ -collapsible  $X$ -subgraph of  $G$ . Since  $I_X(G)$  has a unique collection of vertex-disjoint maximal collapsible subgraphs  $W_1, \dots, W_k$  such that  $\bigcup_{i=1}^k V(W_i) = V(I_X(G))$ , the graph  $G$  has a unique collection of pairwise vertex-disjoint maximal  $X$ -collapsible  $X$ -subgraphs  $H_1, \dots, H_k$  such that  $\bigcup_{i=1}^k V(H_i) \cup R(X) = V(G)$ . The  $X$ -reduction of  $G$ , denoted by  $G'_X$ , is the graph obtained from  $G$  by contracting  $H_1, \dots, H_k$ . The graph  $G$  is  $X$ -reduced if there exists a graph  $G_1$  and an independent subset  $X_1$  of  $D(G_1)$  such that  $X = R(X_1)$  and  $G$  is the  $X_1$ -reduction of  $G_1$ . An  $X$ -subgraph  $H$  of  $G$  is called  $X$ -reduced if  $H$  is  $(X \cap V(H))$ -reduced.

Note that if  $X = \emptyset$ , the refinement method is just the original method. A graph  $G$  is  $\emptyset$ -collapsible if and only if  $G$  is collapsible. Theorem 2.1(c) is a special case of Theorem 2.3 below.

**Theorem 2.3** ([76]). Let  $G$  be a connected simple graph,  $X$  an independent subset of  $D(G)$ , and

$G'_X$  the  $X$ -reduction of  $G$ . Then  $G$  has a dominating eulerian subgraph if and only if  $G'_X$  has a dominating eulerian subgraph containing all nontrivial vertices of  $G'_X$ .  $\square$

### III. Reductions of 4-cycles

Suppose that a graph  $G$  contains an induced 4-cycle  $H$ . Since  $H$  is not collapsible, Theorem 2.1 cannot apply in this case. However, there are two ways which provide the extensions of the reduction method to subgraphs that are 4-cycles.

- Catlin's  $\pi$ -collapsibility reduction method.

Catlin [19] introduces  $\pi$ -collapsible graphs to deal with 4-cycles. Let  $H$  be a graph and let  $\pi = \langle V_1, V_2 \rangle$  be a partition of  $V(H)$ . Then  $H$  is called  $\pi^-$ -collapsible (resp.  $\pi^+$ -collapsible) if for every subset  $R \subseteq V(H)$  with  $|R|$  even, the following holds:

(i) if  $|R \cap V_1|$  is odd (reps., is even), then  $H$  has an  $R$ -subgraph;

(ii) if  $|R \cap V_1|$  is even (reps., is odd), then  $H + e$  has an  $R$ -subgraph  $\Gamma_e \subseteq H$ , for any added edge  $w = w_1w_2$  with  $w_i \in V_i$ , ( $1 \leq i \leq 2$ ).

If  $H$  is either  $\pi^-$ -collapsible or  $\pi^+$ -collapsible, then  $H$  is  $\pi$ -collapsible.

Here we describe a special case of this more general  $\pi$ -collapsibility reduction method. Let  $G$  be a graph containing a 4-cycle  $xyzwx$ , and define  $E = \{xy, yz, zw, wx\}$ . Define  $G/\pi$  to be the graph obtained from  $G - E$  by identifying  $x$  and  $z$  to form a vertex  $v_1$ , by identifying  $w$  and  $y$  to form a vertex  $v_2$ , and by adding a new edge  $v_1v_2$ .

**Theorem 3.1** ([19]). For the graphs  $G$  and  $G/\pi$  defined above, each of the following holds:

- If  $G/\pi \in \mathcal{CL}$ , then  $G \in \mathcal{CL}$ ;
- If  $G/\pi \in \mathcal{SL}$ , then  $G \in \mathcal{SL}$ .  $\square$

- Lai's reduction method for  $m$ -cycles.

Let  $C_m = x_1x_2 \cdots x_mx_1$  be an  $m$ -cycle of  $G$  with  $m \geq 4$ . Let  $e$  and  $f$  be two non-adjacent edges of  $C_m$ . Without loss of generality, we may assume that  $e = x_1x_2$  and  $f = x_ix_{i+1}$ , where  $m > i > 2$ . Define

$$G^* = (G - [E(C_m) - \{e, f\}]) / \{e, f\}.$$

Let  $w$  and  $z$  denote the two vertices of  $G^*$  to which the edges  $e$  and  $f$  are contracted, respectively. For convenience, we regard

$$V(G^*) = (V(G) - \{x_1, x_2, x_i, x_{i+1}\}) \cup \{w, z\}, \quad E(G^*) = E(G) - E(C_m).$$

**Theorem 3.2** ([54]). For the graphs  $G$  and  $G^*$  defined above, each of the following holds:

- If  $G^* \in \mathcal{CL}$ , then  $G \in \mathcal{CL}$ ;
- If  $G^* \in \mathcal{SL}$ , then  $G \in \mathcal{SL}$ .  $\square$

These techniques have been used to study the structures of small order reduced graphs ([35], [30]). Using these two theorems with Catlin's reduction technique, Lai ([51], [53]) proved two conjectures of Paulraja ([68], [69]) and a conjecture of Catlin [19]. One of the theorems in [51] is the following: If a 2-connected graph  $G$  has  $\delta(G) \geq 3$  and each edge lies in a cycle of length at most 4, then  $G \in \mathcal{CL}$ . Since a spanning eulerian subgraph in a cubic graph is a Hamilton cycle, a

corollary of the result above (in [51]) is that every cubic graph whose edges are lying in a cycle of length at most 4 is hamiltonian. (Proof: such graphs are 2-connected).

There are a few other results related to Catlin's  $\pi$ -collapsible graphs.

**Theorem 3.3** ([54]). Let  $G'$  denote the reduction of  $G$ . Then  $G$  is  $\pi$ -collapsible for some partition  $\pi$  of  $V(G)$  if and only if  $G'$  is  $\pi'$ -collapsible for some partition  $\pi'$  of  $V(G')$ .  $\square$

The converse of Theorem 3.1 is not true, as showed by Example 2 in [61]. However, the following gives a conditional converse of Theorem 3.1.

**Theorem 3.4** ([61]). Let  $G$  be a graph of diameter at most 2 and let  $C_4$ , the 4-cycle, be a nonspanning subgraph of  $G$ . Let  $\pi$  denote the bipartition of  $C_4$ . Then  $G$  is collapsible if and only if one of the following holds:

- (i)  $G/\pi$  is collapsible.
- (ii)  $G$  is spanned by a subgraph  $H \cong K_{2,n-2}$  such that there are two vertices of degree 2 adjacent to each other in  $G$ .  $\square$

#### IV. Reduced Graphs

Besides the theorems mentioned above, a number of useful results on reduced graphs are summarized in this section.

**Theorem 4.1** ([14]). Let  $G$  be a connected graph. Then each of the following holds.

- (a)  $G$  is reduced if and only if  $G$  has no nontrivial collapsible subgraphs.
- (b) Any subgraph of a reduced graph is reduced.
- (c) If  $G$  is reduced, then  $G$  is  $K_3$ -free with  $\delta(G) \leq 3$ .
- (d) If  $G$  is reduced and  $G \notin \{K_1, K_2\}$ , then  $|E(G)| \leq 2|V(G)| - 4$ .  $\square$

For  $X$ -reduced graph, Veldman [76] proved the following:

**Theorem 4.2** ([76]). Let  $G$  be a connected simple graph and  $X$  an independent subset of  $D(G)$ . Then each of the following holds.

- (a)  $G$  is  $X$ -reduced if and only if  $I_X(G)$  is reduced.
- (b)  $G$  is  $X$ -reduced if and only if  $G$  contains no nontrivial  $X$ -collapsible  $X$ -subgraphs.
- (c) If  $G$  is  $X$ -reduced, then every  $X$ -subgraph of  $G$  is  $X$ -reduced.
- (d) If  $G$  is  $X$ -reduced, then  $d(x) = 2$  for all  $x \in X$  and exactly one of the following holds.
  - (d1)  $G \in \{K_1, K_2\}$  and  $X = \emptyset$ .
  - (d2)  $G = P_3$  and  $|X| = 1$ .
  - (d3)  $|E(G)| \leq 2|V(G)| - |X| - 4$ .  $\square$

For the size of a maximum matching in a reduced graph, we have the following:

**Theorem 4.3** ([34] [27]). Let  $M(G)$  be the size of a maximum matching of a connected reduced graph  $G$  of order  $n$ . Let  $D_2(G) = \{v \in V(G) \mid d(v) = 2\}$ . If  $\delta(G) \geq 2$  and  $|D_2(G)| = l$ , then

$$M(G) \geq \min \left\{ \frac{n-1}{2}, \frac{n+4-l}{3} \right\}. \square$$

The reduced graphs of diameter two were characterized in [52]. For the independence number

$\alpha(G)$  of a reduced graph  $G$ , we have  $(\delta(G)\alpha(G) + 4)/2 \leq n \leq 4\alpha(G) - 5$  if  $\alpha(G) \geq 4$  ([28]).

By Theorem 2.1, the problem of determining whether a graph  $G \in \mathcal{CL}$  (or  $\mathcal{SL}$ ) can often be reduced to determining whether a much smaller graph (the reduction of  $G$ ) is in  $\mathcal{CL}$  (or  $\mathcal{SL}$ ). Thus, knowing the structure of reduced graphs is very important even for small reduced graphs, when applying the reduction method. The smallest 2-edge-connected nonsupereulerian reduced graph is  $K_{2,3}$ , and the smallest 3-edge-connected nonsupereulerian reduced graph is the Petersen graph. In the following, we let  $P$  denote the Petersen graph.

**Theorem 4.4** ([35], [31], [30], [48]). Let  $G$  be a graph of order  $n$  and  $G'$  the reduction of  $G$ .

- (a) If  $n \leq 11$  and  $\delta(G) \geq 3$  then either  $G \in \mathcal{CL}$  or  $G' \in \{K_2, P\}$ .
- (b) If  $n \leq 13$  and  $\delta(G) \geq 3$  then either  $G \in \mathcal{SL}$  or  $G' \in \{K_2, K_{1,2}, P\}$ .
- (c) If  $n \leq 17$  and  $\kappa'(G) \geq 2$ , then either  $G \in F_4$  or  $G$  can be contracted to the Petersen graph.  $\square$

Let  $P - v$  be the Petersen graph minus a vertex  $v$ . Let  $G$  be the graph on 14 vertices obtained from  $P - v$  and  $K_{2,3}$  by joining those vertices of degree 2 from  $P - v$  to  $K_{2,3}$  such that  $G$  is 3-edge-connected. Then  $G$  is a nonsupereulerian reduced graph. This shows that Theorem 4.4(b) is best possible. Catlin [16] conjectured that any 3-edge-connected simple graph of order at most 17 is either supereulerian, or it is contractible to the Petersen graph. Note that the contraction here is not necessary to be a contraction of collapsible subgraphs. The two Blanuša snarks (see [79]) of 18 vertices show that "17" is best possible in the conjecture and in Theorem 4.4(c).

## V. Supereulerian Graphs.

There are many results on various sufficient conditions for the existence of spanning eulerian subgraphs (see [4], [9]-[12], [14]-[22], [27]-[35], [38], [49]-[53], [65], [66], [76]-[78], [80]). Several conjectures and open problems ([4], [2], [3], [9], [68], [69]) have been solved by using the reduction method ([14],[15], [22], [29], [49], [51], [76]). For sufficient condition in terms of a lower bound on the number of edges, Catlin and Chen proved the following:

**Theorem 5.1** ([22], [33]). Let  $n$ ,  $m$  and  $p$  be natural numbers,  $m, p \geq 2$ . Let  $G$  be a 2-edge-connected simple graph on  $n > p + 6$  vertices containing no  $K_{m+1}$ . If

$$|E(G)| \geq \binom{n-p+1-k}{2} + (m-1) \binom{k+1}{2} + 2p - 4, \quad (1)$$

where  $k = \lfloor \frac{n-p+1}{m} \rfloor$ , then either  $G$  is supereulerian, or  $G$  can be contracted to a nonsupereulerian graph of order less than  $p$ , or equality holds in (1) and  $G$  can be contracted to  $K_{2,p-2}$  ( $p$  is odd) by contracting a complete  $m$ -partite graph  $T_{m,n-p+1}$  of order  $n-p+1$  in  $G$ .  $\square$

The case  $p = 5$  and  $m \geq n - 4$  of Theorem 5.1 was proved by Cai in [9]. The case  $p = 10$  and  $m \geq n - 9$  for 3-edge-connected graphs [22] solved a conjecture of Cai [9]. For sufficient conditions in terms of lower bounds on degrees of nonadjacent vertices, we have the following:

**Theorem 5.2** ([15],[29]). Let  $G$  be a  $k$ -edge-connected graph of order  $n$  with girth  $g$ , where

$k \in \{2, 3\}$  and  $g \in \{3, 4\}$ . Let  $G'$  be the reduction of  $G$ . If  $n$  is sufficiently large and if

$$d(u) + d(v) \geq \frac{2}{g-2} \left( \frac{n}{(k-1)5} - 4 + g \right),$$

whenever  $uv \notin E(G)$ , then exactly one of the following holds:

- (a)  $G \in \mathcal{CL}$ ;
- (b)  $k = 2$  and  $n = (g-2)5s$  ( $s \geq 20$ ), and  $G' = K_{2,3}$ , and either
  - (b1)  $g = 3$ , the preimage of each vertex of  $K_{2,3}$  is  $K_s$  or  $K_s - e$ , or
  - (b2)  $g = 4$ , the preimage of each vertex of  $K_{2,3}$  is  $K_{s,s}$  or  $K_{s,s} - e$ .
- (c)  $k = 3$  and  $n = (g-2)10s$  ( $s \geq 40$ ), and  $G' = P$ , and either
  - (c1)  $g = 3$ , the preimage of each vertex of  $P$  is  $K_s$  or  $K_s - e$ ; or
  - (c2)  $g = 4$ , the preimage of each vertex of  $P$  is  $K_{s,s}$  or  $K_{s,s} - e$ .  $\square$

For degree conditions of adjacent vertices, we have the following:

**Theorem 5.3** ([34], [27]). Let  $G$  be a  $k$ -edge-connected simple graph of order  $n$ , where  $k \in \{2, 3\}$ . Let  $G'$  be the reduction of  $G$  and  $l = |D_2(G)|$ . If for any  $uv \in E(G)$ ,

$$d(u) + d(v) \geq \frac{2n}{(k-1)5} - 2,$$

then exactly one of the following holds:

- (a)  $G \in \mathcal{CL}$ ;
- (b)  $k = 2$  and  $l < n/5 - 19$ , and either
  - (b1)  $G' = K_{2,c-2}$ , where  $c \leq \max\{5, 3 + l\}$ , or
  - (b2)  $n = 5s$  ( $s > 19$ ),  $G' = C_5$ , and the preimage of each vertex of  $C_5 \in \{K_s, K_s - e\}$ .
- (c)  $k = 3$ ,  $n = 10s$  ( $s > 24$ ),  $G' = P$ , and the preimage of each vertex of  $P \in \{K_s, K_s - e\}$ .  $\square$

A graph  $G$  is almost bridgeless if every cut edge of  $G$  is incident with a vertex of degree 1. Veldman [76] proved the following theorem that solved a conjecture in [4].

**Theorem 5.4** ([76]). Let  $G$  be a connected almost bridgeless simple graph of order  $n$ . If  $n$  is sufficiently large and for any  $uv \in E(G)$ ,

$$d(u) + d(v) \geq \frac{2n}{5} - 2,$$

then  $L(G)$  is hamiltonian.  $\square$

There are other degree condition results obtained by using the reduction technique ([28], [32], [58]) which improved some previous sufficient conditions for a graph to have a spanning or dominating eulerian subgraph. The general idea in using the reduction method to prove degree condition problems for the existence of spanning eulerian subgraphs or dominating eulerian subgraphs is the following: First, prove that the order of the reduction of a graph  $G$  with given degree conditions is small, or is independent on the order of  $G$ , or prove that the number of nontrivial vertices in the reduction graph of  $G$  is small. To do so, one may need to make use of some properties of reduced graphs, such as the size of matchings or independent numbers, together with the given degree condition. Then show that the order of the reduction graph is not more than 5 (or 13 if



$G$  is 3-edge-connected). The former may require some structure results of reduced graphs. The latter may require some results on small reduced graphs. This part may be routine but it may also be tedious. Some other type of degree conditions (e.g., [82]) for spanning eulerian graphs may be improved by using the reduction method. However, it has been noted that the degree conditions for spanning eulerian subgraphs are somewhat restrictive. One may hope to study more structural conditions instead of degree conditions for the existence of spanning eulerian subgraphs.

Other types of sufficient conditions for supereulerian graphs have been investigated, as showed by the following theorems.

**Theorem 5.5** ([59]). Let  $G$  be a simple graph with at least 61 vertices, and let  $G^c$  denote the complement of  $G$ . One of the following holds.

- (a)  $G$  is supereulerian.
- (b)  $G^c$  is supereulerian.
- (c) Both  $G$  and  $G^c$  has a vertex of degree 1.
- (d) One of  $G$  or  $G^c$  is contractible to a  $K_{2,t}$  for some odd integer  $t \geq 3$ , and the other has either one or two vertices of degree 1.
- (e) One of  $G$  and  $G^c$  is contractible to  $K_{1,p}$  for some integer  $p \geq 1$ , and the other has exactly one isolated vertex.  $\square$

A wheel  $W_n$  is the graph obtained from the  $n$ -cycle  $C_n = v_1v_2 \cdots v_nv_1$ , where  $n \geq 2$ , by adding an extra vertex  $v$  and new edges  $\{vv_i : 1 \leq i \leq n\}$ . Define the subdivided wheel  $W_n^*$  to be the graph obtained from  $W_n$  by replacing each edge  $v_iv_{i+1}$ , ( $1 \leq i \leq n, (\text{mod } n)$ ) by a path of length 2,  $v_iv'_i v_{i+1}$  (say), where  $\{v'_1, \dots, v'_n\} \cap V(W_n) = \emptyset$ . Note that  $W_2^* \cong K_{2,3}$ . Let  $\mathcal{F} = \{W_n^* : n \geq 2\}$ .

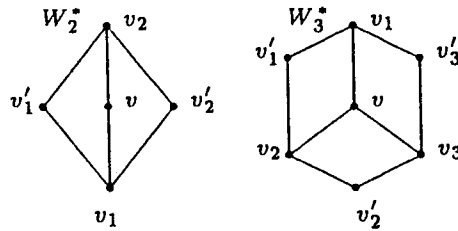


Figure 1: the graphs  $W_2^*$  and  $W_3^*$

A graph  $H$  is a minor of  $G$  if  $H$  is isomorphic to the contraction image of a subgraph of  $G$ . We call  $H$  an induced minor of  $G$  if  $H$  is isomorphic to the contraction image of an induced subgraph of  $G$ .

**Theorem 5.6** ([60]). Let  $G$  be a 2-edge-connected graph. If  $G$  has no induced minor isomorphic to a member in  $\mathcal{F}$ , then  $G$  is supereulerian.  $\square$

## VI. Integer Flows and Cycle Covers

Flows have been discussed by many. See [44] for a comprehensive survey by Jaeger. The idea

of reduction had also been applied to the problems involving integer flows and cycle covers. Tutte has three fascinating conjectures on flows.

**3-Flow Conjecture** ([44]). Every 4-edge-connected graph is in  $F_3$ .

**4-Flow Conjecture** ([44]). Every 2-edge-connected graph either is in  $F_4$ , or has a subgraph contractible to the Petersen graphs.

**5-Flow Conjecture** ([44]). Every 2-edge-connected graph is in  $F_5$ .

It has been proved by Seymour [73] that every 2-edge-connected graph is in  $F_6$ . Fix an integer  $k \geq 3$ . Let  $F_k^o$  denote the collection of connected graphs  $H$  such that for any graph  $G$  that has  $H$  as a subgraph,

$$G \in F_k \iff G/H \in F_k.$$

Catlin [16] considers  $F_4^o$ , and he proves the following.

**Theorem 6.1** ([16]). All collapsible graphs and the 4-cycle are in  $F_4^o$ .  $\square$

Catlin used this theorem ([16], [17]) to show that a 3-edge-connected graph with at most 10 edge-cuts of size 3 is either in  $F_4$ , or contractible to the Petersen graph. Theorem 6.1 and Theorem 4.4(c) were used ([48]) to show that if a 2-edge-connected simple graph  $G$  with  $n \geq 18$  vertices satisfies

$$|E(G)| \geq \binom{n-17}{2} + 34,$$

then either  $G \in F_4$ , or  $G$  is contractible to the Petersen graph. This bound is asymptotically best possible.

For  $F_3^o$ , we have the following result.

**Theorem 6.2** ([55]). Every complete graph  $K_m$  ( $m \geq 5$ ) are in  $F_3^o$ .  $\square$

It is well known that  $K_4$  is the smallest 2-edge-connected graph that is not in  $F_3$ . Among other applications, Theorem 6.2 is used to show that if a 2-edge-connected simple graph  $G$  with  $n \geq 7$  vertices satisfies

$$|E(G)| \geq \binom{n-5}{2} + 49,$$

then either  $G \in F_3$ , or  $G$  is contractible to  $K_4$ . This bound is also asymptotically best possible.

A collection  $\mathcal{C}$  of cycles of  $G$  is called a cycle cover (CC) if every edge of  $G$  lies in a cycle of  $\mathcal{C}$ . If every edge of  $G$  is in exactly  $k$  members  $\mathcal{C}$ , then  $\mathcal{C}$  is called a cycle  $k$ -cover of  $G$ . Bermond, Jackson and Jaeger [5] have shown that every 2-edge-connected graph has a cycle 4-cover. A cycle 2-cover is also called a cycle double cover (CDC).

**Cycle 2-Cover Conjecture** ([72], [74]). Every 2-edge-connected graph has a cycle 2-cover.

For cycle 2-covers, see [45] for a survey. For a 2-edge-connected graph  $G$ , define

$$cc(G) = \min\{|\mathcal{C}| : \mathcal{C} \text{ is a CC of } G\},$$

and

$$sc(G) = \min\{|\mathcal{C}| : \mathcal{C} \text{ is a CDC of } G\}.$$

Bondy ([6], [7]) conjectures the following.

**Conjecture SCDC** ([6], [7]). If  $G$  is a 2-edge-connected simple graph with  $n$  vertices, then

$$sc(G) \leq n - 1.$$

**Conjecture SCC** ([6], [7]). If  $G$  is a 2-edge-connected simple graph with  $n$  vertices, then

$$cc(G) \leq \frac{2n - 1}{3}.$$

No much has been done towards these two conjectures. Bondy and Seyffarth [7] proved Conjecture SCDC for planar triangulations. It was proved in [62] that Conjecture SCDC holds for graphs that do not have a subgraph contractible to  $K_4$ . Let  $\mathcal{F}_{cc}$  denote the collection of graphs satisfying Conjecture SCC, and  $\mathcal{F}_{sc}$  the collection of graphs satisfying Conjecture SCDC. It is still open whether all planar 2-edge-connected simple graphs are in  $\mathcal{F}_{sc}$ .

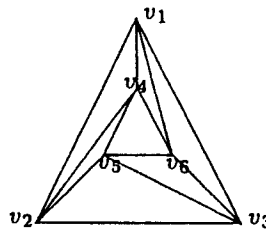
There are some contraction ideas used to attack Conjecture SCC. It is noticed ([63],[64]) that a certain configurations  $H$  satisfies the following: for any planar graphs that contain  $H$  as a subgraph,

$$G \in \mathcal{F}_{cc} \iff G/H \in \mathcal{F}_{cc}.$$

The following lemma is a typical example.

**Lemma 6.3** ([64]). Let  $G$  be a graph and  $H = \Gamma_1$  (see Figure 2) be a subgraph of  $G$  such that the vertices of attachment of  $H$  in  $G$  are lying in  $\{v_1, v_2, v_3\}$ . Let  $e \notin E(G)$  be an edge parallel to  $v_2v_3$  and  $G' = (G - V_H) + e_2$ . Then

$$cc(G) \leq cc(G') + 1. \quad \square$$



$\Gamma_1$

Figure 2: The graph  $\Gamma_1$

Other similar reduction lemmas can be found in [64]. These “reducible” graphs seem to be very sporadic, and are described in [64]. These reduction ideas were used ([63], [64]) to show that for a 2-edge-connected simple graph  $G$  with  $n$  vertices, if  $G$  does not have a subgraph contractible to  $K_4$ , then

$$cc(G) \leq \frac{2n - 2}{3},$$

and if  $G$  is a planar triangulation, then

$$cc(G) \leq \frac{2n - 3}{3}.$$

Both bounds are best possible.

## VII. Hamiltonian Line Graphs

A graph is essentially  $k$ -edge-connected if for any  $X \subseteq E(G)$  with  $|X| \leq k - 1$ , either  $G - X$  is connected or at most one component of  $G - X$  has an edge. It is easy to see that the line graph is  $k$ -connected if and only if  $G$  is essentially  $k$ -edge-connected. Theorem 1.1 gives a criterion for a line graph to be hamiltonian. It is known that any 4-edge-connected graph has a spanning eulerian subgraph [46]. Therefore, the line graph of any 4-edge-connected graph is hamiltonian. Thomassen [75] posed the following conjecture:

**Conjecture ([75]).** All 4-connected line graphs are hamiltonian.

Conjectures which are related to Thomassen Conjecture were discussed in [39]. In [1], Jackson made the following conjecture:

**Conjecture ([1]).** Let  $G$  be a 2-edge-connected graph, then  $G$  has an eulerian subgraph  $H$  with  $|V(H)| \geq 2$  such that for each component  $F$  of  $G - V(H)$ , there are at most three edges between  $F$  and  $H$ .

It was proved in [37] that Jackson's conjecture implies Thomassen's conjecture. Several authors have proved Thomassen's conjecture for some special classes of graphs. Based on a similar idea of [56], Chen and Lai proved the following:

**Theorem 7.1 ([36]).** A 4-connected line graph of a projective planar graph is hamiltonian.  $\square$

Zhan [81] and Jackson [43] independently proved that:

**Theorem 7.2 ([81]).** If  $L(G)$  is 7-connected, then  $L(G)$  is hamiltonian.  $\square$

Here we present a proof of Theorem 7.2 that is based on a unpublished manuscript of Lai [57]. We prove the following theorem first.

**Theorem 7.3.** If  $G$  is 3-edge-connected and essentially 7-edge-connected, then  $G$  is collapsible.

**Proof.** Let  $G$  be a smallest counterexample. Since contraction preserves edge-connectivity, by the minimality of  $G$ ,  $G$  must be reduced. For a vertex  $v \in V(G)$ , define  $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ . Let  $D_i = \{v \in V(G) \mid d(v) = i\}$  and  $D_i^* = \{v \in V(G) \mid d(v) \geq i\}$ . Since  $G$  is essentially 7-edge-connected,  $N(v) \subseteq D_6^*$  if  $v \in D_3$ . Let  $E^* = \{uv \in E(G) \mid u \in D_3 \text{ and } v \in D_6^*\}$ . Let  $H = G[E^*]$ . Then by Theorem 4.1,  $H$  is a reduced bipartite graph with bipartition  $V(H) = D_3 \cup D_6^*$ . By Theorem 4.1(d),  $|E(H)| \leq 2|V(H)| - 4$ . Hence,

$$\begin{aligned} 3|D_3| &\leq |E(H)| \leq 2(|D_3| + |D_6^*|) - 4, \\ |D_3| &\leq 2|D_6^*| - 4. \end{aligned} \tag{2}$$

Since  $G$  is 3-edge-connected,  $D_1 = D_2 = \emptyset$ . By  $\sum_{v \in V(G)} d(v) = 2|E(G)|$  and Theorem 4.1(d) again, we have

$$3|D_3| + 4|D_4| + 5|D_5| + 6|D_6^*| \leq 2|E(G)| \leq 4(|D_3| + |D_4| + |D_5| + |D_6^*|) - 8. \tag{3}$$

By (2) and (3), we have

$$|D_5| + 2|D_6^*| \leq |D_3| - 8 \leq (2|D_6^*| - 4) - 8.$$

Thus,  $|D_5| \leq -12$ , a contradiction.  $\square$

**Proof of Theorem 7.2.** Since  $L(G)$  is 7-connected,  $G$  is essentially 7-edge-connected graph. Let  $X = D_1 \cup D_2$ . Then  $X$  is an independent set of  $G$ . Let  $I_X(G)$  be the graph obtained from  $G$  as defined in section 2.2. Since  $G$  is essentially 7-edge-connected,  $I_X(G)$  must be 3-edge-connected and essentially 7-edge-connected. By Theorem 7.3,  $I_X(G)$  is collapsible. By Theorem 2.1 and then by Theorem 2.3,  $G$  has a dominating eulerian subgraph. Therefore, by Theorem 1.1,  $L(G)$  is hamiltonian.  $\square$

The following result is best possible in the sense that there are graphs  $G$  such that  $L^2(G)$  is 3-connected and  $L^2(G)$  is not hamiltonian.

**Theorem 7.4** ([37]). If  $L^2(G)$  is 4-connected, then  $L^2(G)$  is hamiltonian.  $\square$

The reduction method had also been applied in the study of the hamiltonian index ([23],[50]). A recent article [71] corrected an error of Theorem 20 in [50].

### VIII. Final Remark and Open Problems

In this paper, we primarily focus on Catlin's reduction method, its applications and some recent developments. We choose our topics on how the reduction technique has been applied. It is not our intention to cover all the development of each of our topics. This article supplements, not replaces, Catlin's survey ([18]). We present the following open problems and conjectures to conclude this article.

**Conjecture 8.1** ([19]). Let  $H$  be a graph. If  $H$  is not collapsible, then  $H$  has a supergraph  $G$  such that this equivalence is false:

$$G \in \mathcal{SL} \iff G/H \in \mathcal{SL}.$$

**Conjecture 8.2** ([19]). Let  $G(e)$  denote the graph obtained from  $G$  with an edge  $e \in E(G)$  replaced by a simple path of length 2. If  $G \in \mathcal{CL}$ , then either

$$|E(G)| = \frac{3}{2}(|V(G)| - 1),$$

or for some edge  $e \in E(G)$ ,  $G(e)$  is collapsible.

**Conjecture 8.3** ([19]). Let  $G$  be a graph. If  $H$  is a  $K_2$  in  $G$  such that  $G/H$  is collapsible, then  $G$  is  $\pi$ -collapsible, for some partition  $\pi$  of  $V(G)$ .

**Conjecture 8.4** ([19]). If a collapsible graph  $G$  has a vertex  $v$  of degree 2, then  $G - v$  is  $\pi$ -collapsible, for some partition  $\pi$  of  $V(G - v)$ .

**Conjecture 8.5** ([35]). Let  $G$  be a simple 3-edge-connected graph with  $n$  vertices. If for every edge  $e = uv \in E(G)$ ,

$$d(v) + d(u) > \frac{n}{9} - 2,$$

then when  $n$  is large, either  $G$  is in  $SC$ , or  $G$  is contractible to the Petersen graph.

**Conjecture 8.6.** Every 3-edge-connected, essentially 6-edge-connected graph is collapsible.

**Conjecture 8.7.** Every 3-edge-connected, essentially 5-edge-connected graph is supereulerian.

**Problem 8.8.** Determine

$$\min_{G \in SC} \max \left\{ \frac{|E(H)|}{|E(G)|} : H \text{ is a spanning eulerian subgraph of } G \right\}.$$

It is known [31] that for the independence number  $\alpha(G)$  of a reduced graph  $G$  of order  $n$  with  $\delta(G) \geq 3$ ,  $\frac{3\alpha(G) + 4}{2} \leq n \leq 4\alpha(G) - 5$ , and so  $1/4 \leq \underline{\lim}_{n \rightarrow \infty} \frac{\alpha(G)}{n} \leq \overline{\lim}_{n \rightarrow \infty} \frac{\alpha(G)}{n} \leq 2/3$ . However, there are no examples to show that those bounds are best possible.

**Problem 8.9.** Let  $G$  be a connected reduced graph of order  $n$  with  $\delta(G) \geq 3$ . What is the value of  $\underline{\lim}_{n \rightarrow \infty} \frac{\alpha(G)}{n}$ ? What is the value of  $\overline{\lim}_{n \rightarrow \infty} \frac{\alpha(G)}{n}$ ?

For the chromatic number of a reduced graph, Catlin [24] conjectured that

**Conjecture 8.10** ([24]). Any nontrivial connected reduced graph has chromatic number 2 or 3.

By Theorem 4.1(c) and Grötzsch's Theorem on three-coloring  $K_3$ -free planar graph [40], one can see that this conjecture is true for reduced planar graphs. This conjecture implies that for a reduced graph  $G$  of order  $n$ ,  $n \leq 3\alpha(G)$ , and so  $\underline{\lim}_{n \rightarrow \infty} \frac{\alpha(G)}{n} \geq 1/3$ .

Let  $G$  be a connected reduced graph of order  $n$  with  $\delta(G) \geq 3$ , and  $M(G)$  the size of maximum matching of  $G$ . Then by Theorem 4.3,  $1/2 \geq \underline{\lim}_{n \rightarrow \infty} \frac{M(G)}{n} \geq 1/3$ . Note that all the snarks are reduced and have a perfect matching (e.g., the family of flower snarks [79]), and so  $\overline{\lim}_{n \rightarrow \infty} \frac{M(G)}{n} = 1/2$ . However, the following is unknown.

**Problem 8.11.** Let  $G$  be a connected reduced graph of order  $n$  with  $\delta(G) \geq 3$ . Determine

$$\underline{\lim}_{n \rightarrow \infty} \frac{M(G)}{n}?$$

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