DISCRETE MATHEMATICS

ELSEVIER

Discrete Mathematics 143 (1995) 141–151

Strength and fractional arboricity of complementary graphs☆

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Received 17 February 1993; revised 27 October 1993

Abstract

Let G be a simple graph and G^c be the complement of G. Let $\omega(G)$ denote the number of components of G. As in Catlin et al. (1992), for a nontrivial graph G, the *strength* of G is

$$\eta(G) = \min_{S \subseteq E(G)} \frac{|S|}{\omega(G-S) - \omega(G)},$$

where the minimum is over all subsets $S \subseteq E(G)$ such that $\omega(G-S) > \omega(G)$. The *fractional arboricity* of a nontrivial graph G is

$$\gamma(G) = \max_{\phi \neq H \subseteq G} \frac{|E(H)|}{|V(H)| - \omega(H)},$$

where the maximum runs over all subgraphs H with $|V(H)| > \omega(H)$.

In this note, we shall present Nordhaus–Gaddum types of inequalities on the strength and the fractional arboricity of a graph G. In particular, we show that if G is a simple graph on $n \geqslant 4$ vertices, then each of the following holds:

(a) $\frac{n}{2} \leqslant \gamma(G) + \gamma(G^c) \leqslant \frac{5n}{8} + \frac{n}{8(n-1)}$ if n is even,

(b) $\frac{n}{2} \leqslant \gamma(G) + \gamma(G^c) \leqslant \frac{5n}{8} + \frac{1}{8}$ if n is odd,

(c) $2 \leqslant \eta(G) + \eta(G^c) \leqslant \frac{5n}{8} + \frac{n}{8(n-1)}$ if n is even,

(d) $2 \leqslant \eta(G) + \eta(G^c) \leqslant \frac{5n}{8} + \frac{1}{8}$ if n is odd,

(e) $0 \leqslant \gamma(G)\gamma(G^c) \leqslant \frac{n(n-1)d - d^2(d-1)}{4(n-1)}$,

(f) $0 \leqslant \eta(G)\eta(G^c) \leqslant \frac{n(n-1)d - d^2(d-1)}{4(n-1)}$,

☆ Partially supported by ONR grant N00014-91-J-1699.

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where $d \in \{[y_0], [y_0]+1\}$ in (e) and (f), and $y_0 = (1+\sqrt{1+3n(n-1)})/3$. Moreover, all the bounds are best possible.

1. Introduction

Graphs in this note are finite and simple. We shall use the notation of Bondy and Murty [1], unless otherwise stated. The *density* of a graph G is

$$g(G) = \begin{cases} \dfrac{|E(G)|}{|V(G)| - \omega(G)} & \text{if } E(G) \neq \phi, \\ 0 & \text{if } E(G) = \phi, \end{cases}$$

where $\omega(G)$ denote the number of components of G. A graph G is *uniformly dense* if for any subgraph H of G, $g(H) \leqslant g(G)$. The *strength* of G is defined as

$$\eta(G) = \begin{cases} \min\limits_{S \subseteq E(G)} \dfrac{|S|}{\omega(G-S) - \omega(G)} & \text{if } E(G) \neq \phi, \\ 0 & \text{if } E(G) = \phi, \end{cases}$$

where the minimum is over all subsets $S \subseteq E(G)$ such that $\omega(G-S) > \omega(G)$. The *fractional arboricity* of G is defined as

$$\gamma(G) = \begin{cases} \max\limits_{\phi \neq H \subseteq G} \dfrac{|E(H)|}{|V(H)| - \omega(H)} = \max\limits_{\phi \neq H \subseteq G} g(H) & \text{if } E(G) \neq \phi, \\ 0 & \text{if } E(G) = \phi, \end{cases}$$

where the maximum runs over all subgraphs H with $|V(H)| > \omega(H)$.

From these definitions, it is not difficult to see that for any graph G,

$$\eta(G) \leqslant g(G) \leqslant \gamma(G). \tag{1}$$

It has been indicated by several authors ([2–8], among others) that these parameters can be used as measures of invulnerability of networks. Other applications of these parameters in electrical network analysis can be found in [10, 11], among others.

The *complement* of a simple graph G, denoted by G^c, has $V(G^c) = V(G)$ and $E(G^c) = \{uv \mid u, v \in V(G) \text{ and } uv \notin E(G)\}$.

As in [1], $G + H$ is a graph defined as follows:

$$V(G+H) = V(G) \cup V(H),$$

$$E(G+H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G),\ v \in V(H)\}.$$

For a real number x, $[x]$ denotes the largest integer not bigger than x. And we let K_n denote the complete graph on n vertices. Let $y_0 = (1+\sqrt{1+3n(n-1)})/3$ and

$$f(y) = \frac{n(n-1)y - y^2(y-1)}{4(n-1)}.$$

Define $d = [y_0]$ if $f([y_0]) \geqslant f([y_0]+1)$, $d = [y_0]+1$ if $f([y_0]) < f([y_0]+1)$, i.e. d is an integer which maximizes $f(y)$. A computer search has been used to see if there is a pattern in determining d. Unfortunately such a pattern has not been found and so we have to define d in its current way.

In [9], the famous Nordhaus–Gaddum theorem states

$$2\sqrt{n} \leqslant \chi(G) + \chi(G^c) \leqslant n+1, \qquad n \leqslant \chi(G)\chi(G^c) \leqslant \left(\frac{n+1}{2}\right)^2,$$

where $\chi(G)$ denotes the chromatic number of G. Since then, whenever analogous relations between a graph and its complement are obtained, they are called the Nordhaus–Gaddum inequalities. In this note, we shall present some Nordhaus–Gaddum inequalities for the strength and the fractional arboricity of a graph G.

2. Preliminaries

In this section, we recall some prior results. Lemma 2.1 is a well-known observation and so we state it without proof.

Lemma 2.1. *If G is a simple graph, then either G or G^c is connected.*

Theorem 2.2 (Catlin et al. [3, Theorem 6]). *Let G be a graph. The following are equivalent*:
(1) *G is uniformly dense* (i.e. $\gamma(G) = g(G)$),
(2) $\gamma(G) = \eta(G)$,
(3) $\eta(G) = g(G)$.

Lemma 2.3. *If a graph G is not uniformly dense, then G has a induced uniformly dense subgraph H such that $g(H) = \gamma(G)$ and $|V(H)| < |V(G)|$.*

Lemma 2.4. *If G is uniformly dense and $\omega(G) \geqslant 2$, then there exists a component H of G such that $g(H) = \gamma(G)$.*

Proofs of Lemmas 2.3 and 2.4 follow from the definition of $\gamma(G)$.

Lemma 2.5 (Catlin et al. [3, Theorem 12]). *If the only nontrivial component H of G is a uniformly dense graph, then G is also uniformly dense and*

$$\eta(G) = \gamma(G) = \eta(H) = \gamma(H). \tag{2}$$

Lemma 2.6 (Catlin et al. [3, Corollary 8]). *Complete graphs are uniformly dense.*

Lemma 2.7. (Peng et al. [12, Corollary of Lemma 2]). *If H is connected and uniformly dense, then $K_m + H$ is also uniformly dense.*

Corollary 2.8. *Let G be the disjoint union $K_{n-m} \cup K_m^c$ where $n > m > 0$. Then G^c is uniformly dense.*

Proof. Note that $G^c = K_m + K_{n-m}^c$ and that K_{n-m}^c is uniformly dense. Therefore, Corollary 2.8 follows from Lemma 2.7. □

Lemma 2.9. *If $E(G) \neq \phi$, then $\eta(G) \geqslant 1$. Moreover, if G has a cut-edge, then $\eta(G) = 1$.*

Proof. Since G is not an edgeless graph, then for any $S \subseteq E(G)$ with $\omega(G - S) > \omega(G)$, we must have $\omega(G - S) - \omega(G) \leqslant |S|$ and so

$$\frac{|S|}{\omega(G - S) - \omega(G)} \geqslant 1.$$

It follows that $\eta(G) \geqslant 1$. On the other hand, let e be a cut edge of G. Then

$$\eta(G) \leqslant \frac{|\{e\}|}{\omega(G - e) - \omega(G)} = \frac{1}{1} = 1,$$

and so $\eta(G) = 1$ follows. □

3. Main result

In this paper, we shall prove the following main result.

Theorem 3.1. Let G be a graph on $n \geqslant 4$ vertices. Then

$$\frac{n}{2} \leqslant \gamma(G) + \gamma(G^c) \leqslant \begin{cases} \dfrac{5n}{8} + \dfrac{n}{8(n-1)} & \text{if } n \text{ is even,} \\ \dfrac{5n}{2} + \dfrac{1}{8} & \text{if } n \text{ is odd,} \end{cases} \tag{3}$$

$$2 \leqslant \eta(G) + \eta(G^c) \leqslant \begin{cases} \dfrac{5n}{8} + \dfrac{n}{8(n-1)} & \text{if } n \text{ is even,} \\ \dfrac{5n}{8} + \dfrac{1}{8} & \text{if } n \text{ is odd,} \end{cases} \tag{4}$$

$$0 \leqslant \gamma(G)\gamma(G^c) \leqslant \frac{n(n-1)d - d^2(d-1)}{4(n-1)} = f(d), \tag{5}$$

$$0 \leqslant \eta(G)\eta(G^c) \leqslant \frac{n(n-1)d - d^2(d-1)}{4(n-1)} = f(d), \tag{6}$$

where d is defined as in the Section 1. All the bounds in Theorem 3.1 are best possible in the following sense.

Example 1. Let $H_1 = K_n$. By Lemma 2.6, $\gamma(H_1) + \gamma(H_1^c) = g(H_1) + g(H_1^c) = n/2$, and so the lower bound of (3) can be attained.

Example 2. Let H_2 be a graph of order n such that

$$H_2 = \begin{cases} K_{\frac{n}{2}} \cup K_{\frac{n}{2}}^c, & n \text{ even}, \\ K_{\frac{n-1}{2}} \cup K_{\frac{n+1}{2}}^c, & n \text{ odd}. \end{cases}$$

Then by Lemmas 2.5, 2.6 and Corollary 2.8, $\gamma(H_2) + \gamma(H_2^c) = \eta(H_2) + \eta(H_2^c)$ reaches the upper bounds of (3) and (4).

Example 3. When $n = 4$, we define $H_3 = K_4$. When $n \geqslant 5$, we define H_3 as follows. Let u_1, u_2, u_3 and u_4 be distinct vertices. Let P be a path with $V(P) = \{u_1, u_2, u_3, u_4\}$ and $E(P) = \{u_1u_2, u_2u_3, u_3u_4\}$. Let $Q = K_{n-4}$ be a complete graph on $n - 4$ vertices such that $V(Q) \cap V(P) = \phi$. Then we define H_3 by

$$V(H_3) = V(P) \cup V(Q),$$

$$E(H_3) = E(P) \cup E(Q) \cup \{u_2v \mid v \in V(Q)\} \cup \{u_4v \mid v \in V(Q)\}.$$

Note that when $n \geqslant 5$ both H_3 and H_3^c have cut edges u_1u_2 and u_2u_4, respectively. Hence by Lemma 2.9, $\eta(H_3) = 1$ and $\eta(H_3^c) = 1$. Since $\eta(K_4) = 2$ and $\eta(K_4^c) = 0$, $\eta(H_3) + \eta(H_3^c) = 2$, for $n \geqslant 4$. Therefore, the lower bound of (4) can be attained.

Example 4. Let $H_4 = K_n$. Then both $\gamma(H_4^c) = 0$ and $\eta(H_4^c) = 0$, and so $\gamma(H_4)\gamma(H_4^c) = \eta(H_4)\eta(H_4^c) = 0$. Therefore, the lower bound of both (5) and (6) can be reached.

Example 5. Let $H_5 = K_d \cup K_d^c$. Then by Lemmas 2.5–2.7 and Corollary 2.8,

$$\gamma(H_5)\gamma(H_5^c) = \eta(H_5)\eta(H_5^c) = \frac{n(n-1)d - d^2(d-1)}{4(n-1)},$$

and so the upper bounds of both (5) and (6) can be reached.

4. Proof of the main result

By Lemma 2.1, among all the graph G on n vertices with

$$\gamma(G) + \gamma(G^c) \quad \text{maximized,} \tag{7a}$$

we can choose one such that

$$G^c \text{ is connected,} \tag{8a}$$

and subject to (7a) and (8a), such that

$$|E(G^c)| \text{ is maximized.} \tag{9a}$$

Lemma 4.1. G^c is uniformly dense.

Proof. Suppose that G^c is not uniformly dense. By Lemma 2.3, there exists a proper induced subgraph H of G^c such that $g(H) = \gamma(G^c)$. Let $e \in E(G^c) - E(H)$. Consider $G_1 = G + e$ and $G_1^c = G^c - e$. Since $H \subseteq G_1^c$, $g(H) \leqslant \gamma(G_1^c) \leqslant \gamma(G^c) = g(H)$, and so

$$\gamma(G_1^c) = \gamma(G^c). \tag{10}$$

Since G is a spanning subgraph of G_1, any subgraph of G is also a subgraph of G_1. Therefore,

$$\gamma(G_1) \geqslant \gamma(G). \tag{11}$$

By (10) and (11),

$$\gamma(G_1) + \gamma(G_1^c) \geqslant \gamma(G) + \gamma(G^c),$$

contrary to (9a). Therefore, G^c is uniformly dense. □

Lemma 4.2. *G is not a connected uniformly dense graph, and G is the disjoint union $H \cup K_m^c$ with $g(H) = \gamma(G)$, where $m = |V(G)| - |V(H)|$ is a positive integer.*

Proof. Suppose G is connected and uniformly dense. By Lemma 4.1, G_1^c is uniformly dense. Then

$$\gamma(G) + \gamma(G^c) = g(G) + g(G^c) = \frac{|E(G)|}{n-1} + \frac{\binom{n}{2} - |E(G)|}{n-1} = \frac{n}{2}$$

contrary to (7a) by Example 2. Thus G is not a connected uniformly dense graph. Therefore, G is either not uniformly dense, or uniformly dense with $\omega(G) \geqslant 2$. It follows by Lemma 2.3 in the former case and by Lemma 2.4 in the latter case that G has a proper uniformly dense subgraph H with $g(H) = \gamma(G)$. If $E(G) - E(H) \neq \phi$, then there exists $e \in E(G) - E(H)$. Denote $G_1 = G - e$ and $G_1^c = G^c + e$. Since $H \subseteq G - e = G_1$, $g(H) \leqslant \gamma(G_1) \leqslant \gamma(G) = g(H)$ and so

$$\gamma(G_1) = \gamma(G). \tag{12}$$

By Lemma 4.1, $g(G^c) = \gamma(G^c)$ and so by (8a)

$$\gamma(G_1^c) \geqslant g(G_1^c) = \frac{|E(G^c)| + 1}{|V(G)| - 1} > \frac{|E(G^c)|}{|V(G)| - 1} > g(G^c) = \gamma(G^c). \tag{13}$$

Since G_1^c is also connected, it follows by (12) and (13) that $\gamma(G_1)+\gamma(G_1^c)>\gamma(G)+\gamma(G^c)$, contrary to (7a). Therefore, $E(G)-E(H)=\phi$, and so $G=H\cup K_m^c$. This proves the conclusion of Lemma 4.2. □

Lemma 4.3. H is a complete graph.

Proof. By Lemma 4.1, G^c is uniformly dense. By Lemma 4.2, $g(H)=\gamma(G)$. By (8), G^c is connected. All these imply the following:

$$\begin{aligned}\gamma(G)+\gamma(G^c)&=\frac{|E(H)|}{|V(H)|-1}+\frac{\binom{n}{2}-|E(H)|}{n-1}\\&=\frac{\binom{n}{2}}{n-1}+\left(\frac{1}{|V(H)|-1}-\frac{1}{n-1}\right)|E(H)|.\end{aligned}\tag{14}$$

Since $|V(H)|<n$,

$$\frac{1}{|V(H)|-1}-\frac{1}{n-1}>0.$$

It follows by (14) that $\gamma(G)+\gamma(G^c)$ is an increasing function on $|E(H)|$ for fixed $|V(H)|$. Note that by Corollary 2.8, when $H=K_{n-m}$, G^c is also uniformly dense. Therefore, by (7a), H must be a complete graph. □

Denote $m=n-|V(G)|$. By Lemmas 4.1–4.3, we have $|V(H)|=n-m$ and

$$\begin{aligned}\gamma(G)+\gamma(G^c)&=\frac{|E(H)|}{|V(H)|-1}+\frac{\binom{n}{2}-|E(H)|}{n-1}\\&=\frac{\binom{n-m}{2}}{n-m-1}+\frac{\binom{n}{2}-\binom{n-m}{2}}{n-1}\\&=\frac{n^2-n+mn-m^2}{2(n-1)}.\end{aligned}\tag{15}$$

The right end of (15) is a quadratic polynomial of m for fixed n, so we use standard techniques to obtain its maximum value. If n is odd, then $\gamma(G)+\gamma(G^c)$ has a maximum value

$$\frac{5n}{8}+\frac{1}{8}\quad\text{at } m=\frac{n-1}{2},$$

i.e.

$$\gamma(G) + \gamma(G^c) \leqslant \frac{5n}{8} + \frac{1}{8}.$$

If n is even, then $\gamma(G) + \gamma(G^c)$ has a maximum value

$$\frac{5n}{8} + \frac{n}{8(n-1)} \quad \text{at } m = \frac{n}{2},$$

i.e.

$$\gamma(G) + \gamma(G^c) \leqslant \frac{5n}{8} + \frac{n}{8(n-1)}.$$

By (1), $\eta(G) \leqslant \gamma(G)$, $\eta(G^c) \leqslant \gamma(G^c)$, and so we have proved the upper bounds of both (3) and (4).

By (1),

$$\gamma(G) \geqslant g(G) = \frac{|E(G)|}{n - \omega(G)} \geqslant \frac{|E(G)|}{n-1} \tag{16}$$

and

$$\gamma(G^c) \geqslant g(G^c) = \frac{\binom{n}{2} - |E(G)|}{n - \omega(G^c)} \geqslant \frac{\binom{n}{2} - |E(G)|}{n-1}. \tag{17}$$

Combine (16) and (17) to get

$$\gamma(G) + \gamma(G^c) \geqslant \frac{|E(G)|}{n-1} + \frac{\binom{n}{2} - |E(G)|}{n-1} = \frac{n}{2}$$

and so the lower bound of (3) is obtained.

To see the lower bound of (4) is valid, it suffices to note that if $E(G) \neq \phi$ and $E(G^c) \neq \phi$, then by Lemma 2.9, $\eta(G) + \eta(G^c) \geqslant 2$. When either $E(G) = \phi$ or $E(G^c) = \phi$, Example 1 in Section 3 indicates that $\eta(G) + \eta(G^c) = n/2 \geqslant 2$. Therefore, the lower bound of (4) holds in both cases.

To prove (5) and (6), we argue similarly. Thus, among all the graph G of $n \geqslant 4$ vertices with

$$\gamma(G)\gamma(G^c) \quad \text{maximized.} \tag{7b}$$

we can choose one so that

$$G^c \text{ is connected,} \tag{8b}$$

and subject to (7b) and (8b),

$$|E(G^c)| \text{ is maximized.} \tag{9b}$$

With fairly similar arguments in the proofs of Lemmas 4.1 and 4.2, one can easily prove the following two lemmas.

Lemma 4.4. *G^c is uniformly dense.*

Lemma 4.5. *G is not a connected uniformly dense graph and G has form $H \cup K_m^c$ and $g(H) = \gamma(G)$, where m is a positive integer.*

Lemma 4.6. *H is a complete graph.*

Proof. By Lemma 4.4, G^c is uniformly dense. Since G^c is connected,

$$\gamma(G)\gamma(G^c) = \frac{|E(H)|}{|V(H)|-1}\,\frac{\binom{n}{2} - |E(G)|}{n-1}$$

$$= \frac{x\left(\binom{n}{2} - x\right)}{(n-1)(y-1)}, \tag{18}$$

where $x = |E(H)|$, $y = |V(H)|$. Note that by Lemma 4.5, $|V(H)| < n$. We consider two cases.

Case 1: $x > \frac{1}{2}\binom{n}{2} = n(n-1)/4$.

We first claim that H is not a tree. By contradiction, we assume that H is a tree and so $x = y - 1$. Since $x > n(n-1)/4$, we have

$$4(y-1) > n(n-1). \tag{19}$$

If $4 \geqslant y$, then by (19), $4(4-1) \geqslant 4(y-1) > n(n-1)$, contrary to the assumption that $n \geqslant 4$. If $y > 4$, then by (19), $y(y-1) > n(n-1)$, contrary to the fact that $y = |V(H)| < n$. Therefore, H must not be a tree and so there is an edge $e \in E(G)$ suh that $H - e$ is also connected. It follows that $\omega(G-e) = \omega(G)$. Let $G_1 = G - e$. Note that $x = |E(H)| = |E(G)|$ and $y = |V(H)| = |V(G)| - \omega(G) + 1$. By (1),

$$\gamma(G_1) \geqslant g(G_1) = \frac{|E(G)|-1}{|V(G)|-\omega(G)} = \frac{|E(H)|-1}{|V(H)|-1} = \frac{x-1}{y-1}. \tag{20}$$

$$\gamma(G_1^c) \geqslant g(G_1^c) = \frac{|E(G^c)|+1}{|V(G)|-1} = \frac{\binom{n}{2} - (x-1)}{n-1}. \tag{21}$$

Since $x > 1/2\binom{n}{2}$, $\gamma(G)\gamma(G^c)$ is an decreasing function on x for fixed n and y. Note that x is positive integer. If $x - 1 \geqslant \frac{1}{2}\binom{n}{2}$, combine (20) and (21) to get

$$\gamma(G)\gamma(G^c) = \frac{\binom{n}{2}x - x^2}{(n-1)(y-1)} < \frac{\binom{n}{2}(x-1) - (x-1)^2}{(n-1)(y-1)} \leqslant \gamma(G_1)\gamma(G_1^c),$$

contrary to (7b).

If $x - 1 < \frac{1}{2}\binom{n}{2}$, then since $x > \frac{1}{2}\binom{n}{2}$ and x is an integer, we have $x - 1 = \frac{1}{2}\binom{n}{2} - \frac{1}{2}$, and $x = \frac{1}{2}\binom{n}{2} + \frac{1}{2}$. Combine (20) and (21) to get

$$\gamma(G)\gamma(G^c) = \frac{\binom{n}{2}x - x^2}{(n-1)(y-1)} = \frac{\binom{n}{2}(x-1) - (x-1)^2}{(n-1)(y-1)} \leqslant \gamma(G_1)\gamma(G_1^c),$$

contrary to (9b).

Case 2: $x \leqslant \frac{1}{2}\binom{n}{2}$.

For each fixed y, the right end of (18) is an increasing function of x and so by (7b) and by Corollary 2.8, we must have $x = \binom{y}{2}$ and H must be a complete subgraph of G. □

By Lemmas 4.4–4.6, and by $x = |E(H)| = \binom{y}{2}$,

$$f(y) = \gamma(G)\gamma(G^c) = \frac{|E(H)|}{|V(H)| - 1}\,\frac{\binom{n}{2} - |E(H)|}{n-1}$$

$$= \frac{x\left(\binom{n}{2} - x\right)}{(n-1)(y-1)} = \frac{n(n-1)y - y^2(y-1)}{4(n-1)}.$$

Note that the only positive critical point of $f(y)$ is

$$y_0 = \frac{1 + \sqrt{1 + 3n(n-1)}}{3}.$$

When $y \in (0, (1 + \sqrt{1 + 3n(n-1)})/3)$, $f'(y) > 0$. When $y \in ((1 + \sqrt{1 + 3n(n-1)})/3, n)$, $f'(y) < 0$. Therefore, when $y = d$, $\gamma(G)\gamma(G^c)$ has the maximum value $[n(n-1)d - d^2(d-1)]/4(n-1)$. By (1), $\eta(G) \leqslant \gamma(G)$, $\eta(G^c) \leqslant \gamma(G^c)$, so

$$\eta(G)\eta(G^c) \leqslant \gamma(G)\gamma(G^c) \leqslant \frac{n(n-1)d - d^2(d-1)}{4(n-1)}.$$

This proves the upper bounds of (5) and (6). The lower bounds of (5) and (6) are trivial. This completes the proof of Theorem 3.1. □

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