



Every matroid is a submatroid of a uniformly dense matroid

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Abstract

For a graph G with at least one edge, define

$$d(G) = \frac{|E(G)|}{|V(G)|} \quad \text{and} \quad m(G) = \max_{H \subseteq G} d(H).$$

Karoński and Ruciński (1982) conjectured that every connected graph G is a subgraph of a graph G' with $m(G') = d(G') = m(G)$. This conjecture has been proved by Györi et al. (1985) and, independently by Payan (1986). The following is related. Define

$$g(G) = \frac{|E(G)|}{|V(G)| - 1} \quad \text{and} \quad \gamma(G) = \max_{H \subseteq G} g(H).$$

Payan (1986) proves that every connected graph G is a subgraph of a graph G' with $g(G') = \gamma(G') = \gamma(G)$.

In this paper, we shall show that both theorems above are related by matroid elongations, and we shall also extend these results to their versions in binary matroids and regular matroids.

1. Introduction

The graphs in this paper are finite and undirected. Multiple edges are allowed but loops are forbidden. The matroids considered in this paper are loopless matroids on finite nonempty sets. We use $M = (S, \mathcal{I}(M))$ to denote a matroid M with ground set S and the collection of independent sets $\mathcal{I}(M)$. When no confusion occurs, we use \mathcal{I} for $\mathcal{I}(M)$. For undefined terms, see Bondy and Murty [1] (for graphs) and Welsh [11] (for matroids).

Let G be a graph with $E(G) \neq \emptyset$ and let $X \subseteq E(G)$. The *contraction* G/X is the graph obtained by identifying the ends of each edge in X and deleting the resulting loops. For convenience, we define $G/\emptyset = G$.

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Let $M = (S, \mathcal{I})$ be a matroid. Let $N \subseteq S$ be a subset of S . Recall that in [11], the closure of N in M is denoted by $\sigma(N)$. If $S = \sigma(N)$, then we say that N spans M , and say that N is a *spanning set*. For a subset $X \subseteq S$, $M|X$ denotes the *restriction* of M to X , and $M.X$ denotes the *contraction* of M to X . Thus a subset $X' \subseteq X$ is independent in $M|X$ if and only if X' is independent in M , and X' is independent in $M.X$ if and only if for any independent subset $T \subseteq S - X$, $T \cup X'$ is independent in M (see [11, Ch. 4]). The set of all spanning sets of M is denoted by $\mathcal{S}(M)$. If N does not span M , then we define the (*loopless*) *contraction* by

$$M/N = M.(M - \sigma(N)). \quad (1)$$

Thus if ρ, ρ_c denote the rank functions of M and M/N , respectively, then by a formula in [11, p. 62],

$$\rho_c(X) = \rho(X \cup N) - \rho(N) \quad \text{for any subset } X \subseteq M/N. \quad (2)$$

Let G be a graph. Define

$$d(G) = \frac{|E(G)|}{|V(G)|} \quad \text{and} \quad m(G) = \max_{H \subseteq G} d(H). \quad (3)$$

A graph G is *balanced* if $m(G) = d(G)$. In 1982, Karoński and Ruciński [7] conjectured that every connected graph G is a subgraph of a balanced graph H with $m(H) = m(G)$. This conjecture has been proved by Györi et al. [5] and, independently by Payan [9].

Theorem 1.1 (Györi et al. [5] and Payan [9]). *Every connected graph G is a subgraph of a balanced graph G' with $m(G') = m(G)$.*

The following is related. Define

$$g(G) = \frac{|E(G)|}{|V(G)| - 1} \quad \text{and} \quad \gamma(G) = \max_{H \subseteq G} g(H). \quad (4)$$

A graph G is *uniformly dense* if $g(G) = \gamma(G)$. (We follow [3] for using the term uniformly dense graphs. In [10], a uniformly dense graph is called a *strongly balanced* graph, and in [9], a uniformly dense graph is called a *decomposable* graph.)

Theorem 1.2 (Payan [9]). *For every graph G , there is a uniformly dense graph G' containing G as a subgraph such that $\gamma(G) = \gamma(G')$.*

Let $M = (S, \mathcal{I})$ be a loopless matroid with rank function ρ and let

$$g(M) = \frac{|S|}{\rho S} \quad \text{and} \quad \gamma(M) = \max_{\emptyset \neq X \subseteq S} g(M|X). \quad (5)$$

A loopless matroid $M = (S, \mathcal{I})$ is *uniformly dense* if $\gamma(M) = g(M)$, and the quantity $g(M)$ is the *density* of the matroid M . In this paper, we shall show that both Theorems

1.1 and 1.2 are related by matroid elongations. We shall also extend these theorems to their versions in regular matroids and binary matroids.

In Section 2, the elongations of a matroid will be discussed, and a relationship between a balanced graph G and an elongation of the cycle matroid $M(G)$ will be shown. In Section 3, we shall exhibit some extensions of matroids. In Section 4, some prior results on uniformly dense graphs are recalled. In the last section, we shall follow an idea of Payan [9] and use a matroid extension in Section 3 in place of the vertex-identification technique used by Payan in [9] to extend Theorems 1.1 and 1.2 to their matroid versions.

2. Elongations of a matroid

Let G be a graph with $E(G) \neq \emptyset$. The *cycle matroid* of G , denoted by $M(G)$, has all edge subsets that induce forests as its independent sets (see [11, page 28]).

Let $M = (S, \mathcal{I})$ be a matroid with rank function ρ such that $S \notin \mathcal{I}$. For any integer i with $1 \leq i \leq |S| - \rho(S)$, we define

$$\mathcal{F}(M^i) = \{X \subseteq S: X \in \mathcal{I}(M) \text{ and } |X| = \rho(S) + i\}. \quad (6)$$

Theorem 2.1 (Welsh [11, Theorem (4.1.2)]). *The family of subsets $\mathcal{F}(M^i)$ of S is the bases of a matroid M^i on S . (We shall call M^i the i -elongation of M).*

We are most interested in M^1 . Let ρ and ρ^1 denote the rank functions of M and M^1 , respectively. It follows by the definition of M^1 that for any $T \subseteq S$,

$$\rho^1(T) = \begin{cases} \rho(T) = |T| & \text{if } T \in \mathcal{I}(M), \\ \rho(T) + 1 & \text{if } T \notin \mathcal{I}(M). \end{cases} \quad (7)$$

Lemma 2.2 is needed in the proofs below.

Lemma 2.2 (Hardy et al. [6, Theorem 1, p. 14]). *Let a_1, a_2, \dots, a_m and b_1, b_2, \dots, b_m be positive numbers. Then*

$$\frac{a_1 + a_2 + \dots + a_m}{b_1 + b_2 + \dots + b_m} \leq \max_{1 \leq i \leq m} \frac{a_i}{b_i}.$$

Theorem 2.3. *Let G be a connected graph with $E(G) \neq \emptyset$ such that G is not a tree. Let $M = M(G)$ denote the cycle matroid of G and let $M^1 = M^1(G)$. Then M^1 is uniformly dense if and only if G is balanced. Moreover, when M^1 is uniformly dense, the density of M^1 is equal to $d(G)$.*

Proof. Let G, M and M^1 satisfy the hypothesis of Theorem 2.3. Let ρ^1 denote the rank function of M^1 . Let g denote the density function of M^1 . If H is a subgraph of G , then for convenience we write $M^1(H)$ for the restricted submatroid $M^1|E(H)$.

Suppose first that G is balanced. Then for any subgraph L of G ,

$$d(L) = \frac{|E(L)|}{|V(L)|} \leq \frac{|E(G)|}{|V(G)|} = d(G). \quad (8)$$

Since G is connected, and is not a tree, we have by (7), $\rho^1(E(G)) = |V(G)|$, and so

$$d(G) = \frac{|E(G)|}{|V(G)|} = \frac{|E(G)|}{\rho^1(E(G))} = g(M^1) \geq 1. \quad (9)$$

To show that M^1 is uniformly dense, we must show that for any edge-induced subgraph H , $g(M^1(H)) \leq g(M^1(G))$.

Let H be a subgraph of G with $E(H) \neq \emptyset$, and let H_1, H_2, \dots, H_c be the components of H such that for some l with $1 \leq l \leq c$, H_1, \dots, H_l are trees and H_{l+1}, \dots, H_c are not trees, and such that

$$\frac{|E(H_{l+1})|}{|V(H_{l+1})|} \leq \frac{|E(H_{l+2})|}{|V(H_{l+2})|} \leq \dots \leq \frac{|E(H_c)|}{|V(H_c)|}. \quad (10)$$

Note that for any j ($l \leq j \leq c$), by (7),

$$g(M^1(H_j)) = \frac{|E(H_j)|}{\rho^1(E(H_j))} = \frac{|E(H_j)|}{|V(H_j)|} = d(H_j) \geq 1. \quad (11)$$

If H is a forest, then by (7), $\rho^1(E(H)) = |V(H)| - c$, and so by (9),

$$g(M^1(H)) = \frac{|E(H)|}{|V(H)| - c} = 1 \leq g(M^1(G)).$$

Therefore, we may assume that H is not a forest, and so H_c is not a tree. Hence by (7),

$$\rho^1(E(H)) = (|V(H_1)| - 1) + \dots + (|V(H_l)| - 1) + |V(H_{l+1})| + \dots + |V(H_c)|,$$

and so by (10), by (11), by Lemma 2.2, and by (8) (with $L = H_c$ in (8)) and (9),

$$\begin{aligned} g(M^1(H)) &= \frac{|E(H)|}{\rho^1(E(H))} & (12) \\ &= \frac{|E(H_1)| + |E(H_2)| + \dots + |E(H_c)|}{(|V(H_1)| - 1) + \dots + (|V(H_l)| - 1) + |V(H_{l+1})| + \dots + |V(H_c)|} \\ &= d(H_c) \quad (\text{by (10), by (11) and by Lemma 2.2}) \\ &\leq d(G) = g(M^1(G)) \quad (\text{by (8) and (9)}). \end{aligned}$$

Therefore M^1 is uniformly dense, by (12).

Conversely, we assume that M^1 is uniformly dense. Since G is connected and is not a tree, we have by (7), $\rho^1(E(G)) = |V(G)|$ and for any nontrivial edge-induced

subgraph H of G , $\rho^1(E(H)) \leq |V(H)|$. It follows that

$$d(G) = \frac{|E(G)|}{|V(G)|} = g(M^1(G)) \geq g(M^1(H)) \geq \frac{|E(H)|}{|V(H)|} = d(H), \tag{13}$$

and so G is balanced. \square

Theorem 2.4. *Let $M = (S, \mathcal{J})$ be a matroid such that $S \notin \mathcal{J}$. If M is uniformly dense, then for any i with $0 < i \leq |S| - \rho(S)$, M^i is also uniformly dense.*

Proof. By the definition of elongations, if $0 < i < |S| - \rho(S)$, then $M^{i+1} = (M^i)^1$. Therefore, to prove Theorem 2.4, it suffices to show that if M is uniformly dense, then M^1 is also uniformly dense. Let ρ, ρ^1 denote the rank functions of M and M^1 , respectively.

By way of contradiction, we assume that M is uniformly dense but M^1 is not uniformly dense. Thus there is a subset $X \subset S$ such that

$$\frac{|X|}{\rho^1(X)} > \frac{|S|}{\rho^1(S)} = \frac{|S|}{\rho(S) + 1}. \tag{14}$$

The last equality follows from (7) and the fact that $S \notin \mathcal{J}(M)$.

We shall claim that $X \notin \mathcal{J}(M)$. By way of contradiction, we assume that $X \in \mathcal{J}(M)$. Thus $|X| = \rho(X)$, and so by (7),

$$\frac{|X|}{\rho^1(X)} \leq \frac{|X|}{\rho(X)} \leq 1. \tag{15}$$

By the assumption that $S \notin \mathcal{J}$, and by (7), we have $|S| \geq \rho(S) + 1$, and so by (14) and (15),

$$1 = \frac{|X|}{\rho(X)} \geq \frac{|X|}{\rho^1(X)} > \frac{|S|}{\rho^1(S)} = \frac{|S|}{\rho(S) + 1} \geq 1,$$

a contradiction.

Therefore, $X \notin \mathcal{J}(M)$, and so by (7),

$$\rho^1(X) = \rho(X) + 1. \tag{16}$$

But then (14) and (16) together with $|S| > |X|$ imply that

$$\frac{|X|}{\rho(X) + 1} > \frac{|S|}{\rho(S) + 1} \quad \text{and so} \quad \frac{|X|}{\rho(X)} > \frac{|S|}{\rho(S)},$$

contrary to the assumption that M is uniformly dense. \square

3. Some extensions of matroids

We start with a definition of parallel extension.

Definition. Let M be a matroid with a distinguished element $e \in M$. Let $e' \notin M$ be an element. Define, as a set, $M' = M \cup \{e'\}$, and define

$$\begin{aligned} \mathcal{I}(M') = & \{I: e' \notin I \text{ and } I \in \mathcal{I}(M)\} \\ & \cup \{I: e' \in I, e \notin I \text{ and } (I - \{e'\}) \cup \{e\} \in \mathcal{I}(M)\}. \end{aligned} \quad (17)$$

Proposition 3.1. $M' = (M', \mathcal{I}(M'))$ is a matroid that contains M as a submatroid.

Proof. The proof is straightforward and so it is omitted. \square

We notice that the matroid M' in Proposition 3.1 is obtained from M by adding an element e' parallel to e .

Definition. Let p, q be two positive integers, and let $e \in S$ be a fixed element of a matroid $M = (S, \mathcal{I})$. Denote by $M(e, p, q)$ the matroid obtained from M by adding $p - 2$ new elements parallel to e and by adding $q - 1$ new elements parallel to x for every $x \in S - \{e\}$. When $p = t + 1$ and $q = t$, we use $M_{(t)}$ for $M(e, t + 1, t)$. Note that in $M_{(t)}$, any element in S can be the element e , and therefore no element in $M_{(t)}$ is distinguished. We shall call $M_{(t)}$ the t -parallel extension of M .

For every $x \in S$, let $[x]$ denote the set of all parallel elements in M that are parallel to x , including x . Thus if M is a simple matroid (one that does not have loops nor parallel elements), then the ground set of $M_{(t)}$ is $\bigcup_{x \in S} [x]$ with each $|[x]| = t$. Note that a subset X is independent in $M_{(t)}$ if and only if $|X \cap [x]| \leq 1$ for every $x \in S$ and the set $\{x: [x] \cap X \neq \emptyset\}$ is in $\mathcal{I}(M)$. For this reason, we shall regard an independent subset X in $M_{(t)}$ as a independent subset in M in the proof of Theorem 5.3, when no confusion arises.

Let $M_1 = (S_1, \mathcal{I}(M_1))$ and $M_2 = (S_2, \mathcal{I}(M_2))$ be two matroids on disjoint sets. Fix two elements $x_1 \in M_1$ and $x_2 \in M_2$ and let x denote an element not in $S_1 \cup S_2$. Let \mathcal{I}'' denote the collection of subsets of $S = (S_1 - \{x_1\}) \cup (S_2 - \{x_2\}) \cup \{x\}$ such that $U \in \mathcal{I}''$ if and only if one of the following holds:

(A1) $x \notin U$; and both $U \cap S_1$ and $U \cap S_2$ are independent in M_1 and M_2 , respectively; and either $(U \cap S_1) \cup \{x_1\}$ is independent in M_1 or $(U \cap S_2) \cup \{x_2\}$ is independent in M_2 .

(A2) $x \in U$; and both $[(U - \{x\}) \cup \{x_1\}] \cap S_1$ and $[(U - \{x\}) \cup \{x_2\}] \cap S_2$ are independent in M_1 and M_2 , respectively.

Proposition 3.2 (Brylawski [2]). *The collection \mathcal{I}'' is the set of independent sets of a matroid on S .*

Let $S = (S_1 - \{x_1\}) \cup (S_2 - \{x_2\}) \cup \{x\}$, and let $M_1 \otimes_{(x_1, x_2)} M_2$ denote the matroid on S with \mathcal{I}'' as its collection of independent sets. Then $M_1 \otimes_{(x_1, x_2)} M_2$ is called

a matroid connection of M_1 and M_2 (see [2]). By regarding x as x_i ($1 \leq i \leq 2$), we can see that M_i is a restricted submatroid of $M_1 \otimes_{(x_1, x_2)} M_2$.

Let G_1 and G_2 be two graphs with disjoint vertex sets. Let $e_i \in E(G_i)$ ($1 \leq i \leq 2$). Denote by $G_1 \otimes_{(e_1, e_2)} G_2$ a graph obtained from the union of G_1 and G_2 by identifying e_1 with e_2 .

Proposition 3.3. *Each of the following holds.*

(i) *If G_1 and G_2 are two graphs with $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$, then*

$$M(G_1) \otimes_{(e_1, e_2)} M(G_2) \cong M(G_1 \otimes_{(e_1, e_2)} G_2). \quad (18)$$

(ii) *If $M_1 = (S_1, \mathcal{I}(M_1))$ and $M_2 = (S_2, \mathcal{I}(M_2))$ are two regular matroids with $e_1 \in S_1$ and $e_2 \in S_2$, then $M_1 \otimes_{(e_1, e_2)} M_2$ is also a regular matroid.*

(iii) *If $M_1 = (S_1, \mathcal{I}(M_1))$ and $M_2 = (S_2, \mathcal{I}(M_2))$ are two binary matroids with $e_1 \in S_1$ and $e_2 \in S_2$, then $M_1 \otimes_{(e_1, e_2)} M_2$ is also a binary matroid.*

Sketch of proof. For (i), it suffices to show that both sides of (18) have the same set of independent sets, which is an immediate consequence of the definition \mathcal{I} ". The other two conclusions (ii) and (iii) are consequences of Theorem (10.2.1) of [11] and Theorem (10.4.1) of [11], respectively. \square

4. Prior results on uniformly dense graphs

We shall quote some of the prior results that will be used in this paper.

Theorem 4.1 (Catlin et al. [4, Theorem 4]). *Let $M = (S, \mathcal{I})$ be a matroid. M is uniformly dense with $\gamma(M) = s/t$ if and only if $M_{(t)}$ is the disjoint union of s bases.*

Theorem 4.2 (Nash-Williams [8]). *Let M be a matroid. Then $M_{(t)}$ is the disjoint union of s independent sets if and only if $\gamma(M) \geq s/t$.*

Theorem 4.3 (Payan [9]). *For any fraction $s/t \geq 1$, there is a simple graph G with a distinguished edge $e \in E(G)$ such that $M(e, s, t)$ is uniformly dense with $\gamma(M(e, s, t)) = s/t$, where $M = M(G)$ is the cycle matroid of G .*

5. Uniformly dense extensions of matroids

Let $M = (S, \mathcal{I})$ be a matroid. A matroid $M' = (S', \mathcal{I}')$ is a *uniformly dense extension* of M if $S \subseteq S'$ and $\mathcal{I} \subseteq \mathcal{I}'$ and if M' is uniformly dense with $\gamma(M) = \gamma(M')$. Thus Theorems 1.1 (in view of Theorem 2.3) and 1.2 can be restated as follows.

Theorem 5.1. *Every 1-elongation of a graphic matroid has a uniformly dense extension which is also a 1-elongation of a graphic matroid.*

Theorem 5.2. *Every graphic matroid has a uniformly dense extension which is also graphic.*

The main result in this section is the following.

Theorem 5.3. *Every matroid M has a uniformly dense extension M' . Moreover, each of the following holds:*

- (i) *If M is regular, then M' is also regular.*
- (ii) *If M is binary, then M' is also binary.*

Proof. Let $M = (S, \mathcal{I})$ be a matroid and let $\gamma(M) = s/t$, where s and t are two positive integers. By Theorem 4.2, $M_{(t)}$ can be covered by s mutually disjoint independent sets T_1, T_2, \dots, T_s . We argue by induction on the quantity

$$f(M) = \min \sum_{i=1}^s (\rho(S) - |T_i|), \quad (19)$$

where the minimum is taken over all coverings of s independent sets of $M_{(t)}$ such that $\gamma(M) = s/t$. We assume that

$$f(M) = \min \sum_{i=1}^s (\rho(S) - |T_i|) \quad (20)$$

for some independent sets T_1, \dots, T_s that cover $M_{(t)}$. If $f(M) = 0$, then each T_i is a base of M and so by Theorem 4.1, M is uniformly dense. In this case, we set $M' = M$.

Assume that Theorem 5.3 holds for smaller value of $f(M)$ and that $f(M) > 0$. Assume again $\gamma(M) = s/t$, for some positive integers s and t that (20) holds for some independent sets T_1, T_2, \dots, T_s of $M_{(t)}$ that cover $M_{(t)}$. Since $f(M) > 0$, we may assume that $\rho(S) - |T_1| > 0$. By the independent axiom (I3) [11, p. 7], there is an element $x_1 \in S - T_1$ such that $T_1 \cup \{x_1\} \in \mathcal{I}$. By Theorem 4.3, there is a graphic matroid $M_2 = (S_2, \mathcal{I}(M_2))$ with a distinguished element $x_2 \in S_2$ such that $M_2(x_2, s, t)$ is uniformly dense with $\gamma(M(e, s, t)) = s/t$. By Theorem 4.1, $M_2(x_2, s, t)$ is the disjoint union of bases F_1, F_2, \dots, F_s . Since $|[x_2]| = s - 1$ in $M_2(x_2, s, t)$, we may assume that $x_2 \notin F_1$, and $x_2 \in F_i$ ($2 \leq i \leq s$). Let

$$M'' = M \otimes_{(x_1, x_2)} M_2 \quad \text{and} \quad S'' = S \cup (S_2 - \{x_2\}),$$

regarding $x = x_1$ in M'' . Let ρ, ρ_2 and ρ'' denote the rank functions of M, M_2 and M'' , respectively.

Note that now $(M'')_{(t)}$ are covered by mutually disjoint subsets T'_1, T'_2, \dots, T'_s , where

$$T'_1 = T_1 \cup F_1 \quad \text{and} \quad T'_i = T_i \cup (F_i - \{x_2\}) \quad (2 \leq i \leq s). \quad (21)$$

Let B be a base of M with $x_1 \in B$, and let B_2 be a basis of M_2 with $x_2 \in B_2$. Then by (A1) and (A2) in the definition of $M \otimes_{(x_1, x_2)} M_2$, $B \cup (B_2 - \{x_2\})$ is a basis of M'' and so

$$\rho''(S'') = \rho(S) + \rho_2(S_2) - 1. \tag{22}$$

By (A1) and (A2) in the definition of $M \otimes_{(x_1, x_2)} M_2$, each T'_i is independent in M'' , and so by (20) and (22), and by the facts that $|F_i| = \rho_2(S_2)$ ($1 \leq i \leq s$), we have

$$\begin{aligned} f(M'') &\leq \sum_{i=1}^s (\rho''(S'') - |T'_i|) \\ &= \rho(S) + \rho_2(S_2) - 1 - (|T_1| + |F_1|) \\ &\quad + \sum_{i=2}^s (\rho(S) + \rho_2(S_2) - 1 - (|T_i| + |F_i| - 1)) \\ &= \sum_{i=1}^s (\rho(S) - |T_i|) - 1 = f(M) - 1. \end{aligned} \tag{23}$$

As M'' is covered by independent sets T'_1, T'_2, \dots, T'_s , it follows by Theorem 4.2 that

$$\gamma(M'') = \frac{s}{t} = \gamma(M). \tag{24}$$

Note that a graphic matroid is both regular and binary (see [11, Ch. 10]). It follows by Proposition 3.3 that if M is regular (binary), then M'' is also regular (binary). Therefore by (23) and by induction, M'' has a uniformly dense extension M' with

$$\gamma(M'') = \gamma(M'), \tag{25}$$

such that if M'' is regular (binary), then so is M' . As M'' is an extension of M , M' is also an extension of M . By (24) and (25),

$$\gamma(M') = \gamma(M'') = \gamma(M).$$

Therefore, M' is a uniformly dense extension of M , and so the proof of Theorem 5.3 is now completed. \square

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