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Fractional arboricity, strength, and principal partitions in graphs and matroids

Paul A. Catlin
Wayne State University, Detroit, MI 48202, USA

Jerrold W. Grossman
Oakland University, Rochester, MI 48309, USA

Arthur M. Hobbs*
Texas A & M University, College Station, TX 77843, USA

Hong-Jian Lai**
West Virginia University, Morgantown, WV 26506, USA

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Abstract

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In a 1983 paper, D. Gusfield introduced a function which is called (following W.H. Cunningham, 1985) the *strength* of a graph or matroid. In terms of a graph G with edge set $E(G)$ and at least one link, this is the function $\eta(G) = \min_{F\subseteq E(G)} |F|/(\omega(G-F) - \omega(G))$, where the minimum is taken over all subsets F of $E(G)$ such that $\omega(G-F)$, the number of components of $G-F$, is at least $\omega(G)+1$. In a 1986 paper, C. Payan introduced the *fractional arboricity* of a graph or matroid. In terms of a graph G with edge set $E(G)$ and at least one link this function is $\gamma(G) = \max_{H\subseteq G} |E(H)|/(|V(H)| - \omega(H))$, where H runs over

Correspondence to: Professor A.M. Hobbs, Department of Mathematics, Texas A & M University, College Station, TX 77843, USA.

* Part of this work was done at Oakland University, Rochester, MI, and part at Indiana-Purdue at Fort Wayne, IN.

** Part of this work was done at Wayne State University, Detroit, MI, and part at the University of Waterloo, Ont.

all subgraphs of G having at least one link. Connected graphs G for which $\gamma(G) = \eta(G)$ were used by A. Ruciński and A. Vince in 1986 while studying random graphs.

We characterize the graphs and matroids G for which $\gamma(G) = \eta(G)$. The values of γ and η are computed for certain graphs, and a recent result of Erdös (that if each edge of G lies in a C_3, then $|E(G)|\geq\frac{3}{2}(|V(G)|-1)$) is generalized in terms of η.

The *principal partition* of a graph was introduced in 1967 by G. Kishi and Y. Kajitani, by T. Ohtsuki, Y. Ishizaki, and H. Watanabe, and by M. Iri (all of these were published in 1968). It has been used since then for the analysis of electrical networks in which the two Kirchhoff laws and Ohm's law hold, because it often allows the currents and voltage drops in the network to be completely computed with fewer measurements than are required for either of the Kirchhoff laws used alone. J. Bruno and L. Weinberg generalized the principal partition to matroids in 1971, and their generalization was refined independently by N. Tomizawa (1976) and by H. Narayanan and M.N. Vartak (1974, 1981). Here we demonstrate that γ and η are closely related to the principal partition and can be used to give a simple definition of both the principal partition and the more recent refinements of it.

We use the notation of Welsh [26] and (for graphs) Bondy and Murty [1]. In particular, given a matroid M on set S, for any subset X of S, we use $M.X$ for the contraction of M to X, and $M|X$ for the restriction of M to X. In addition, we adopt M/X for $M.X^{\mathrm{C}}$. We let $\mathbb{N}$ stand for the set of positive integers and $\mathbb{R}$ stand for the set of real numbers. To avoid unnecessary repetition, a matroid labeled M will always be on a set S and will always have rank function ϱ. For simplicity, we suppose all matroids and graphs in this paper are loopless.

In a matroid M with $\varrho M>0$, we define

$$g(M)=\frac{|S|}{\varrho S} \quad \text{and} \quad g(X)=\frac{|X|}{\varrho X} \quad \text{for any } X\subseteq S \text{ with } \varrho X>0.$$

Following Narayanan and Vartak [16], we call $g(M)$ the *density* of the matroid M. We let

$$\gamma(M)=\max_{X\subseteq S} g(X),$$

where the maximum is taken over all subsets $X\subseteq S$ for which $\varrho X>0$. Let us say that $X'\subseteq S$ *achieves the value* $\gamma(M)$ if $g(X)=\gamma(M)$. We note that, if $X'\subseteq S$ achieves the value $\gamma(M)$, then $\sigma X'=X'$, for otherwise expanding from X' to its closure would increase $|X'|$ without changing $\varrho X'$.

We further define

$$\eta(M)=\min_{X\subseteq S}\frac{|S\setminus X|}{\varrho S-\varrho X},$$

where the minimum is taken over all subsets $X\subseteq S$ for which $\varrho X<\varrho S$. In cases where no confusion is possible, for any $X\subseteq S$, we use $\gamma(X)$ and $\eta(X)$, respectively, to denote $\gamma(M|X)$ and $\eta(M|X)$.

The function η was introduced for graphs in reciprocal form by Gusfield in 1983

[9]. He said that $\eta(G)$ "can be used as a measure of vulnerability"; the smaller the value of $\eta(G)$, "the more vulnerable is the graph to large amounts of disconnection for few edge deletions". This function was generalized and extended to matroids by Cunningham [6]. Followig Cunningham, we refer to $\eta(M)$ as the *strength* of matroid M.

The function γ was introduced implicitly by Tomizawa [23] and independently (and more explicitly) by Narayanan and Vartak [15,16]. Extending the Narayanan and Vartak term "density", we say that a matroid M for which $\gamma(H) \le g(M)$ for every restriction H of M (equivalently, for which $\gamma(M) = g(M)$) is *uniformly dense.* Uniformly dense graphs were discussed in a 1986 paper by Ruciński and Vince [21], where they were used to help prove a theorem about random graphs. They proved that, for every rational number $r \ge 1$, there is a uniformly dense graph G for which $\gamma(G) = r$. A similar result was shown at about the same time by Payan [20].

The two functions γ and η are closely connected through the dual of the matroid, as is shown by our first theorem.

Theorem 1. *For any loopless matroid M on set S, having loopless dual M^*,*

$$\eta(M^*) = \frac{\gamma(M)}{\gamma(M) - 1},$$

and equivalently,

$$\gamma(M^*) = \frac{\eta(M)}{\eta(M) - 1}.$$

Proof. Applying the formula $\varrho^* X = |X| - \varrho S + \varrho X^{\mathrm{C}}$ (see [26, p. 35]), we have

$$\begin{aligned}
\eta(M^*) &= \min_{\substack{X \subseteq S \\ \varrho^* X \ne \varrho^* S}} \frac{|S \setminus X|}{\varrho^* S - \varrho^* X} \\
&= \min_{\substack{X \subseteq S \\ \varrho^* X \ne \varrho^* S}} \frac{|X^{\mathrm{C}}|}{|X^{\mathrm{C}}| - \varrho X^{\mathrm{C}}} \\
&= \min_{\substack{X \subseteq S \\ \varrho X^{\mathrm{C}} > 0}} \frac{|X^{\mathrm{C}}| / \varrho X^{\mathrm{C}}}{|X^{\mathrm{C}}| / (\varrho X^{\mathrm{C}}) - 1} \\
&= \min_{\substack{X \subseteq S \\ \varrho X > 0}} \left(1 + \frac{1}{|X| / (\varrho X) - 1}\right) \\
&= 1 + \frac{1}{(\max_{X \subseteq S, \varrho X > 0} |X| / (\varrho X)) - 1} \\
&= \frac{\gamma(M)}{\gamma(M) - 1}.
\end{aligned}$$

The other formula is algebraically equivalent to the first one. □

Let t be a natural number, and let M be a loopless matroid. A *family* is a set in which elements may appear more than once. Let us define a *t-packing* of M to be a family $\mathcal{F}$ of bases of M such that each element of M is in at most t bases of $\mathcal{F}$. We let $\eta_t(M)$ be the cardinality of the largest t-packing of M. Dually, let us define a *t-covering* of M to be a family $\mathcal{F}$ of independent sets of M such that each element of M is in at least t members of $\mathcal{F}$. Then we let $\gamma_t(M)$ be the cardinality of the smallest t-covering of M.

In 1961, Nash-Williams [17] and Tutte [24] independently proved:

Theorem 2. *If G is a connected loopless graph with at least two vertices, then*

$$\eta_1(G) = \min_{F \subseteq E(G)} \left\lfloor \frac{|F|}{\omega(G-F)-1} \right\rfloor,$$

where the minimum is over all subsets F of E(G) for which $\omega(G-F)>1$.

In 1964, Nash-Williams published the dual theorem [18]:

Theorem 3. *If G is a connected graph with at least two vertices, then*

$$\gamma_1(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil,$$

where the maximum runs over all subgraphs H of G having at least one link.

These two theorems were extended to matroids in 1965 by Edmonds [7] (but see also Lehman [13, p. 710]). By replacing each element of the matroid M with t parallel elements, and then applying the Edmonds extension, we have immediately:

Theorem 4. *Let M be a matroid on S with rank function* ϱ. *Let* $s,t \in \mathbb{N}$, *with* $s \geq t$. *Then*

(i) *M has a t-packing of cardinality s if and only if* $\eta(M) \geq s/t$; *and*
(ii) *M has a t-covering of cardinality s if and only if* $\gamma(M) \leq s/t$.

Corollary 5. *Let M be a loopless matroid on S and let* $t \in \mathbb{N}$. *Then*

$$\gamma_t(M) = \lceil t\gamma(M) \rceil \tag{1}$$

and

$$\eta_t(M) = \lfloor t\eta(M) \rfloor. \tag{2}$$

The definitions of $\gamma(M)$ and $\eta(M)$ give

$$\eta(M) \leq \frac{|S|}{\varrho S} \leq \gamma(M). \tag{3}$$

In our next theorem, we shall determine when equality holds in (3).

Nash-Williams [18] introduced γ_1, which is now called *arboricity*. Following this lead, we call $\gamma(M)$ the *fractional arboricity* of M. This corresponds to Payan's *arboricité rationnelle* [20]. We note that $\gamma(M)$ is a fraction, and further it is an important dividing point in the following sense: If $s/t \geq \gamma(M)$, then there is a family $\mathscr{F}$ of s bases of M such that each element of S is in at least t of the bases in $\mathscr{F}$. However, if $s/t < \gamma(M)$, then no such family $\mathscr{F}$ exists.

In the following definition, we shall use S_i to denote the underlying set of matroid H_i. Given the matroid M, we construct a sequence of matroids $(H_1, H_2, \ldots, H_k)$ by the following rules:

(i) $H_1 = M$;

(ii) for $i \geq 1$, if the set S_i has a subset X_i with $\eta(X_i) > \eta(H_i)$, then let $H_{i+1} = H_i/(\sigma X_i)$, where (σX_i) is the closure of X_i in H_i;

(iii) H_k has no loops and no subset X such that $\eta(X) > \eta(H_k)$.

Since $\eta(X_i)$ is defined, $X_i \neq \emptyset$ and so S_{i+1} is strictly contained in S_i. But $\varrho X_i = \varrho(\sigma(X_i))$; since $\eta(X_i) > \eta(H_i)$, it follows that $\sigma X_i \neq S_i$. Hence $S_{i+1} \neq \emptyset$, and thus H_k exists by the finiteness of M. We call H_k an *η-reduction* of M, and we use M_0 to denote any η-reduction of M (we show later that M_0 is unique). When $M = M_0$, we say that M is *η-reduced*.

Theorem 6. *Let M be a loopless matroid on set S with rank function ϱ. The following are equivalent*:

(a) $\gamma(M)\varrho S = |S|$ *(i.e., M is uniformly dense).*

(b) $\eta(M)\varrho S = |S|$.

(c) $\gamma(M) = \eta(M)$.

(d) *M is η-reduced.*

(e) *There is a function $f: \{1, 2, \ldots, \varrho S\} \to \mathbb{R}$ such that*

  (i) *$f(r)/r \leq f(\varrho S)/(\varrho S)$ for $1 \leq r \leq \varrho S$,*

  (ii) *$f(\varrho S) = |S|$, and*

  (iii) *$|X| \leq f(\varrho X)$ for every $X \subseteq S$ with $\varrho X > 0$.*

(f) *For any positive integers s and t such that $\gamma(M) = s/t$, there is a family $\mathscr{F}$ of s bases of M such that each element of S is in exactly t bases in $\mathscr{F}$.*

(g) *For any positive integers s and t such that $\eta(M) = s/t$, there is a family $\mathscr{F}$ of s bases of M such that each element of S is in exactly t bases in $\mathscr{F}$.*

(h) *There is a $t \in \mathbb{N}$ and a family $\mathscr{F}$ of bases of M such that $\mathscr{F}$ is both a t-covering and a t-packing.*

Proof. ((c) $\Rightarrow$ (a) and (c) $\Rightarrow$ (b)). These follow from (3).

((f) $\Rightarrow$ (h) and (g) $\Rightarrow$ (h)). These implications are immediate from the definitions.

Next, choose $t \in \mathbb{N}$ such that $t\gamma(M)$ and $t\eta(M)$ are integers, and set $g = \gamma_t(M)$ and $h = \eta_t(M)$. By (1) and (2),

$$t\gamma(M) = \gamma_t(M) \quad \text{and} \quad t\eta(M) = \eta_t(M). \tag{4}$$

Thus, by the definitions of $\gamma_t(M)$ and $\eta_t(M)$, there is a family $\mathscr{F}=\{B_1,B_2,\ldots,B_g\}$ of bases of M such that each $x\in S$ is in at least t members of $\mathscr{F}$, and there is a family $\mathscr{F}'=\{B_1',B_2',\ldots,B_h'\}$ of bases of M such that each $x\in S$ is in at most t members of $\mathscr{F}'$.

((a) $\Rightarrow$ (h)). Suppose (a) holds. By (4) and (a),

$$t\gamma(M)\varrho S=\gamma_t(M)\varrho S=\sum_{i=1}^{g}|B_i|\geq t\,|S|=t\gamma(M)\varrho S.$$

Equality must hold, and so $\mathscr{F}$ is a family of g bases such that each $x\in S$ is in exactly t members of $\mathscr{F}$. Thus (h) holds.

((b) $\Rightarrow$ (h)). The proof of this is similar to that of the previous case.

((h) $\Rightarrow$ (c)). By (h), $\mathscr{F}$ is both a t-covering and a t-packing of cardinality s. Then by Theorem 4 and (3),

$$\eta(M)\geq\frac{s}{t}\geq\gamma(M)\geq\eta(M),$$

which implies equality. Thus (c) holds.

((h) $\Rightarrow$ (f)). Let s and t be positive integers such that $\gamma(M)=s/t$. Then by formula (4), $\gamma_t=s$. Thus there is a family $\mathscr{F}$ of $g=s$ bases of M such that each element of S is in at least t of the bases. But by (h) $\Rightarrow$ (c) $\Rightarrow$ (a), $\gamma(M)=|S|/\varrho S$. Hence each element of S is in exactly t bases of $\mathscr{F}$, which is (f).

((h) $\Rightarrow$ (g)). The proof of this is similar to that of the previous case.

((c) $\Rightarrow$ (d)). For the sake of contradiction, suppose that (c) holds but that S has a subset X such that $\eta(X)>\eta(M)$. By definition and (3), we have

$$\gamma(M)\geq\gamma(X)\geq\eta(X)>\eta(M),$$

contrary to (c).

((d) $\Rightarrow$ (b)). Suppose that M is η-reduced. Choose $X\subseteq S$ such that $|X|$ is minimized with respect to the condition that

$$\eta(M)=\frac{|S\setminus X|}{\varrho S-\varrho X}.$$

For the sake of contradiction, suppose $\varrho X\neq 0$. Then we can find $X_1\subseteq X$ such that

$$\eta(X)=\frac{|X\setminus X_1|}{\varrho X-\varrho X_1}.$$

Since M is η-reduced, $\eta(X)\leq\eta(M)$ and so we have

$$\begin{aligned}|S\setminus X_1|&=|S\setminus X|+|X\setminus X_1|\\&=\eta(M)(\varrho S-\varrho X)+\eta(X)(\varrho X-\varrho X_1)\\&\leq\eta(M)(\varrho S-\varrho X+\varrho X-\varrho X_1)\\&=\eta(M)(\varrho S-\varrho X_1).\end{aligned}$$

Since we have $\sigma(X_1) \subseteq \sigma(X) \neq S$, it follows from the definition of $\eta(M)$ that

$$\eta(M) = \frac{|S \setminus X_1|}{\varrho S - \varrho X_1},$$

contrary to the minimality of $|X|$. Thus we must have $\varrho X = 0$. Since M is loopless, it follows that $X = \emptyset$, and thus (b) holds.

((a) $\Rightarrow$ (e)). We have $|S| = \gamma(M)\varrho S$. Define f on $\{1, 2, \ldots, \varrho S\}$ by

$$f(r) = \max_{\varrho X = r} |X|.$$

Then by definition, $|S| = f(\varrho S)$ and $|X| \leq f(\varrho X)$, for all $X \subseteq S$. But also

$$\frac{f(r)}{r} = \max_{\varrho X = r} \frac{|X|}{\varrho X} \leq \gamma(M) = \frac{|S|}{\varrho S} = \frac{f(\varrho S)}{\varrho S}.$$

((e) $\Rightarrow$ (a)). By (e), we have, for each $X \subseteq S$ with $\varrho X > 0$,

$$\frac{|X|}{\varrho X} \leq \frac{f(\varrho X)}{\varrho X} \leq \frac{f(\varrho S)}{\varrho S} = \frac{|S|}{\varrho S}.$$

Since $\gamma(M)$ is the maximum of these $|X|/(\varrho X)$, we have

$$\gamma(M) = \frac{|S|}{\varrho S},$$

and (a) follows. □

In the corollaries, we usually apply condition (e) with $f(r)/r$ nondecreasing.

Corollary 7. *Let M be a matroid on set S. If there are constants a and b such that $a > 0$, $a + b \leq 0$, and*

$$|S| = a\varrho S + a + b, \tag{5}$$

and such that every nonempty subset $X \subseteq S$ satisfies

$$|X| \leq a\varrho X + a + b, \tag{6}$$

then $\gamma(M) = \eta(M) = |S|/\varrho S$.

Proof. Apply (e) $\Rightarrow$ (c) of Theorem 6 with $f(r) = ar + a + b$. Then

$$\frac{f(r)}{r} = \frac{ar + a + b}{r} = a + \frac{a + b}{r}.$$

Since $a > 0$ and $a + b \leq 0$, we have $f(r)/r$ is nondecreasing. □

Corollary 7 is particularly valuable when applied to graphs, as in part (a) of Corollary 8. Recall that the rank of the cycle matroid of a graph G with vertex set $V(G)$, edge set $E(G)$, and number of components $\omega(G)$ is $|V(G)| - \omega(G)$.

Corollary 8. *The set of graphs G satisfying*

$$\gamma(G)=\eta(G)=\frac{|E(G)|}{|V(G)|-\omega(G)}$$

includes:

(a) *plane triangulations*;

(b) *nontrivial complete graphs*; *and*

(c) *cycles*.

Proof. (a) A plane triangulation G (i.e., a plane graph whose faces are all triangles) satisfies (5) and (6) with $a=3$, $b=-6$ (see [1, pp. 144–145]).

(b) Apply (e) $\Rightarrow$ (c) of Theorem 6 with $f(r)=r(r+1)/2$.

(c) Apply (e) $\Rightarrow$ (c) of Theorem 6 with $f(r)=r$ when $r<n-1$ and $f(n-1)=n$, where $n=|V(G)|$. □

We next determine the behavior of γ and η under restriction and contraction, respectively. The purpose of the closure operation σ here is to ensure that contractions produce no loops.

Lemma 9. *Let M be a matroid on S, and let* $X\subseteq S$.

(a) *If* $\varrho X>0$, *then* $\gamma(M)\geq\gamma(M\mid X)$; *and*

(b) *if* $\sigma X\neq S$, *then* $\eta(M)\leq\eta(M/(\sigma X))$.

Proof. By the definition of γ, we have

$$\gamma(M)=\max_{\varrho T>0}\frac{|T|}{\varrho T}\geq\max_{\substack{\varrho T>0\\ T\subseteq X}}\frac{|T|}{\varrho T}=\gamma(M\mid X).$$

Further, using [26, p. 62, formula (1)],

$$\begin{aligned}\eta(M)&=\min_{T\subseteq S}\frac{|S\setminus T|}{\varrho S-\varrho T}\\ &\leq\min_{\sigma X\subseteq T\subseteq S}\frac{|S\setminus T|}{\varrho S-\varrho T}\\ &=\min_{\sigma X\subseteq T\subseteq S}\frac{|S\setminus\sigma X|-|T\setminus\sigma X|}{[\varrho S-\varrho(\sigma X)]-[\varrho(T\cup\sigma X)-\varrho(\sigma X)]}\\ &=\eta(M/(\sigma X)),\end{aligned}$$

where all of the minima are taken over those subsets $T\subseteq S$ such that the denominators are not zero. □

It is easy to see, for any set $X\subseteq S$ with $\varrho X>0$, that $\eta(X)\leq\eta(\sigma X)$. Using this and the previous lemma, we have:

Lemma 10. *If $X \subseteq S$ and if $\eta(X) > \eta(M)$, then*

$$\eta(M/(\sigma X)) = \eta(M). \tag{7}$$

Proof. By the previous observation, we may assume that $X = \sigma X$. Since $\eta(S) = \eta(M)$, by hypothesis $X \neq S$. By (b) of Lemma 9, we have

$$\eta(M/X) \geq \eta(M). \tag{8}$$

Choose $T \subseteq S$ such that $\varrho S > \varrho T$ and $\eta(M)(\varrho S - \varrho T) = |S \setminus T|$. Define $T_1 = T \cap X$ and $T_2 = T \cap X^{\mathrm{C}}$. Then

$$\eta(M)(\varrho S - \varrho T) = |S \setminus T| = |X^{\mathrm{C}} \setminus T_2| + |X \setminus T_1|. \tag{9}$$

Since the rank function is semimodular,

$$\varrho(T \cup X) - \varrho X \leq \varrho T - \varrho T_1. \tag{10}$$

If $\varrho X > \varrho T_1$, then by the definition of $\eta(X)$ and by hypothesis,

$$|X \setminus T_1| \geq \eta(X)(\varrho X - \varrho T_1) > \eta(M)(\varrho X - \varrho T_1).$$

If $\varrho X = \varrho T_1$, then

$$|X \setminus T_1| \geq \eta(X)(\varrho X - \varrho T_1) = \eta(M)(\varrho X - \varrho T_1).$$

Thus

$$|X \setminus T_1| \geq \eta(M)(\varrho X - \varrho T_1),$$

with equality only if $\varrho X = \varrho T_1$. We substitute this into (9) to get

$$\eta(M)(\varrho S - \varrho T - \varrho X + \varrho T_1) \geq |X^{\mathrm{C}} \setminus T_2|, \tag{11}$$

with equality only if $\varrho X = \varrho T_1$.

Let ϱ_X denote the rank function on the contraction M/X. Then by (10), we have

$$\begin{aligned} \varrho_X X^{\mathrm{C}} - \varrho_X T_2 &= \varrho S - \varrho X - (\varrho(T \cup X) - \varrho X) \\ &\geq \varrho S - \varrho X - \varrho T + \varrho T_1. \end{aligned} \tag{12}$$

By (11) and (12),

$$\eta(M)(\varrho_X X^{\mathrm{C}} - \varrho_X T_2) \geq |X^{\mathrm{C}} \setminus T_2|, \tag{13}$$

with equality only if $\varrho X = \varrho T_1$. If $\varrho X = \varrho T_1$, then since $\varrho S > \varrho T$, (12) yields $\varrho_X X^{\mathrm{C}} - \varrho_X T_2 > 0$, so (13) and the definition of $\eta(M/X)$ imply

$$\eta(M) \geq \frac{|X^{\mathrm{C}} \setminus T_2|}{\varrho_X X^{\mathrm{C}} - \varrho_X T_2} \geq \eta(M/X). \tag{14}$$

If $\varrho X > \varrho T_1$, then the inequality in (13) is strict, so $\varrho_X X^{\mathrm{C}} - \varrho_X T_2 > 0$, and

$$\eta(M) > \frac{|X^{\mathrm{C}} \setminus T_2|}{\varrho_X X^{\mathrm{C}} - \varrho_X T_2} \geq \eta(M/X).$$

But this second case is impossible by (8), so $\varrho X = \varrho T_1$ and (14) and (8) together imply (7) and thus the lemma. □

Theorem 11. *There is only one η-reduction M_0 of matroid M. Further, M_0 satisfies*

$$\eta(M)=\eta(M_0). \tag{15}$$

Proof. First, (15) is immediate from Lemma 10. Now, suppose X and Y are subsets of S and suppose $\eta(Y)>\eta(M)$. Then, letting ϱ_1 be the rank in the minor $M/\sigma X$, in $M/\sigma X$

$$\begin{aligned}\eta(Y\setminus\sigma X)&=\min_{Z\subsetneq Y\setminus\sigma X}\frac{|(Y\setminus\sigma X)\setminus Z|}{\varrho_1(Y\setminus\sigma X)-\varrho_1 Z}\\&=\min_{Z\subsetneq Y\setminus\sigma X}\frac{|Y\setminus(\sigma X\cup Z)|}{\varrho Y-\varrho(\sigma X\cup Z)}\\&\geq\eta(Y)\\&>\eta(M)\\&=\eta(M/\sigma X).\end{aligned}$$

Now suppose an η-reduced minor M_1 of matroid M is produced by contracting subsets of S in the order of a sequence $\mathscr{C}=(X_1,X_2,\ldots,X_k)$. To show that the η-reduced matroid is unique, it suffices to show that every element of $\bigcup X_i$ must be contracted in any sequence $\mathscr{C}'$ of contractions leading to an η-reduced minor M_2 of M, for then by symmetry every element in the subsets of S in $\mathscr{C}'$ must be contracted in forming M_1. So suppose Y is a maximal subset of X_i which remains uncontracted in M_2. Since $\eta(X_i)>\eta(M)$ in $M/X_1/X_2/\cdots/X_{i-1}$, the above computation shows that $\eta(Y)>\eta(M)$, contrary to the definition of the η-reduced matroid M_2. The theorem follows. □

Using Theorem 4, it is also possible to show that the only elements contracted in forming M_0 are those in subsets X of S for which $\eta(X)>\eta(M)$ before any contractions are carried out.

Suppose loopless matroid M has loopless dual M^* and has components $M_1,M_2,\ldots,M_k$ such that $M=M_1+M_2+\cdots+M_k$ (see [26, pp. 70–73]). In the following proof we partition S into sets $S_1,S_2,\ldots,S_k$ such that M_i is a matroid on S_i for each i, and for any $X\subseteq S$, use X_i to represent $X\cap S_i$. Then we have

$$\varrho X=\varrho X_1+\varrho X_2+\cdots+\varrho X_k. \tag{16}$$

Theorem 12. *For a loopless matroid M with loopless dual and with components $M_1,M_2,\ldots,M_k$, both*

(a) $$\gamma(M)=\max_{1\leq i\leq k}\{\gamma(M_i)\},$$

and

(b) $$\eta(M)=\min_{1\leq i\leq k}\{\eta(M_i)\}.$$

Proof. We may suppose $k \geq 2$. Let $I=\{1,2,\dots,k\}$. By definition, we can find $X \subseteq S$ with

$$\eta(M)=\frac{|S|-|X|}{\varrho S-\varrho X}.$$

Set $a_i=|S_i|-|X_i|$ and $b_i=\varrho S_i-\varrho X_i$ for all i. Clearly $a_i \geq 0$ and $b_i \geq 0$ for all i. Further, $b_i>0$ for some value of i since $\varrho S-\varrho X>0$. Let $A=\{i \in I\colon b_i>0\}$. Without loss of generality, we may suppose that $\eta(M_1)=\min\{\eta(M_i)\}$. Pick $m \in A$ to minimize a_m/b_m . Using (16) and [10, Theorem 1, p. 14],

$$\eta(M)=\frac{\sum_{i\in I} a_i}{\sum_{i\in I} b_i} \geq \frac{\sum_{i\in A} a_i}{\sum_{i\in A} b_i} \geq \frac{a_m}{b_m} \geq \eta(M_m) \geq \eta(M_1).$$

On the other hand, by (16) we can find $Y_1 \subseteq S_1$ such that

$$\eta(M_1)=\frac{|S_1|-|Y_1|}{\varrho S_1-\varrho Y_1}=\frac{|S|-|Y_1\cup(S\setminus S_1)|}{\sum_{i=1}^{k}\varrho S_i-\varrho Y_1-\sum_{i=2}^{k}\varrho S_i}$$
$$=\frac{|S|-|Y_1\cup(S\setminus S_1)|}{\varrho S-\varrho(Y_1\cup(S\setminus S_1))} \geq \eta(M).$$

This proves (b). The proof of (a) is immediate using Theorem 1 and duality. □

Recently, Erdős [8] asked for a brief proof of the result that if every edge of a graph G is in a triangle of the graph, then $|E(G)| \geq \frac{3}{2}(|V(G)|-\omega(G))$. The next theorem provides a generalization of Erdős' result.

Theorem 13. *Let M be a matroid on a set S, let $\mathscr{F}$ be a family of matroids, and let B be a base of M. If each element of B is contained in a restriction of M isomorphic to a member of $\mathscr{F}$, then*

$$\eta(M) \geq \inf_{H\in\mathscr{F}} \eta(H).$$

Proof. Let M, B and $\mathscr{F}$ satisfy the hypotheses of the theorem. Let

$$b=\inf_{H\in\mathscr{F}} \eta(H), \tag{17}$$

and suppose for the sake of contradiction that

$$b>\eta(M). \tag{18}$$

Let M_0 be the η-reduction of M, and pick an element e_0 of M_0 such that $e_0 \in B$. Then e_0 lies in some restriction H of M isomorphic to an element of $\mathscr{F}$. In M_0, let H_0 be the image of the restriction H under the sequence of contractions that map M to M_0 (and that thus define the η-reduced matroid M_0). Then $e_0 \in S_0$, where S_0 is the underlying set of M_0. Thus, by Lemma 9,

$$\eta(H_0) \geq \eta(H). \tag{19}$$

By (19), (17), (18), and (15), $\eta(H_0) \geq \eta(H) \geq b > \eta(M) = \eta(M_0)$. Hence, H_0 contradicts part (iii) of the definition of M_0, for H_0 is a restriction of M_0. Therefore (18) is false, and the theorem follows. □

We have immediately from this and the definition of η:

Corollary 14. *Let M be a matroid, let B be a base of M, and let $\mathscr{F}$ be a family of matroids. If each element of B is contained in a restriction of M isomorphic to a member of $\mathscr{F}$, then*

$$|S| \geq \left(\inf_{H \in \mathscr{F}} \eta(H) \right) \varrho S.$$

The result of Erdős is immediate from Corollary 14, since $\eta(K_3) = 3/2$.

We next describe the principal partition of a matroid. This has been described before, and the description of it given by Tomizawa [23] implicitly employed the function γ. However, we believe the following development is of value because the coordination of η with γ which we use produces an easier and clearer explanation of the partition.

There are four different partitions of set S of matroid M that have been called "the principal partition". They are:

(1) A partition of $E(G)$ into three parts relating to values of γ greater than 2, equal to 2, and less than 2.

(2) An extension of (1) to matroids and simultaneously a refinement of the part relating to values of γ greater than 2 into parts relating to values of γ greater than each of the integers $k \geq 2$.

(3) A further refinement of (2) to allow fractional values of γ.

(4) A final refinement of each of the parts produced in (3) in a way to be described at the end of this paper.

We will fully describe the third of these, called "the principal partition into irreducible minors" by Tomizawa [23] and called "the P-sequence" by Narayanan and Vartak [16]. For convenience, we will call this third partition the *principal partition* of the matroid. Roughly, the principal partition of matroid M is a decomposition of S into subsets, each of which becomes the underlying set X of a uniformly dense matroid upon contraction of a particular subset of S followed by a restriction to X. (The principal partition will be more completely defined after Theorem 19.) We will describe the other three versions of principal partition in terms of this one after we have completed its definition.

To begin this development, we will connect γ with the work of Tomizawa [23]. For a matroid M on set S, let us define $\Gamma_k(M)$ by

$$\Gamma_k(M) = \max_{X \subseteq S} (|X| - k\varrho X).$$

For $X \subseteq S$, we let $\Gamma_k(X) = \Gamma_k(M \mid X)$. Since $|\emptyset| - k\varrho\emptyset = 0$, clearly $\Gamma_k(M) \geq 0$.

Lemma 15. *Let M be a loopless matroid, and let $k=\gamma(M)$. Then $\Gamma_k(M)=0$. Further, for any set $X\subseteq S$, $g(X)=k$ if and only if X achieves the maximum in $\Gamma_k(M)$ and $X\neq\emptyset$.*

Proof. Suppose $\Gamma_k(M)>0$. Then there is a subset X' of S such that $|X'|-k\varrho X'>0$. Since M is loopless, $\varrho X'>0$, so $|X'|/\varrho X'>k$, contrary to the choice of k. Next, let $g(X)=k$. Then $|X|/\varrho X=k$, so $|X|-k\varrho X=0$ and $X\neq\emptyset$. Conversely, suppose that $X\neq\emptyset$ achieves the maximum in $\Gamma_k(M)$, and suppose that $g(X)\neq k$. Then $g(X)<k=\gamma(M)$ by the definition of γ. Thus $|X|/\varrho X<k$, which gives $|X|-k\varrho X<0$. But then X does not achieve the maximum, a contradiction. □

Next we show that g and γ can be used to provide a lattice of restrictions of matroid M. This theorem and its proof exactly parallel a similar theorem and proof of Tomizawa [23, p. 3]; they are included here, first, because their setting in Tomizawa is complicated and, second, because they are needed here for completeness.

Theorem 16 [23]. *Let M be a loopless matroid on set S, and suppose $g(X_1)=g(X_2)=\gamma(M)$ for subsets $X_1, X_2\subseteq S$. Then*

$$g(X_1\cup X_2)=\gamma(M).$$

Further, if $\varrho(X_1\cap X_2)>0$, then

$$g(X_1\cap X_2)=\gamma(M).$$

Proof. By hypothesis, X_1 and X_2 are nonempty and both achieve the maximum in the definition of $\gamma(M)$. Let $k=\gamma(M)$. By Lemma 15, for $i\in\{1,2\}$, we have $\Gamma_k(X_i)=|X_i|-k\varrho X_i$ and $\Gamma_k(X_1)=\Gamma_k(X_2)=\Gamma_k(M)=0$. Hence, applying the submodularity of ϱ,

$$\begin{aligned}
&\Gamma_k(X_1\cup X_2)+\Gamma_k(X_1\cap X_2)\\
&\quad=\max_{X\subseteq X_1\cup X_2}\{|X|-k\varrho X\}+\max_{X\subseteq X_1\cap X_2}\{|X|-k\varrho X\}\\
&\quad\geq|X_1\cup X_2|-k\varrho(X_1\cup X_2)+|X_1\cap X_2|-k\varrho(X_1\cap X_2)\\
&\quad=|X_1|+|X_2|-k(\varrho(X_1\cup X_2)+\varrho(X_1\cap X_2))\\
&\quad\geq|X_1|+|X_2|-k(\varrho(X_1)+\varrho(X_2))\\
&\quad=\Gamma_k(X_1)+\Gamma_k(X_2)\\
&\quad=0.
\end{aligned}$$

But $\Gamma_k(X_1\cup X_2)=\max_{X\subseteq X_1\cup X_2}(|X|-k\varrho X)\leq\max_{X\subseteq S}(|X|-k\varrho X)=\Gamma_k(M)=0$. Similarly, $\Gamma_k(X_1\cap X_2)\leq 0$. Thus we have $0\geq\Gamma_k(X_1\cup X_2)+\Gamma_k(X_1\cap X_2)\geq 0$, and equality throughout follows. The theorem follows by applying Lemma 15 again. □

It follows from Theorem 16 that the subsets X of S for which $g(X)=\gamma(M)$ form a distributive lattice with (perhaps) the zero removed. Thus there is only one maximal subset U of S for which $g(U)=\gamma(M)$. We will use the symbol U for this set from now on.

Theorem 17. *Let M be a matroid on set S. Let X be any subset of S such that $\gamma(M)=|X|/\varrho X$. Then $\eta(X)=\gamma(X)=\gamma(M)$.*

Proof. Notice that

$$\gamma(X)=\max_{Y\subseteq X}\{|Y|/\varrho Y\}\leq\max_{Y\subseteq S}\{|Y|/\varrho Y\}=\gamma(M)=|X|/\varrho X\leq\gamma(X).$$

Thus we have equality throughout, and $\eta(X)=\gamma(X)$ by Theorem 6. □

Corollary 18 (Catlin [5]). *For any matroid M on set S,*

$$\gamma(M)=\max_{\substack{X\subseteq S\\ \varrho X<\varrho S}}\eta(M\,|\,X).$$

Also,

$$\eta(M)=\min_{\substack{X\subseteq S\\ \varrho(M.X)>0}}\gamma(M.X).$$

Proof. By Theorem 17, $\gamma(M)=\eta(U)$, so $\max\eta(X)\geq\gamma(M)$. But $\gamma(M)\geq\gamma(X)\geq\eta(X)$ for all subsets X of S, and the first part of the corollary follows. The second part is just the dual statement of the first part. □

Theorem 19. *Let M be a matroid on set S, and let U be the maximal subset of S such that $g(U)=\gamma(M)$. If $U\neq S$, then $\gamma(M/U)<\gamma(M)$.*

Proof. Suppose otherwise. Then in M/U there is a maximal set X achieving the value of $\gamma(M/U)$. By Theorem 17, X is a subset of U^{C} such that $\eta((M/U)\,|\,X)=\gamma((M/U)\,|\,X)=\gamma(M/U)$. Let $H=(M/U)\,|\,X$. Suppose $\gamma(H)=s/t$ for positive integers s and t, and (using Theorem 6(f)) let $\mathscr{F}$ be a set of s bases of H such that each element of X is in exactly t bases in $\mathscr{F}$. Suppose $\gamma(M)=u/v$ for positive integers u and v, and let $\mathscr{F}'$ be a set of u bases of $M\,|\,U$ such that each element of U is in exactly v of the bases.

Given a fraction a/b and integer $k>0$, if we have a set of a bases such that each element is in exactly b bases, then by duplicating bases we can get a set of ak bases such that each element is in exactly bk bases. Hence, let t' be a common multiple of t and v, and let s' and u' be positive integers for which there is a set $\mathscr{F}''$ of s' bases of H such that each element of X is in exactly t' of these bases, and there is a set $\mathscr{F}'''$ of u' bases of $M\,|\,U$ such that each element of U is in exactly t' of these bases. Because $\gamma(M/U)\geq\gamma(M)$, we have that $s'\geq u'$.

Make a one to one assignment of u' bases B_i'' of $\mathscr{F}''$ to the bases B_i''' in $\mathscr{F}'''$. Then, in $M\,|\,(U\cup X)$ the sets $B_i''\cup B_i'''$ with $i\in\{1,2,\dots,u'\}$ constitute a set of u' bases of $M\,|\,(U\cup X)$ such that each element of $M\,|\,(U\cup X)$ is contained in at most t' of these bases. Hence, by (3) and Theorems 4 and 6,

$$g(M\,|\,(U\cup X))\geq\eta(M\,|\,(U\cup X))\geq\frac{u'}{t'}=\frac{u}{v}=\gamma(M)=g(M\,|\,U),$$

contrary to the choice of U. This contradiction establishes that $\gamma(M/U)<\gamma(M)$. □

Given a loopless matroid M on a set S, define sequences $(U_1,U_2,\dots,U_n)$ and $(N_1,N_2,\dots,N_n)$ in two steps as follows: Let $M_1=M$ and $S_1=S$. Let $i=1$.

Step 1. Let U_i be the maximal subset of S_i such that $g(U_i)=\gamma(M_i)$. Let $N_i=M_i\,|\,U_i$.

Step 2. If $U_i\neq S_i$, then let $M_{i+1}=M_i\,.\,(S_i\setminus U_i)$, and let $S_{i+1}=S_i\setminus U_i$. Replace i with $i+1$ and go to Step 1. But if $U_i=S_i$, then let $n=i$ and stop.

The sequence $(U_n,U_{n-1},\dots,U_1)$ is precisely Tomizawa's "partition of M into irreducible minors".

According to Narayanan and Vartak [16], a *P-sequence* of matroid M on set S is an ordered partition of S into a sequence $(P_1,P_2,\dots,P_n)$ such that

(a) $(M\,|\,\bigcup_{i\leq k}P_i)\,.\,P_k$ is uniformly dense for $k=1,2,\dots,n$; and

(b) $g((M\,|\,\bigcup_{i\leq k}P_i)\,.\,P_k)>g((M\,|\,\bigcup_{i\leq r}P_i)\,.\,P_r)$ whenever $k<r$.

They showed [16, Theorem 11] that, if the P-sequence exists, then it is unique. We show next that the sequence $(U_1,U_2,\dots,U_n)$ is a P-sequence for any matroid M.

Theorem 20. *Let M be a matroid on set S. The sequence $(U_1,U_2,\dots,U_n)$ defined above is a P-sequence, in which $P_i=U_i$ for each $i\in\{1,2,\dots,n\}$.*

Proof. As defined, $(U_1,U_2,\dots,U_n)$ is a partition of S. Thus it will suffice to verify (a) and (b) of the definition of P-sequence. But $(M\,|\,\bigcup_{i\leq k}P_i)\,.\,P_k=N_k$ for each k, and N_k is uniformly dense, verifying (a). Further, by Theorems 6 and 19, $g(N_k)=\gamma(N_k)>\gamma(N_r)=g(N_r)$. □

The term "principal partition" was first used and defined for graphs in 1967 (see the survey paper [25, pp. 118–119]). Three distinct approaches to it were investigated in 1967 and published in 1968 by Kishi and Kajitani [12], by Iri [11], and by Ohtsuki, Ishizaki and Watanabe [19]. For a particularly clear description of the Kishi and Kajitani approach, see [22, Chapter 11].

To describe their principal partition, labeled (1) in our eaerlier list, let M be the cycle matroid of a graph G, and define subgraphs G_0, G_1, and G_2 as edge-generated subgraphs of G as follows: Let

$$G_1=G\left[\bigcup U_i\right],$$

where the union is over all U_i for which

$$\gamma\left(M.\left(\bigcup_{j=i}^{n} U_j\right)\middle| U_i\right)>2.$$

If there is a k such that

$$\gamma\left(M.\left(\bigcup_{j=k}^{n} U_j\right)\middle| U_k\right)=2,$$

let G_0 be formed by contracting the edges of $\bigcup_{j=1}^{k-1} U_j$ and restricting the resulting graph to the edges in U_k. Finally, form G_2 by contracting the edges of $\bigcup U_i$ where the union is over all U_i for which

$$\gamma\left(M.\left(\bigcup_{j=i}^{n} U_j\right)\middle| U_i\right)<2.$$

In any case for which the stated set is empty, let the corresponding G_i, $i\in\{0,1,2\}$ be the empty graph.

The significance of this principal partition, in the case that G is the graph of an electrical network is this: We wish to measure currents in some branches and voltage drops across other branches and to choose these branches in such a way that as few measurements as possible are made while still being able to use Kirchhoff's laws and Ohm's law to completely determine the currents in and voltage drops across all network branches. Now, in G_1, the complement of a spanning forest is larger than the forest. Thus the number of measurements in this part is minimized by measuring voltage drops on a spanning forest and using Kirchhoff's voltage law and Ohm's law to determine the remaining voltage drops and currents in G_1. In G_2, the complement of a spanning forest is smaller than the forest, and in G_0 they are the same. Hence the number of measurements in $G_0 \cup G_2$ is minimized by measuring currents in the complement of a spanning forest and using these and the already determined values in G_1 with Kirchhoff's current law and Ohm's law to determine the remaining voltage drops and currents. If both G_1 and G_2 are nonempty, the total number of measurements as described is less than is needed for either of the Kirchhoff laws used alone throughout the network.

Bruno and Weinberg generalized a refinement of the Kishi and Kajitani principal partition to matroids in 1970 and 1971 [3,4]. Their work thus provided the second version of the principal partition, in which, instead of dividing the sequence $(U_1, U_2, \ldots, U_n)$ at 2, as in Kishi and Kajitani, the division points are at all possible integer values. Bruno and Weinberg also showed how the principal partition can be used to classify the graphs for the Shannon two-person switching game [2], and they used their refinement to give an alternative statement of Edmonds' Cospanning-Sets Theorem [7].

The Bruno and Weinberg generalization was further refined independently by Tomizawa (1976) [23] and Narayanan and Vartak [14–16], as has already been described. They refined it further, in a manner that is best described by paraphrasing Tomizawa, and thus gave the fourth and final version of the principal partition. We say that a matroid M on set S is *atomic* [16], or *strongly irreducible* [23], if $\gamma(M \mid X) < \gamma(M)$ for every proper subset $X \subsetneq S$. The Tomizawa and Narayanan and Vartak refinement subdivides each of the matroids based on the sets in the sequence $(U_1, U_2, \ldots, U_n)$ into a sequence of strongly irreducible matroids.

To describe this refinement, we first recall that, by Theorem 16 those sets $X \subseteq U_i$, for which $g(X) = \gamma(N_i)$ form a distributive lattice under union and intersection, with perhaps the zero omitted. Adjoin zero to this lattice and let $(\emptyset = S_0, S_1, \ldots, S_k = U_i)$ be a longest chain in the lattice. For each $j \in \{1, 2, \ldots, k\}$, let $T_j = S_j \setminus S_{j-1}$. Narayanan and Vartak [16, p. 230] show that $N_j \mid S_j \, . \, T_j$ is uniformly dense for each j. By this and the maximality of the chain, each of the matroids $N_j \mid S_j \, . \, T_j$ is strongly irreducible. Further, the resulting partition of U_i is unique if the ordering is not considered. By carrying this partition out on each of the matroids N_i, we refine the principal partition of the matroid M into strongly irreducible minors.

We have thus shown the connection between the functions γ and η, and we have shown how these two functions can be used together to describe and clarify the principal partition of a matroid.

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References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Elsevier, New York, 1976).
[2] J. Bruno and L. Weinberg, A constructive graph-theoretic solution of the Shannon switching game, IEEE Trans. Circuit Theory 17 (1970) 74–81.
[3] J. Bruno and L. Weinberg, Principal partition and principal minors of a matroid, with application, in: Annals of the New York Academy of Sciences 175 (New York Acad. Sci., New York, 1970) 49–65.
[4] J. Bruno and L. Weinberg, The principal minors of a matroid, Linear Algebra Appl. 4 (1971) 17–54.
[5] P.A. Catlin, The reduction of graph families closed under contraction, to appear.
[6] W.H. Cunningham, Optimal attack and reinforcement of a network, J. ACM 32 (1985) 549–561.
[7] J. Edmonds, Lehman's switching game and a theorem of Tutte and Nash-Williams, J. Res. Nat. Bur. Standards 69B (1965) 73–77.

[8] P. Erdős, Elementary problem E3255, Amer. Math. Monthly 95 (1988) 259.
[9] D. Gusfield, Connectivity and edge-disjoint spanning trees, Inform. Process. Lett. 16 (1983) 87–89.
[10] G. Hardy, J.E. Littlewood and G. Polya, Inequalities (Cambridge University Press, Cambridge, 1952).
[11] M. Iri, A min-max theorem for the ranks and term-ranks of a class of matrices—An algebraic approach to the problem of the topological degrees of freedom of a network, Electron. and Comm. Japan 51A (5) (1968) 18–25.
[12] G. Kishi and Y. Kajitani, Maximally distinct trees in a linear graph, Electron. and Comm. Japan 51A (5) (1968) 35–42.
[13] A. Lehman, A solution of the Shannon switching game, J. SIAM 12 (1964) 687–725.
[14] H. Narayanan, Theory of matroids and network analysis, Doctoral Thesis, Indian Institute of Technology, Bombay (1974).
[15] H. Narayanan and N. Vartak, On molecular and atomic matroids, in: S.B. Rao, ed., Combinatorics and Graph Theory, Proceedings, Calcutta, 1980, Lecture Notes in Mathematics 885 (Springer, Berlin, 1981) 358–364.
[16] H. Narayanan and N. Vartak, An elementary approach to the principal partition of a matroid, Trans. IECE Japan E64 (1981) 227–234.
[17] C.St.J.A. Nash-Williams, Edge-disjoint spanning trees of finite graphs, J. London Math. Soc. 36 (1961) 445–450.
[18] C.St.J.A. Nash-Williams, Decompositions of finite graphs into forests, J. London Math. Soc. 39 (1964) 12.
[19] T. Ohtsuki, Y. Ishizaki and H. Watanabe, Network analysis and topological degrees of freedom, Electron. and Comm. Japan 51A (6) (1968) 33–40.
[20] C. Payan, Graphes équilibrés et arboricité rationnelle, European J. Combin. 7 (1986) 263–270.
[21] A. Ruciński and A. Vince, Strongly balanced graphs and random graphs, J. Graph Theory 10 (1986) 251–264.
[22] M.N.S. Swamy and K. Thulasiraman, Graphs, Networks and Algorithms (Wiley, New York, 1981).
[23] N. Tomizawa, Strongly irreducible matroids and principal partition of a matroid into strongly irreducible minors, Electron. and Commun. Japan 59A (2) (1976) 1–10.
[24] W.T. Tutte, On the problem of decomposing a graph into n connected factors, J. London Math. Soc. 36 (1961) 221–230.
[25] H. Watanabe, Activities on circuit theory in Japan, IEEE Trans. Circuits and Systems 31 (1984) 112–123.
[26] D.J.A. Welsh, Matroid Theory (Academic Press, London, 1976).