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Every 4-Connected Line Graph of a Planar Graph is Hamiltonian

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Abstract. Let G be a graph with $E(G) \neq \varnothing$. The line graph of G, written $L(G)$, has $E(G)$ as its vertex set, where two vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. Thomassen conjectured that all 4-connected line graphs are hamiltonian [2]. We show that this conjecture holds for planar graphs.

1. Introduction

We follow the notation of [1] except otherwise noted. Graphs in this note are finite and simple. Let G be a graph. The line graph of G, written $L(G)$, has $E(G)$ as its vertex set, where two vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. In [2], Thomassen poses the following conjecture.

Conjecture 1 (Thomassen [2]). *If $L(G)$ is 4-connected, then $L(G)$ is hamiltonian.*

The following theorem was proved by Zhan [4] and, independently, by B. Jackson [private communication].

Theorem 1.1 (Zhan [4]). *If $L(G)$ is 7-connected, then $L(G)$ is hamiltonian.*

In [4], Zhan proves a stronger result that if $L(G)$ is 7-connected, then $L(G)$ is hamiltonian connected.

For planar graphs, Tutte has a well known result:

Theorem 1.2 (Tutte [3]). *Every 4-connected planar graph is hamiltonian.*

In general, the line graph of a planar graph may not be planar. In this note, we shall prove Conjecture 1 for planar graphs.

Theorem 1.3. *Let G be a planar simple graph. If $L(G)$ is 4-connected, then $L(G)$ is hamiltonian.*

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The proof of Theorem 1.3 will appear in Section 3. In Section 2, for each 2-connected simple planar graph G, we construct a new planar graph $W(G)$ so that if $W(G)$ is hamiltonian, then $L(G)$ is also hamiltonian. We shall show that if $L(G)$ is 4-connected, then $W(G)$ is also 4-connected. Thus by Theorem 1.2, $W(G)$ is hamiltonian, and so $L(G)$ is hamiltonian. In Section 3, we shall show that the 2-connectedness imposed in Section 2 can be lifted.

2. A Special Case

Let G be a simple planar graph. We consider the case when G is a simple 2-connected planar graph. For each vertex $u \in V(G)$, denote

$$E_G(u) = \{e \in E(G): e \text{ is incident with } u \text{ in } G\}.$$

When there is no confusion, we use $E(u)$ for $E_G(u)$. We assume that G is embedded in the plane. Define a new graph $W(G)$ using the following 5 steps:

(W1) Replace each $e \in E(G)$ by a path P_e of length 2 with a new vertex $v(e)$ as the only internal vertex of P_e.

(W2) For each vertex $u \in V(G)$ with $d_G(u) = m \geq 3$, let $e_1, e_2, \ldots, e_m$ denote the edges in $E(u)$, where the subscripts indicate the cyclic ordering of the planar embedding of G. Add new edges $v(e_i)v(e_{i+1})$, for all $1 \leq i \leq m \pmod m$.

(Thus each u together with the new vertices $\{v(e): e \in E(u)\}$ will induce a wheel with a rim cycle of length $d_G(u)$. We call this wheel the *wheel associated with* u, and u the *center* of the wheel).

(W3) For each $u \in V(G)$ with $d_G(u) = 3$, delete the center of the wheel associated with u, (namely, the vertex u).

(W4) For each $u \in V(G)$ with $d_G(u) = 2$, join $v(e)$ and $v(e')$ and delete u, where e and e' are the edges of G incident with u in G.

(W5) For each $u \in V(G)$ with $d_G(u) = 1$, delete u.

The resulting graph will be denoted by $W(G)$. Fig. 1 shows some local correspondence between G and $W(G)$.

Lemma 2.1. *$V(W(G))$ is the disjoint union of $V(L(G))$ and $\{u \in V(G): d_G(u) \geq 4\}$.*

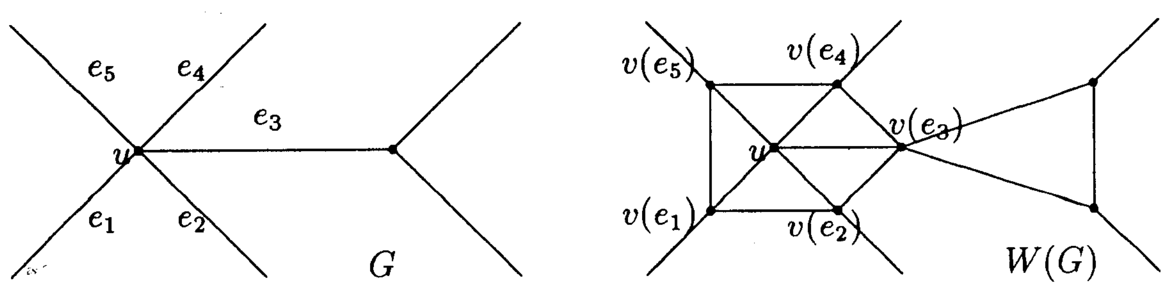

Fig. 1. The graphs G and $W(G)$

Proof. The vertices $v(e)$, $e \in E(G)$, introduced in (W1), constitute $V(L(G))$. By (W2), all vertices $u \in V(G)$ with $d_G(u) \geq 4$ are in $V(W(G))$. By (W3)–(W5), all other vertices of $V(G)$ are deleted. □

Lemma 2.2. *The line graph $L(G)$ can be obtained from $W(G)$ by the following operation: For each $u \in V(G)$ with $d_G(u) = m \geq 4$,*

(2A) *remove u from $W(G)$, and*
(2B) *add $m(m-1)/2$ new edges joining the vertices $\{v(e)\colon e \in E(u)\}$ so that $\{v(e)\colon e \in E(u)\}$ induces a complete subgraph K_m in the resulting graph.*

Proof. Let W' denote the resulting graph obtained in Lemma 2.2. By Lemma 2.1 and by (2A) of Lemma 2.2,

$$V(W') = V(W(G)) - V(G) = V(L(G)). \tag{1}$$

Therefore it suffices to show that $E(W') = E(L(G))$.

Assume first that $v(e)v(e') \in E(L(G))$. Then there is a $u \in V(G)$ such that e, $e' \in E(u)$. If $d_G(u) \geq 3$, then $v(e)v(e') \in E(W')$ by (W2). If $d_G(u) = 2$, then $v(e)v(e') \in E(W')$ by (W4). In any case, $v(e)v(e') \in E(W')$, and so $E(L(G)) \subseteq E(W')$.

Conversely, we assume that $v(e)v(e') \in E(W')$. If $v(e)v(e')$ is an edge newly added in (2B) of Lemma 2.2, then e, $e' \in E(u)$ for some $u \in V(G)$ with $d_G(u) \geq 4$, and so $v(e)v(e') \in E(L(G))$, by the definition of $L(G)$. Thus we assume that e and e' are not incident with a vertex $u \in V(G)$ with $d_G(u) \geq 4$. It follows by (W2)–(W4) that e, e' are both incident with a vertex $v \in V(G)$ with $2 \leq d_G(v) \leq 3$, and so $v(e)v(e') \in E(L(G))$. Therefore $E(W') \subseteq E(L(G))$, and so $E(W') = E(L(G))$. This proves Lemma 2.2. □

Lemma 2.3. *$W(G)$ is planar.*

Proof. This follows from the facts that G is a plane graph and that the operations (W1)–(W5) preserve planarity. □

Lemma 2.4. *If G is 2-connected and if $L(G)$ is 4-connected, then $W(G)$ is 4-connected.*

Proof. Suppose that $R \subset V(W(G))$ is a vertex cut with $|R| \leq 3$. If R does not contain a vertex in $\{u \in V(G)\colon d_G(u) \geq 4\}$, then by Lemma 2.1, $R \subseteq V(L(G))$ and so by Lemma 2.2, R is a vertex cut of $L(G)$, contrary to the assumption that $L(G)$ is 4-connected. Therefore, we assume that R contains a vertex $u \in V(G)$ with $d_G(u) \geq 4$. Since in $W(G)$, u is the center of the wheel associated with u, R must also contain two vertices $v(e)$, $v(e')$ with e, $e' \in E(u)$. This implies that u is a cut vertex of G, contrary to the assumption that G is 2-connected. This proves Lemma 2.4 □

Lemma 2.5. *If $W(G)$ has a Hamilton cycle, then $L(G)$ has a Hamilton cycle C such that for every vertex $u \in V(G)$ with $d_G(u) \geq 4$, there are two edges e, $e' \in E(u)$ that are consecutive vertices in C.*

Proof. Suppose that $W(G)$ has a Hamilton cycle C'. Then by Lemma 2.1, each $u \in V(G)$ with $d_G(u) \geq 4$ must be in $V(C')$. Since in $W(G)$, u is the center of the wheel associated with u, there must be two members $e, e' \in E(u)$ such that $v(e)$ and u are consecutive vertices in C' and such that u and $v(e')$ are consecutive vertices in C'. By Lemma 2.2, $v(e)$ and $v(e')$ are adjacent in $L(G)$, and so a Hamilton cycle of $L(G)$ can be obtained from C' by replacing the section $v(e)uv(e')$ in C' by an edge $v(e)v(e')$, for each $u \in V(G) \cap V(W(G))$. This proves Lemma 2.5. □

Theorem 2.6. *Let G be a simple 2-connected planar graph. If $L(G)$ is 4-connected, then $L(G)$ has a Hamilton cycle C such that for every vertex $u \in V(G)$ with $d_G(u) \geq 4$, there are two edges $e, e' \in E(u)$ that are consecutive vertices in C.*

Proof. By Lemma 2.3, $W(G)$ is planar. By Lemma 2.4, $W(G)$ is 4-connected. Therefore by Theorem 1.2, $W(G)$ has a Hamilton cycle, and so Theorem 2.6 now follows from Lemma 2.5. □

3. Proof of Theorem 1.3

In this section, the 2-connectedness imposed in Theorem 2.6 will be lifted. If v is a cut vertex of G, then the components of $G - v$ will have vertex sets $V_1, V_2, \ldots, V_c$. Each subgraph $G[V_i \cup \{v\}]$ is called a *v-component* of G, (see page 119 of [1]).

We shall prove a slightly stronger version of Theorem 1.3:

Theorem 3.1. *Let G be a simple planar graph with $\kappa(L(G)) \geq 4$. Then $L(G)$ has a Hamilton cycle C such that for every vertex $v \in V(G)$ with $d_G(v) \geq 4$, if v is not a cut vertex of G, then there are two distinct edges $e, e' \in E(v)$ that are consecutive vertices of C in $L(G)$.*

Proof. We argue by induction on the number of cut vertices of G. If G does not have a cut vertex, then $\kappa(G) \geq 2$, and so Theorem 3.1 follows from Theorem 2.6. Therefore we assume that G has k cut vertices with $k \geq 1$, and that Theorem 3.1 holds for graphs with at most $k - 1$ cut vertices.

Let v be a cut vertex of G, and let G_i, $1 \leq i \leq c$, be the v-components of G. Since $L(G)$ is 4-connected, for each i, $1 \leq i \leq c$,

$$\text{either } G_i \cong K_2, \qquad \text{or } v \text{ has degree at least 4 in } G_i \tag{2}$$

By (2), by the definition of v-components of G, and by $\kappa(L(G)) \geq 4$, we have $\kappa(L(G_i)) \geq 4$ for each G_i where $G_i \ncong K_2$.

Therefore by (2) and by induction, for each G_i where $G_i \ncong K_2$, $L(G_i)$ has a Hamilton cycle C_i satisfying the conclusion of Theorem 3.1. Since G_i is a v-component of G, v is not a cut vertex of G_i, $(1 \leq i \leq c)$. Therefore, for each $G_i \ncong K_2$, the v-component G_i has two distinct edges $e_i', e_i'' \in E_{G_i}(v)$ with $v(e_i')$ and $v(e_i'')$ being consecutive vertices in C_i. Let $v(e')P_iv(e'') = C_i - v(e_i')v(e_i'')$ denote the Hamilton $(v(e_i'), v(e_i''))$-path in $L(G_i)$.

For each G_j where $G_j \cong K_2$, let e_j denote the only edge in $E(G_j)$, let $v(e_j') = v(e_j'') = v(e_j)$, and for convenience regard the notation $v(e_j')P_jv(e_j'')$ as the single vertex path $v(e_j)$ (only when $G_j \cong K_2$).

Recall that for all $1 \leq i \leq c$, e_i', e_i'' are in $E_G(v)$, and so in $L(G)$, all the $v(e_i')$'s and the $v(e_j'')$'s are vertices in a complete subgraph of $L(G)$. Thus one can glue these paths and vertices together as follows

$$v(e_1')P_1v(e_1'')v(e_2')P_2v(e_2'')v(e_3')\cdots v(e_c')P_cv(e_c'')v(e_1')$$

and so a Hamilton cycle of $L(G)$ that satisfies the conclusion of Theorem 3.1 is obtained. Theorem 3.1 is now proved by induction. □

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