Collapsible Graphs and Matchings

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ABSTRACT

A graph G is *collapsible* if for every even subset $R \subseteq V(G)$, there is a spanning connected subgraph of G whose set of odd degree vertices is R. A graph is *reduced* if it does not have nontrivial collapsible subgraphs. Collapsible and reduced graphs are defined and studied in [4]. In this article, we obtain a lower bound on the size of a maximum matching in a reduced graph. As an application, we verify and strengthen the Benhocine, Clark, Köhler, and Veldman conjecture [1], when restricted to 3-edge-connected graphs, by showing that for n large, a simple graph G with order n and with $\kappa'(G) \geq 3$ is collapsible or is contractible to the Petersen graph if for each edge $uv \in E(G)$, $d(u) + d(v) \geq (n/5) - 2$. We also characterize the extremal graphs. © 1993 John Wiley & Sons, Inc.

INTRODUCTION

We follow the notation of Bondy and Murty [3], except when otherwise stated. A graph may have multiple edges but not loops. For a graph G, $O(G)$ denotes the set of vertices of odd degree in G. If $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the ends of each edge in X and deleting the resulting loops. If H is a subgraph of G, we use G/H for $G/E(H)$. A vertex v in G/H is *nontrivial* if v is the contraction image of a nontrivial connected subgraph L of G; and the connected subgraph L of G is called the *preimage* of v. Throughout this note, we let P denote the Petersen graph.

A graph is *supereulerian* if it has a spanning eulerian subgraph. A graph G is *collapsible* if for every even subset $R \subseteq V(G)$, there is a spanning connected subgraph H_R of G with $O(H_R) = R$. Thus K_1 is

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both supereulerian and collapsible. Following Catlin [4], we use $\mathcal{CL}$ and $\mathcal{SL}$ to denote the families of collapsible graphs and supereulerian graphs, respectively. Obviously, $\mathcal{CL} \subset \mathcal{SL}$.

In [4], Catlin showed that every graph G has a unique collection of pairwise disjoint maximal collapsible subgraphs $H_1, H_2, \ldots, H_c$. The *reduction* of G is the contraction $G/(\cup_{i=1}^{c} E(H_i))$. A graph is *reduced* if it is the reduction of some other graph.

Theorem A (Catlin [4,5]). Let G be a graph and H be a collapsible subgraph of G.

(a) G is reduced if and only if G has no nontrivial collapsible subgraphs.
(b) G is collapsible if and only if G/H is collapsible.
(c) G is supereulerian if and only if G/H is supereulerian.
(d) If $G \notin \{K_1, K_2\}$ is reduced, then G is K_3-free with $\delta(G) \leq 3$ and

$$|E(G)| \leq 2|V(G)| - 4.$$

(e) If G has 2 edge-disjoint spanning trees, then $G \in \mathcal{CL}$. ∎

For graphs with few vertices, Chen proved:

Theorem B (Chen [7],[8]). If G is a simple graph with at most 11 vertices and with $\kappa'(G) \geq 3$, then either $G \in \mathcal{CL}$ or the reduction of G is isomorphic to the Petersen graph. ∎

A LOWER BOUND ON THE SIZE OF A MAXIMUM MATCHING

Let $M(G)$ denote a maximum matching of G. An *odd component* of G is one that has an odd number of vertices. Let $q(G)$ denote the number of odd components of G.

Theorem C (Berge [2] and Tutte [10]). Let G be a graph of n vertices. If

$$t = \max_{S \subset V(G)} \{q(G - S) - |S|\}, \tag{1}$$

then $|M(G)| = (n - t)/2$. ∎

Theorem 1. Let G be a connected reduced graph with $|V(G)| = n$ and with $\delta(G) \geq 3$. Then

$$|M(G)| \geq \min\left\{\frac{n-1}{2}, \frac{n+4}{3}\right\}. \tag{2}$$

Proof. Define t by (1). Then by Theorem C, it suffices to show that if $t \geq 2$, then

$$t \leq \frac{n-8}{3}. \tag{3}$$

Suppose that $t \geq 2$. Let $S \subset V(G)$ attain the maximum in (1). Since G is connected and since $t \geq 2$, $|S| \geq 1$. Assume that $G - S$ has r odd components containing exactly one vertex.

Assume that $m = q(G - S)$ and let $G_1, G_2, \ldots, G_m$ be the odd components of $G - S$.

Case 1: $r = 0$. Then $|V(G_i)| \geq 3$, $(1 \leq i \leq m)$. We may assume that

$$3 \leq |V(G_1)| \leq |V(G_2)| \leq \cdots \leq |V(G_m)|. \tag{4}$$

(1a) $|V(G_1)| = 3$. Then by (4),

$$n \geq 3m + |S|. \tag{5}$$

By (d) of Theorem A, G is K_3-free and so $G_1 \cong K_{1,2}$. By $\delta(G) \geq 3$, by $\delta(K_{1,2}) = 1$ and since G has no 3-cycles, $|S| \geq 2$ and so by (5),

$$t = m - |S| \leq \frac{n - |S|}{3} - |S| = \frac{n - 4|S|}{3} \leq \frac{n-8}{3}.$$

(1b) $|V(G_1)| \geq 5$. Then by (4),

$$n \geq 5m + |S|. \tag{6}$$

Since $m - |S| = t \geq 2$ and $|S| \geq 1$, we have $m \geq 3$ and so by (6),

$$n \geq 16. \tag{7}$$

By $|S| \geq 1$, (6) and (7),

$$t = m - |S| \leq \frac{n - |S|}{5} - |S| \leq \frac{n - 6|S|}{5} \leq \frac{n-6}{5} \leq \frac{n-8}{3}.$$

Case 2. $r \geq 1$. Let $G_1, G_2, \ldots, G_r$ be the odd components of $G - S$ containing exactly one vertex, say that $V(G_i) = \{v_i\}$, $(1 \leq i \leq r)$. Let $V'' = \{v_1, \ldots, v_r\}$, and let $G'' = G[V'' \cup S]$ and $n'' = |V(G'')|$. Then

$$n'' = r + |S|, \tag{8}$$

and

$$n'' \le n - 3(m - r),$$

which implies

$$m - r \le \frac{n - n''}{3}. \tag{9}$$

Since G'' is also a reduced graph and since all the r vertices in V'' have degree at least 3 in G'', it follows that $G'' \notin \{K_1, K_2\}$ and so by (d) of Theorem A,

$$3r \le |E(G'')| \le 2n'' - 4.$$

By (8), we have

$$6r \le 4n'' - 8 = n'' - 8 + 3(r + |S|),$$

which implies

$$r - |S| \le \frac{n'' - 8}{3}. \tag{10}$$

Combine (9) and (10) to get

$$t = m - |S| = (m - r) + (r - |S|) \le \frac{n - 8}{3}.$$

Hence (3) holds always and so (2) follows. ∎

Corollary 2. If G is a nontrivial connected reduced graph with $|V(G)| = n$ and $\kappa'(G) \ge 3$, then $|M(G)| \ge (n + 4)/3$.

Proof. By Theorem B, either $G \cong P$ or $n \ge 12$. Since $\lceil (n + 4)/3 \rceil \le \lceil (n - 1)/2 \rceil$ for $n \ge 10$, we are done by Theorem 1. ∎

MATCHINGS AND REDUCTIONS

Let G be a graph and let G' be the reduction of G. Define $a''(G)$ to be the maximum cardinality of an independent set of edges in G'. Since G' is uniquely determined by G (see [4]), $a''(G)$ is well defined. Let $X = \{x_i y_i : (1 \le i \le k)\}$ be a set of k edges in G. Define

$$\sum_G (X) = \sum_{i=1}^{k} (d_G(x_i) + d_G(y_i)).$$

When $X = \{e\}$, we use $\sum_G (e)$ for $\sum_G (\{e\})$. For convenience, we regard $\kappa'(K_1) = \infty$ and $a''(K_1) = 0$.

Theorem 3. Let G be a 3-edge-connected simple noncollapsible graph with n vertices and let p be a positive integer. If for every matching M_p of size p in G,

$$\sum_G (M_p) \geq n - 2p\,, \tag{11}$$

and if

$$n > 12p(p - 1)\,, \tag{12}$$

then G', the reduction of G, satisfies

$$a''(G) \leq p, \quad \text{and} \quad |V(G')| \leq 3p - 4\,.$$

Proof. Since $\kappa'(G) \geq 3$, we have $\kappa'(G') \geq 3$ also. Let $c = |V(G')|$ and $m = a''(G)$. Since $\kappa'(G') \geq 3$, we have $m \geq 2$. Since G is not collapsible, $c > 1$. If $m \leq p$, then by Corollary 2, the conclusion of Theorem 3 holds. Hence we assume that

$$m \geq p + 1\,. \tag{13}$$

Let $M = \{e_1, e_2, \ldots, e_m\}$ be a matching in G' with size m. By Corollary 2,

$$m \geq \frac{c + 4}{3}\,. \tag{14}$$

Note that $M \subseteq E(G)$ also. Without loss of generality, we assume that

$$\sum_G (e_1) \leq \sum_G (e_2) \leq \cdots \leq \sum_G (e_m)\,.$$

By (13), $p \leq m - 1$. Thus by (11),

$$\sum_G (e_m) \geq \cdots \geq \sum_G (e_{p+1}) \geq \frac{\sum_G(\{e_1, e_2, \ldots, e_p\})}{p} \geq \frac{n}{p} - 2\,.$$

It follows that

$$\begin{aligned}\sum_G (M) &= \sum_G(\{e_{p+1}, \ldots, e_m\}) + \sum_G(\{e_1, e_2, \ldots, e_p\}) \\ &\geq (m - p)\left(\frac{n}{p} - 2\right) + n - 2p\,,\end{aligned}$$

and so

$$\sum_G(M) \geq m\left(\frac{n}{p} - 2\right). \tag{15}$$

For each i $(1 \leq i \leq m)$, let $e_i = v_i u_i$, and let $H_i(v)$ and $H_i(u)$ denote the preimages of v_i and u_i in G, respectively. Recall that $H_1(v), \ldots, H_m(v), H_1(u), \ldots, H_m(u)$ are pairwise disjoint. Let $S = V(G'[M])$. Since $\kappa'(G') \geq 3$, $G' \neq K_2$ and so by (d) of Theorem A, there are at most $2|E(G')| \leq 4c - 8$ incidences in G of edges in $E(G')$ with S. Hence

$$\begin{aligned}\sum_G(M) &\leq 2|E(G')| + \sum_{i=1}^{m}(|V(H_i(v))| + |V(H_i(u))| - 2)\\ &\leq 4c - 8 - 2m + n\,. \end{aligned}\tag{16}$$

By (15) and (16),

$$\frac{m}{p}(n - 2p) \leq \sum_G(M) \leq 4c - 8 - 2m + n\,. \tag{17}$$

By (14), $c \leq 3m - 4$. Then by (17),

$$\frac{mn}{p} \leq 4(3m - 4) - 8 + n\,. \tag{18}$$

Suppose that $p = 1$. If $n \geq 12$, then (18) implies

$$n - 12 \leq m(n - 12) \leq n - 24\,,$$

a contradiction. If $n \leq 11$, then $G' = P$ by Theorem B, whence we reach a contradiction with (11).

Hence $p \geq 2$ and so by (12), $n > 12p$. It follows from (18) that

$$m \leq \frac{(n - 24)p}{n - 12p}\,. \tag{19}$$

By (12), we have $((n - 24)p)/(n - 12p) < p + 1$ and so by (19), we have $m \leq p$, contrary to (13). This completes the proof of Theorem 3. ∎

The following corollary follows from Theorem 3.

Corollary 4. Let $p > 0$ be an integer and let G be a simple graph with $n > 12p(p - 1)$ vertices and with $\kappa'(G) \geq 3$. If for every $e \in E(G)$,

$$\sum_G(e) \geq \frac{n}{p} - 2\,,$$

then exactly one of the following holds:

(a) G is in $\mathcal{CL}$;
(b) G', the reduction of G, satisfies

$$a''(G) \leq p \quad \text{and} \quad |V(G')| \leq 3p - 4 .$$

AN APPLICATION

The line graph of a graph G, denoted by $L(G)$, has vertex set $E(G)$, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G. In [1], Benhocine, Clark, Köhler, and Veldman conjectured that $L(G)$ is hamiltonian for any 2-edge-connected simple graph G of large order n satisfying

$$\sum_{G}(e) > \frac{2n}{5} - 2,$$

for each edge $e \in E(G)$. Li [8] proved this conjecture with an additional condition that the minimum degree is at least 4.

An eulerian subgraph H of G such that each edge in G is incident with at least one vertex in H is called a *dominating eulerian subgraph.* In [4], Catlin proved:

Theorem D (Catlin [4]). Let G be a graph with $|E(G)| \geq 3$ and let G' be the reduction of G. Then $L(G)$ is hamiltonian if and only if G' has a dominating eulerian subgraph that contains all nontrivial vertices of G'. ∎

Note that every supereulerian graph has a dominating eulerian subgraph, and that by $\mathcal{CL} \subset \mathcal{SL}$, a collapsible graph also admits a dominating eulerian subgraph. For 3-edge-connected graphs, we prove the following.

Corollary 5. Let G be a simple graph with $\kappa'(G) \geq 3$ and $n = |V(G)| > 240$. If for every matching M_5 of size 5 in G,

$$\sum_{G}(M_5) \geq n - 10, \tag{20}$$

then either $G \in \mathcal{CL}$ or the reduction of G is the Petersen graph.

Proof. Let G' denote the reduction of G. By Theorem 3 with $p = 5$, we have $a''(G) \leq 5$ and $|V(G')| \leq 11$, and so by Theorem B, either $G' = K_1$ and so $G \in \mathcal{CL}$, or G' is the Petersen graph. ∎

Corollary 6. Let G be a graph satisfying the hypotheses of Corollary 4. Then either $L(G)$ is hamiltonian or G can be contracted to the Petersen graph in such a way that every vertex of the contraction is nontrivial.

Proof. It follows from Corollary 5 and Theorem D, and from the fact that $P - v \in \mathcal{SL}$, for any vertex $v \in V(P)$. ∎

The following gives more details about the extremal graphs.

Theorem 7. Let G be a simple graph with $\kappa'(G) \geq 3$ and with $n = |V(G)| > 240$ vertices. If for each edge $e \in E(G)$,

$$\sum_G (e) \geq \frac{n}{5} - 2, \tag{21}$$

then exactly one of the following holds:

(a) G is collapsible;
(b) $n = 10s$ for some integer $s > 24$, and G can be contracted to the Petersen graph P in such a way that the preimage of each vertex of P is either K_s or $K_s - e'$ for some $e' \in E(K_s)$.

Proof. The proof is routine and so is omitted. ∎

Chen ([7],[8]) previously proved Theorem 7 with an additional condition $\delta(G) \geq 4$ and without the restriction on the number of vertices.

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