



Note

Large survivable nets and the generalized prisms

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Abstract

For a graph G with vertices labeled $1, 2, \dots, n$ and a permutation α in S_n , the α -generalized prism over G , $\alpha(G)$, consists of two copies of G , say G_x and G_y , along with the edges $(x_i, y_{\alpha(i)})$, for $1 \leq i \leq n$. In [Discrete Appl. Math. 30 (1991) 229–233], the importance of building large graphs by using generalized prisms is indicated, and the connectivity of the generalized prisms is discussed. Let $f(G)$ denote a graphical measure of G , and let $\bar{f}(G)$ denote the maximum value of $f(H)$ taken over all subgraphs H of G . When f is a vulnerability measure, networks G with $f(G) = \bar{f}(G)$ would usually be regarded as *survivable*, for a knowledgeable enemy would find no especially attractive targets. In this note, we investigate sufficient conditions for $f(\alpha(G)) = \bar{f}(\alpha(G))$, for any $\alpha \in S_{|V(G)|}$, where f is the connectivity, edge-connectivity, or the minimum degree. As a result, we obtain a method to produce large survivable networks by repeatedly taking generalized prisms, and extend some results in [Discrete Appl. Math. 30 (1991) 229–233]. Related polynomial algorithms are discussed.

1. Introduction

Graphs in this note are simple and finite. We follow the notation of Bondy and Murty [1] unless otherwise stated. Let S_n denote the permutation group of degree n . Let G be a graph with vertices labeled $1, 2, \dots, n$, and let α be a permutation in S_n . The α -generalized prism over G , $\alpha(G)$, consists of two copies of G , say G_x and G_y , along with the edges $(x_i, y_{\alpha(i)})$, for $1 \leq i \leq n$. Prior results on the generalized prisms can be found in [2, 3, 12–14], among others. As in [1], the connectivity, the edge-connectivity and the minimum degree of G are denoted by $\kappa(G)$, $\kappa'(G)$ and $\delta(G)$, respectively. In [11], Piazza and Ringeisen proved the following.

Theorem 1.1 (Piazza and Ringeisen [12]). *For a connected graph G with n vertices. If $\kappa(G) = \delta(G)$, then for any $\alpha \in S_n$, $\kappa(\alpha(G)) = \delta(\alpha(G))$.*

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Theorem 1.2 (Piazza and Ringeisen [12]). *If G is a nontrivial tree, or an n -cycle, or the wheel W_n , or the n -cube Q_n , or the complete graph K_n , or the complete bipartite graph $K_{m,n}$ with $n \geq m$, then for any $\alpha \in S_{|V(G)|}$, $\kappa(\alpha(G)) = \kappa'(\alpha(G)) = \delta(G) + 1$.*

The edge-connectivity analog of the results above is obtained by Piazza in [11].

Let $f(G)$ be a graphical function and define $\bar{f}(G)$ to be the maximum value of $f(H)$ taken over all subgraphs H of G . Prior results on $\bar{\kappa}$, $\bar{\kappa}'$ and $\bar{\delta}$ can be found in [5–10], among others. When f is a vulnerability measure of a network G , G can be viewed as *survivable* if $f(G) = \bar{f}(G)$, for a knowledgeable enemy would find no especially attractive targets [4]. In this note, we consider results of the form that if G has property P , then for any $\alpha \in S_{|V(G)|}$, $\alpha(G)$ also has property P , and investigate sufficient conditions for $\kappa(\alpha(G)) = \bar{\kappa}(\alpha(G))$ or $\kappa'(\alpha(G)) = \bar{\kappa}'(\alpha(G))$, for any $\alpha \in S_{|V(G)|}$.

2. Main results

We start with a result which extends Theorem 3.1 of [12].

Theorem 2.1. *Let G be a connected graph. Then both of the following hold:*

- (i) $\min\{2\kappa(G), \delta(G) + 1\} \leq \kappa(\alpha(G)) \leq \bar{\kappa}(\alpha(G)) \leq \bar{\delta}(G) + 1$.
- (ii) $\min\{2\kappa'(G), \delta(G) + 1\} \leq \kappa'(\alpha(G)) \leq \bar{\kappa}'(\alpha(G)) \leq \bar{\delta}(G) + 1$.

Proof. Let $U(G)$ denote the minimum value of $|S| + |V(C)|$ taken over all vertex-cuts S of G and all nonempty components C of $G - S$. In [12], it is shown that

$$\min\{2\kappa(G), U(G)\} \leq \kappa(\alpha(G)) \leq U(G). \quad (1)$$

To show Theorem 2.1(i), we first prove the following:

$$U(G) = \delta(G) + 1. \quad (2)$$

Let v be a vertex of G with $d(v) = \delta(G)$, and let S be the set of vertices adjacent to v in G . Then by the definition of $U(G)$, we have $U(G) \leq |\{v\}| + |S| = \delta(G) + 1$. Conversely, let $S \subseteq V(G)$ be a vertex-cut and let C be a component of $G - S$ such that $U(G) = |S| + |V(C)|$. Since $V(C) \neq \emptyset$, one can choose $v \in V(C)$. Since the vertices in G adjacent to v is a subset of $(V(C) - \{v\}) \cup S$, we have $\delta(G) \leq d(v) \leq |S| + |V(C) - \{v\}| = U(G) - 1$, and so (2) holds.

By (1) and (2), we have $\min\{2\kappa(G), \delta(G) + 1\} \leq \kappa(\alpha(G))$ for all $\alpha \in S_n$. What left for Theorem 2.1(i) is to show $\bar{\kappa}(\alpha(G)) \leq \bar{\delta}(G) + 1$. Let G_x and G_y be two copies of G such that $V(\alpha(G)) = V(G_x) \cup V(G_y)$. Let H be a subgraph of $\alpha(G)$ with $\bar{\kappa}(\alpha(G)) = \kappa(H)$, and let $H_x = H \cap G_x$ and $H_y = G_y \cap H$.

If both H_x and H_y are isomorphic to K_1 , then $H \cong K_2$ and so we must have $G = K_1$ and $\alpha(G) = K_2$. In this case, Theorem 2.1(i) follows trivially. Hence we may assume that $|V(H_x)| \geq 2$. It follows that $\bar{\kappa}(\alpha(G)) = \kappa(H) \leq \delta(H) \leq \delta(H_x) + 1 \leq \bar{\delta}(G) + 1$, and so the proof of Theorem 2.1(i) is completed. The proof of Theorem 2.1(ii) is similar. \square

By Theorem 2.1, Theorem 4.2 of [12] can be extended to the following.

Corollary 2.2. *Let G be a connected graph with n vertices. Then each of the following holds:*

- (i) $\kappa(\alpha(G)) = \delta(\alpha(G))$, for any $\alpha \in S_n$ if and only if $2\kappa(G) \geq \delta(G) + 1$.
- (ii) $\kappa'(\alpha(G)) = \delta(\alpha(G))$, for any $\alpha \in S_n$ if and only if $2\kappa'(G) \geq \delta(G) + 1$.

Proof. Note that we always have $\kappa(\alpha(G)) \leq \delta(\alpha(G)) = \delta(G) + 1$, and so if $2\kappa(G) \geq \delta(G) + 1$, then by Theorem 2.1(i), $\kappa(\alpha(G)) = \delta(G) + 1$. Conversely, we suppose that $2\kappa(G) < \delta(G) + 1$. Then for α being the identity map in S_n , we have $\kappa(\alpha(G)) = 2\kappa(G) < \delta(G) + 1 = \delta(\alpha(G))$. This proves Corollary 2.2(i). The proof for Corollary 2.2(ii) is similar. \square

Corollary 2.3. *Let G be a connected graph with n vertices. Each of the following holds:*

- (i) If $\kappa(G) = \bar{\delta}(G)$, then for any $\alpha \in S_n$, $\kappa(\alpha(G)) = \bar{\delta}(\alpha(G))$.
- (ii) If $\kappa'(G) = \bar{\delta}(G)$, then for any $\alpha \in S_n$, $\kappa'(\alpha(G)) = \bar{\delta}(\alpha(G))$.

Proof. Since G is connected, $\kappa(G) = \delta(G) = \bar{\delta}(G) \geq 1$. Thus

$$2\kappa(G) \geq \kappa(G) + 1 = \delta(G) + 1 = \bar{\delta}(G) + 1 = \bar{\delta}(\alpha(G)),$$

and so Corollary 2.3(i) follows from Corollary 2.2(i). The proof for Corollary 2.3(ii) is similar. \square

By Corollary 2.3, Theorem 1.2 can be extended to the following:

Corollary 2.4. *If G is a nontrivial tree, or an n -cycle, or the wheel W_n , or the n -cube Q_n , or the complete graph K_n , or the complete bipartite graph $K_{m,n}$ with $n \geq m$, then for any $\alpha \in S_{|V(G)|}$,*

$$\kappa(\alpha(G)) = \bar{\kappa}(\alpha(G)) = \kappa'(\alpha(G)) = \bar{\kappa}'(\alpha(G)) = \delta(\alpha(G)) = \bar{\delta}(\alpha(G)) = \delta(G) + 1.$$

Proof. By [6] or by [9], it is easy to see that for any G in the list of Corollary 2.4, we have $\bar{\delta}(G) = \kappa(G)$ and so Corollary 2.4 follows from Corollary 2.3. \square

Part (ii) of Corollary 2.3 can be improved to the following result.

Theorem 2.5. *Let G be a connected graph with n vertices. If $\kappa'(G) = \bar{\kappa}'(G)$ and $\delta(G) = \bar{\delta}(G)$, then for any $\alpha \in S_n$, we have both $\kappa'(\alpha(G)) = \bar{\kappa}'(\alpha(G))$ and $\delta(\alpha(G)) = \bar{\delta}(\alpha(G))$.*

The following theorems by Mader will be needed in the proof of Theorem 2.5.

Theorem 2.6 (Mader [7]). *Let G be a simple graph with n vertices and let $k \geq 1$ be an integer. If*

$$|E(G)| > (n - k)k + \frac{k(k - 1)}{2},$$

then $\bar{\kappa}'(G) > k$.

Proof of Theorem 2.5. By Theorem 2.1(ii), it suffices to show that $2\kappa'(G) \geq \delta(G) + 1$. By contradiction, we assume that

$$2\kappa'(G) \leq \delta(G). \quad (3)$$

It follows by (3) that $2|E(G)| \geq n\delta(G) \geq n(2\kappa'(G))$ and so

$$|E(G)| \geq n\kappa'(G) > (n - \kappa'(G))\kappa'(G) + \frac{\kappa'(G)(\kappa'(G) - 1)}{2}. \quad (4)$$

By Theorem 2.5, $\bar{\kappa}'(G) > \kappa'(G)$, contrary to the hypothesis that $\bar{\kappa}'(G) = \kappa'(G)$. This proves Theorem 2.5. \square

It is not known to us whether the connectivity analog of Theorem 2.5 would hold. We present the following examples to show that only one of the two conditions in Theorem 2.5 or in its connectivity analog may not be sufficient.

Example 1. The condition $\delta(G) = \bar{\delta}(G)$ alone cannot guarantee either $\kappa'(\alpha(G)) = \bar{\kappa}'(\alpha(G))$, or $\kappa(\alpha(G)) = \bar{\kappa}(\alpha(G))$ for all $\alpha \in S_n$, as indicated by the example below.

Let n and m be two integers with $n > 2m > 0$. Let H_1 and H_2 be two vertex-disjoint complete graphs each of which is isomorphic to K_{n+1} . Let $G(n, m)$ denote the graph obtained from the disjoint union of H_1 and H_2 by adding m new edges e_1, e_2, \dots, e_m that join m distinct vertices x_1, x_2, \dots, x_m in H_1 to m distinct vertices y_1, y_2, \dots, y_m in H_2 . Then we have $\kappa(G(n, m)) = \kappa'(G(n, m)) = m$ and $\delta(G) = \bar{\delta}(G) = n$. For α being the identity map, $\kappa(\alpha(G)) = \kappa'(\alpha(G)) = 2m$ and $\bar{\kappa}(\alpha(G)) = \bar{\kappa}'(\alpha(G)) = n + 1$.

Example 2. The condition $\kappa'(G) = \bar{\kappa}'(G)$ alone cannot guarantee $\kappa'(\alpha(G)) = \bar{\kappa}'(\alpha(G))$, for all $\alpha \in S_n$, as indicated in the example below. The graph in Fig. 1 also shows that $\kappa(G) = \bar{\kappa}(G)$ alone cannot guarantee $\kappa(\alpha(G)) = \bar{\kappa}(\alpha(G))$, for all $\alpha \in S_n$.

Let G be the graph in Fig. 1. Then we have $\kappa(G) = \bar{\kappa}(G) = 2$. For α being the identity map, we have $\kappa(\alpha(G)) = \kappa'(\alpha(G)) = 3$ and $\bar{\kappa}(\alpha(G)) = \bar{\kappa}'(\alpha(G)) = 4$.

Note that this graph G is obtained from a graph described in [5] (denoted by $K_2(2; 2, 2)$ in Example 2 of [5]) by adding a path of length 3. One can easily obtain an infinite family of graphs with similar properties.

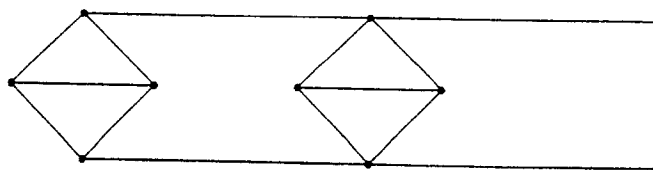


Fig. 1. The graph in Example 2.

3. Applications and algorithms

As an application of Corollary 2.3 and Theorem 2.5, one can start with any graph G with $\kappa(G) = \bar{\delta}(G)$, or with $\kappa'(G) = \bar{\delta}(G)$, or with $\kappa'(G) = \bar{\kappa}'(G)$ and $\delta(G) = \bar{\delta}(G)$, to construct large survivable networks by repeatedly taking generalized prisms.

For practical reasons, one may expect to check the hypotheses of Corollary 2.3 or Theorem 2.5, for any graph G . In the rest, we shall describe some polynomial algorithms that will check the hypotheses of Theorem 2.5, for any graph G .

In [10], Matula presented an algorithm that computes $\kappa'(G)$ in $O(nm)$ time, for any graph G with $n = |V(G)|$ and $m = |E(G)|$. Matula also presented an algorithm to determine $\bar{\kappa}'(G)$ in $O(n^2m)$ time. Since $m = O(n^2)$ in general, direct applying Matula's algorithm for $\bar{\kappa}'(G)$ to check the hypotheses of Theorem 2.4 may require $O(n^4)$ time. However, by Theorem 2.6, one can check the inequality in Theorem 2.6 first, and so it will take only $O(n^3)$ time to determine if $\kappa'(G) = \bar{\kappa}'(G)$. In order to check if $\delta(G) = \bar{\delta}(G)$, one can successively delete vertices of degree less than or equal to $\delta(G)$ from G (if the process stops with an empty graph, then $\delta(G) = \bar{\delta}(G)$, otherwise $\delta(G) < \bar{\delta}(G)$), and so this can be done in $O(n^2)$ time. Therefore, the hypotheses of Theorem 2.5 can be checked in $O(n^3)$ time.

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