Supereulerian Complementary Graphs

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ABSTRACT

Nebeský in [12] show that for any simple graph with $n \geq 5$ vertices, either G or G^c contains an eulerian subgraph with order at least $n - 1$, with an explicitly described class of exceptional graphs. In this note, we show that if G is a simple graph with $n \geq 61$ vertices, then either G or G^c is supereulerian, with some exceptions. We also characterize all these exceptional cases. These results are applied to show that if G is a simple graph with $n \geq 61$ vertices such that both G and G^c are connected, then either G or G^c has a 4-flow, or both G and G^c have cut-edges. © 1993 John Wiley & Sons, Inc.

1. INTRODUCTION

We shall use the notation of Bondy and Murty [1], unless otherwise stated. Graphs may have multiple edges but not loops. For a simple graph G, G^c denotes the *complement* of G. The line graph of G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two edges are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G. Let $O(G)$ denote the set of vertices of odd degree in G. A graph G is *eulerian* if G is connected and $O(G) = \emptyset$. A graph is *supereulerian* if it has a spanning eulerian subgraph. The trivial graph K_1 is regarded as supereulerian. The maximum order of an eulerian subgraph of G is denoted by $\mathrm{eup}(G)$. Thus G is supereulerian if and only if $\mathrm{eup}(G) = |V(G)|$. In [8] and [9], Nebeský proved the following:

Theorem A (Nebeský [11]). Let G be a simple graph of order $n \geq 5$, then either $L(G)$ or $L(G^c)$ is hamiltonian. ∎

Theorem B (Nebeský [12]). Let G be a simple graph of order $n \geq 4$, then either $\mathrm{eup}(G) \geq n - 1$ or $\mathrm{eup}(G^c) \geq n - 1$, with an explicitly described class of exceptional graphs. ∎

In this article, we shall show the following:

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Theorem 1. Let G be a simple graph with order $n \geq 61$. One of the following holds:

(i) G is supereulerian.
(ii) G^c is supereulerian.
(iii) Both G and G^c have a vertex of degree one.
(iv) One of G or G^c can be contracted to a $K_{2,t}$ for some odd integer $t \geq 3$, and the other has either one or two vertices of degree 1.
(v) One of G and G^c can be contracted to $K_{1,p}$ for some integer $p \geq 1$, and the other has exactly one isolated vertex.

2. COLLAPSIBLE GRAPHS AND REDUCTIONS

In this section, we list some of the mechanisms needed in the proofs. Let G be a graph and let $X \subseteq E(G)$. The *contraction* G/X is the graph obtained from G by identifying the ends of each edge in X and deleting the resulting loops. If H is a subgraph of G, then we use G/H for $G/E(H)$.

Let $R \subseteq V(G)$ be a subset with even cardinality. An *R-subgraph* of G is a subgraph Γ of G such that $G - E(\Gamma)$ is connected and such that $O(\Gamma) = R$. A graph G is *collapsible* if for every $R \subseteq V(G)$, with $|R|$ even, G has an R-subgraph. Note that by definition, K_1 is collapsible. Note also that G is supereulerian if and only if G has an $O(G)$-subgraph. Thus any collapsible graph is supereulerian. As examples, we have the following (see page 33 of [2] and Theorem 11 of [3]):

Theorem C (Catlin [2,3]). All complete graphs of at least 3 vertices and all complete bipartite graphs $K_{m,n}$ with $\min\{m,n\} \geq 3$ are collapsible; the 2-cycles and $K_{3,3} - e$ ($K_{3,3}$ minus an edge) are collapsible. The 4-cycle, and more generally, the complete bipartite graphs $K_{2,t}$, are not collapsible. ∎

In [2], Catlin proved that every graph G has a unique collection of maximal collapsible subgraphs, say $H_1, H_2, \ldots, H_c$. Thus the graph $G' = G/(\cup_{i=1}^{c} E(H_i))$ is unique, and is called the *reduction* of G. If a vertex u in the reduction G' is the contraction image of a maximal collapsible subgraph U of G, then U is the *preimage* of u in G. A graph is *reduced* if it is the reduction of some graph. Note that by the definition of G', $E(G')$ can be regarded as a subset of $E(G)$. The next theorem is a summary of Theorems 3–5, 7, 8, and Corollary 1 in [2].

Theorem D (Catlin [2]). Let G be a graph, and let $F(G)$ denote the minimum number of extra edges that must be added to G so that the resulting graph has 2 edge-disjoint spanning trees. Each of the following holds:

(i) Let H be a collapsible subgraph of G. Then G is collapsible if and only if G/H is collapsible; G is supereulerian if and only if G/H is supereulerian.

(ii) G is reduced if and only if G has no nontrivial collapsible subgraphs, if and only if G is the reduction of itself. In particular, a reduced graph contains no cycles of length at most 3.
(iii) G is collapsible if and only if the reduction of G is K_1.
(iv) If G has 2 edge-disjoint spanning trees, that is $F(G) = 0$, then G is collapsible.
(v) If G is reduced and $G \neq K_2$, then $\delta(G) \leq 3$ and $F(G) \geq 2$.
(vi) If H_1 and H_2 are collapsible subgraphs of G and if H_1 and H_2 are not vertex-disjoint, then $H_1 \cup H_2$ is a collapsible subgraph of G.
(vii) If G has a spanning collapsible subgraph, then G is collapsible. ∎

Corollary 1. Let $K_{2,t} + e$ denote a simple graph spanned by a $K_{2,t}$ with $2t + 1$ edges. Then $K_{2,t} + e$ is collapsible.

Proof. Let $H = K_{2,t} + e$ and let e denote the only edge in $E(H) - E(K_{2,t})$. If e is incident with the two nonadjacent vertices of degree t in H, then every edge in H lies in a 3-cycle. Contracting all these 3-cycles one in a time until none left, we obtain a K_1, which is collapsible. By repeated applications of (i) of Theorem D, H is collapsible. If $t \geq 3$ and e is incident with two vertices of degree 2, then e is in common of two 3-cycles. By contracting these two 3-cycles, we obtain a graph in which every edge lies in a 2-cycle. By contracting these 2-cycles and by repeatedly applying (i) of Theorem D again, we conclude that H is also collapsible. ∎

Formula (1) below for a reduced graph G follows from Nash-Williams [10] and Tutte [13]. (A more detailed proof of (1) can be found in [5].)

$$F(G) = 2|V(G)| - |E(G)| - 2. \tag{1}$$

Corollary 2 below is a special case of a recent result in [6], which says that if G is a connected reduced graph with $F(G) = 2$, then $G \cong K_{2,t}$ for some $t \geq 1$. For the sake of completeness, we present a short proof of Corollary 2, using a fresh result of Chen.

Theorem E (Chen [7]). Let G be a reduced graph with at most 11 vertices and with $\kappa'(G) \geq 3$. If $F(G) \leq 3$, then either $G = K_1$ or G is the Petersen graph. ∎

Corollary 2. Let G be a connected reduced graph with at most 11 vertices. If $F(G) \leq 2$, then either $G = K_1$, or $G = K_2$, or $G = K_{2,t}$ for some $t \geq 1$.

Proof. By (ii) of Theorem D and by (1), it is easy to see that Corollary 2 holds for reduced graphs with order at most 4. Inductively, we assume that Corollary 2 holds for graphs with order smaller than G and that $n = |V(G)| \geq 5$. If $\kappa'(G) \geq 3$, then Corollary 2 follows immediately from Theorem E. Assume first that $\kappa'(G) = 2$ and that G has an edge cut X with

$|X| = 2$. Let G_1 and G_2 denote the two components of $G - X$ such that $F(G_1) \leq F(G_2)$. By (ii) of Theorem D, G_1 and G_2 are reduced also, and so by (1),

$$F(G_1) + F(G_2) = 2n - (|E(G)| - 2) - 4 = 2n|E(G)| - 2$$
$$= F(G) \leq 2.$$

Thus either $F(G_1) = 0$ and so $G_1 \cong K_1$, or $F(G_1) = 1$ and so $G_1 \cong K_2$, by (ii) and (v) of Theorem D. If $F(G_1) = 1$, then $F(G_2) = 1$ also and so $n = 4$, contrary to the assumption that $n \geq 5$. Hence we have $F(G_1) = 0$ and so $F(G_2) \leq 2$, By induction and by $n \geq 5$, $G_2 = K_{2,t}$ for some $t \geq 2$. It follows then by Theorem C and (ii) of Theorem D that G cannot contain a $K_{3,3} - e$ as a subgraph. Thus $G = K_{2,t+1}$, and the corollary follows. The proof when $\kappa'(G) = 1$ is similar. ∎

3. THE PROOFS

Note that any collapsible graph is supereulerian. We shall prove the following stronger versions of Theorem 1.

Theorem 2. Let G be a simple graph with $n \geq 61$ vertices. Then one of the following holds:

(i) G is collapsible.
(ii) G^c is collapsible.
(iii) Both G and G^c have a vertex of degree one.
(iv) One of G and G^c has its reduction isomorphic to $K_{2,t}$, for some integer $t \geq 2$, and the other has either one or two vertices of degree 1.
(v) One of G and G^c has its reduction isomorphic to $K_{1,p}$, for some integer $p \geq 1$, and the other has exactly one isolated vertex.

Proof of Theorem 1 (from Theorem 2). Assume the truth of Theorem 2. By (i) of Theorem D, (i) and (ii) of Theorem 1 follow from (i) and (ii) of Theorem 2, respectively. Obviously, (iii) and (v) of Theorem 2 imply (iii) and (v) of Theorem 1, respectively. Suppose that (iv) of Theorem 2 holds. If t is even, then $K_{2,t}$ is eulerian, and so by (i) of Theorem D, G or G^c is supereulerian. Therefore, (i) or (ii) of Theorem 1 must hold. If t is odd, then (iv) of Theorem 1 holds. ∎

To prove Theorem 2, we start with some lemmas.

Lemma 1. If H_1 and H_2 are two vertex-disjoint collapsible subgraphs of G, and if there are two edges e_1, e_2 in G such that each e_i has one end in H_1 and the other end in H_2, $(1 \leq i \leq 2)$, then $L = G[E(H_1) \cup E(H_2) \cup \{e_1, e_2\}]$ is collapsible.

Proof. Since $(L/H_1)/H_2$ is a 2-cycle, the fact that H_2 is collapsible implies that L/H_1 is collapsible, by (i) of Theorem D. Similarly, since H_1 and L/H_1 are collapsible, it follows that L is collapsible. ∎

Lemma 2. Let G be a 2-edge-connected graph with n vertices and let H be a collapsible subgraph of G.

(i) If $|V(H)| \geq n - 2$, then G is collapsible.
(ii) If $|V(H)| = n - 3$ and if each vertex in $V(G) - V(H)$ is adjacent to some vertex in $V(H)$, then G is collapsible.

Proof. Assume first that $|V(H)| \geq n - 2$. Then $|V(G/H)| \leq 3$ and $\kappa'(G/H) \geq \kappa'(G) \geq 2$. It follows that every edge of G/H lies in a 2-cycle or a 3-cycle of G/H. Thus applications of (i) of Theorem D show that G must be collapsible.

Assume then the hypothesis of (ii) of Lemma 2. By (i) of Lemma 2, we may assume further that $|V(H)| = n - 3$ and so $|V(G/H)| = 4$. Hence G/H is a 2-edge-connected graph with 4 vertices such that v_H, the vertex of G/H to which H is contracted, is adjacent to every other vertex in G/H, and so G/H is spanned by a $K_{1,3}$. This, together with $\kappa'(G/H) \geq 2$, implies that every edge of G/H must be lying in a 2-cycle or a 3-cycle of G/H. By applications of (i) of Theorem D, G/H (and therefore G), must be collapsible. ∎

For integer $i \geq 2$, let $D_i(G) = \{v \in V(G)\colon d_G(v) = i\}$.

Lemma 3. Suppose that G is a 2-edge-connected reduced graph with n vertices. Each of the following holds:

(i) $|D_2(G)| + |D_3(G)| \geq 4$.
(ii) If $|D_2(G)| + |D_3(G)| = 4$, then $D_3(G) = \emptyset$.
(iii) If $|D_2(G)| + |D_3(G)| = 5$, then $|D_2(G)| \geq 3$.

Proof. By (1), by (v) of Theorem D and by counting the incidences of G, we have

$$4n - 8 \geq 2|E(G)| \geq 2|D_2(G)| + 3|D_3(G)| + 4(n - |D_2(G)| - |D_3(G)|),$$

and so

$$2|D_2(G)| + |D_3(G)| \geq 8. \tag{2}$$

Thus (i), (ii), and (iii) of Lemma 3 follow from (2). ∎

Lemma 4. Let G be a simple graph with $n \geq 61$ vertices. If

$$\min\{\kappa'(G), \kappa'(G^c)\} \geq 2,$$

then either G or G^c is collapsible.

Proof. Suppose that G is not collapsible. By (iii) of Theorem D, G', the reduction of G, is nontrivial. Let $u_1, u_2, \ldots, u_m$ be the vertices in $D_2(G') \cup D_3(G')$, where $m = |D_2(G') \cup D_3(G')|$, and let U_i be the preimage of u_i in G $(1 \leq i \leq m)$ such that

$$|V(U_1)| \leq |V(U_2)| \leq \cdots \leq |V(U_m)|. \tag{3}$$

Let $X = \cup_{i=1}^{4} V(U_i)$ and $Y = V(G) - X$. Let $Y' \subseteq Y$ denote the set of vertices not adjacent in G to any vertices in X, and let $Y'' \subseteq Y$ denote the set of vertices each of which is adjacent in G to at least 3 vertices in X.

Lemma 5. Each of the following holds:

(i) $|Y''| \leq 3$;
(ii) If $|Y''| = 3$, then each vertex in Y'' is adjacent to exactly 3 vertices in X.

Proof. Suppose that $|Y''| \geq 4$. Let $y_1'', y_2'', y_3'', y_4'' \in Y''$ and let E'' denote the set of edges in $E(G')$ joining a y_i'' to a u_j $(1 \leq i, j \leq 4)$. Since each $y_i'' \in Y''$ is adjacent in G to at least 3 vertices in X, $|E''| \geq 12$. Let $H' = G'[E'']$. By (1), by $|V(H')| \leq 8$ and since G' is reduced, we have $F(H') \leq 2$. Since H' is not a $K_{2,t}$ nor a K_2, it follows by Corollary 2 that H' contains a nontrivial collapsible subgraph, contrary to the fact that G' is reduced, by (ii) of Theorem D. Thus (i) of Lemma 5 must hold. Assume then that $|Y''| = 3$. If there is some vertex $y \in Y''$ that is adjacent to one vertex in each U_i $(1 \leq i \leq 4)$, then the fact that $|Y''| = 3$ implies that $K_{3,3} - e$ must be a subgraph of G', a contradiction again. ∎

Proof of Lemma 4, continued. By (i) of Lemma 3, $m \geq 4$.

Case 1. $m = 4$. By (ii) of Lemma 3, $u_i \in D_2(G')$, $(1 \leq i \leq 4)$.

Suppose first that $|V(U_4)| \geq 5$. Let $V_4 \subseteq V(U_4)$ denote the set of vertices that are adjacent to some vertices in $V(G) - V(U_4)$. Since $u_4 \in D_2(G'')$, $|V_4| \leq 2$ and

$$\text{for any } x \in V(U_4) - V_4,\ x \text{ is not incident to any edge in } G'. \tag{4}$$

It follows by (4) that for all $v \in V(G) - V(U_4)$, and for all $x \in V(U_4) - V_4$, $vx \in E(G^c)$. Thus $G^c - V_4$ is spanned by a complete bipartite subgraph $K_{s,t}$, where $s = |V(U_4) - V_4| \geq 3$ and $t = |V(G) - V(U_4)|$. Since $\cup_{i=1}^{3} V(U_i) \subseteq V(G) - V(U_4)$, $t \geq 3$, and so by Theorem C, $G^c - V_4$ is spanned by a collapsible subgraph. By (vii) of Theorem D, $G^c - V_4$ is collapsible. Since $|V_4| \leq 2$ and by (i) of Lemma 2, G^c is collapsible.

Now suppose that $|V(U_4)| \leq 4$. Recall that $X = \cup_{i=1}^{4} V(U_i)$ and $Y = V(G) - X$. By (3) and by $|V(U_4)| \leq 4$, $4 \leq |X| \leq 16$. Since each $u_i \in D_2(G')$, there are at most 8 vertices in Y that are adjacent to some vertices in X, and so by $|X| \leq 16$ and by $n \geq 61$, $|Y'| \geq n - |X| - 8 \geq n - 24 > 3$. Hence by Theorem C, by $|X| \geq 4$ and $|Y'| \geq 4$, G^c has a maximal collapsible subgraph H containing $X \cup Y'$.

Since Y'' consists of vertices in $V(G) - X$ that are adjacent in G to at least 3 vertices in X, and since $u_i \in D_2(G')$, we have $3|Y''| \leq 2|X| = 8$ and so $|Y''| \leq 2$. Since every vertex in $V(G^c) - Y''$ is adjacent to at least two vertices in $X \subset V(H)$ in G^c, it follows by Lemma 1 and by the maximality of H that $V(G^c) - Y'' \subseteq V(H)$. Thus by $|Y''| \leq 2$, $|V(H)| \geq n - 2$, and so by (i) of Lemma 2, G^c is collapsible.

Case 2. $m \geq 5$.

Assume first that $m \geq 6$. Then $V(U_5) \cup V(U_6) \subset Y$. Thus by (3), $|Y|/2 \geq |V(U_5)| \geq |X|/4$ and so by $|X| + |Y| = n \geq 61$,

$$|Y| \geq \frac{n}{3} \geq 20. \tag{5}$$

Since u_1, u_2, u_3, u_4 are in $D_2(G') \cup D_3(G')$, at most 12 vertices in Y are adjacent to some vertices in X. Hence by (5), $|Y'| \geq 20 - 12 > 3$. Note that in G^c, every vertex in X is adjacent to every vertex in Y'. Since $|X| \geq 4$, it follows by Theorem C that G^c has a maximal collapsible subgraph H containing $X \cup Y'$.

By Lemmas 5 and 1, $|Y''| \leq 3$ and $V(G^c) - Y'' \subseteq V(H)$. If $|Y''| = 2$, then $|V(H)| \geq n - 2$ and so by (i) of Lemma 2, G^c must be collapsible. If $|Y''| = 3$, then by (ii) of Lemma 5, every vertex in $V(G^c - V(H))$ is adjacent to some vertex in H. Thus by (ii) of Lemma 2, G^c must be collapsible also.

Now we consider the case $m = 5$. By (iii) of Lemma 3, $|D_2(G')| \geq 3$. Thus there are at least two vertices in $\{u_1, u_2, u_3, u_4\}$ having degree 2 in G'. This implies that $|Y - Y'| \leq 10$. Since $V(U_5) \subseteq Y$ and by (3), $|Y| \geq |V(U_5)| \geq |X|/4$. This, together with $|X| + |Y| = n$, yields $|Y| \geq n/5$. Since $n \geq 61$, we have

$$|Y| \geq \frac{n}{5} \geq \frac{61}{5} \Longrightarrow |Y| \geq 13. \tag{6}$$

It follows by $|Y - Y'| \leq 10$ and (6) that $|Y'| \geq 3$ and so by imitating the proof for the case $m \geq 6$, we conclude that G^c is collapsible. ∎

Lemma 6. Let G be a simple graph with order $n \geq 61$. If $\kappa'(G) \leq 1$, then one of the following holds:

(i) G^c is collapsible.

(ii) The reduction of G^c is $K_{1,p}$ for some integer $p \geq 1$ and G has exactly one isolated vertex.
(iii) $D_1(G) \neq \emptyset$.

Proof. Let G be a graph satisfying the hypothesis of Lemma 6. We assume that $|D_1(G)| = 0$, since otherwise (iii) of Lemma 6 holds.

Suppose first that $\delta(G) \geq 2$. Thus G has two nontrivial vertex-disjoint subgraphs G_1 and G_2, together spanning G, such that either there are no edges in G joining G_1 to G_2 (when $\kappa'(G) = 0$), or there is exactly one edge e in G joining G_1 and G_2 (when $\kappa'(G) = 1$). Since $\delta(G) \geq 2$ and since G is simple, both $s = |V(G_1)| \geq 3$ and $t = |V(G_2)| \geq 3$. It follows that G^c is spanned by a $K_{s,t}$ or $K_{s,t} - e$, and so by Theorem C and (vii) of Theorem D, G^c is collapsible and (i) of Lemma 6 holds.

Assume then that G has isolated vertices. It is clear that if G has at least two isolated vertices v and u (say), then every vertex $w \in V(G) - \{u, v\}$ lies in a 3-cycle $vuwv$ in G^c. By applying (i) of Theorem D repeatedly, one can contract all these 3-cycles to result in a K_1, which is collapsible. It follows that G^c is collapsible. Hence we may assume that G has only one isolated vertex v (say). Let $Y = V(G) - \{v\}$. Then G^c is spanned by a $K_{1,n-1}$. It follows that every edge in $E(G) - E(K_{1,n-1})$ is either a cut-edge of G or lies in a 3-cycle of G. Hence by applying (i) of Theorem D, we conclude that either G^c is collapsible (when every edge of G lies in a 3-cycle), or the reduction of G^c is $K_{1,p}$ for some $p \geq 1$ (when G has some cut-edges). ∎

Lemma 7. Let G be a simple graph with $n \geq 61$ vertices. If $|D_1(G)| > 0$ and if $\kappa'(G^c) \geq 2$, then one of the following holds:

(i) G^c is collapsible.
(ii) The reduction of G^c is a $K_{2,t}$ for some integer $t \geq 2$ and G has either one or two vertices of degree 1.

Proof. Assume that G satisfies the hypothesis of Lemma 7. Choose a subset $X \subseteq D_1(G)$ and let $N(X)$ denote the set of vertices in $V(G) - X$ that are adjacent to some vertex of X in G. Since $X \subseteq D_1(G)$,

$$|N(X)| \leq |X|. \tag{7}$$

Let $Y' = V(G) - (X \cup N(X))$ be the set of vertices in G that are not incident with vertices in X and let $s = |Y'|$. Consider the following cases.

Case 1. $|D_1(G)| \geq 3$.

Choose $X = \{v_1, v_2, v_3\} \subseteq D_1(G)$. Since $n \geq 61$ and by (7), $s = |Y'| \geq n - 6 \geq 55$. Thus G^c contains a subgraph H spanned by a $K_{3,s}$, which is collapsible, by Theorem C and by (vii) of Theorem D. Since $X \subseteq D_1(G)$, either $|N(X)| \leq 2$ or every vertex in $N(X)$ is adjacent in G^c to at least two vertices in $X \subseteq V(H)$, and so by $\kappa'(G^c) \geq 2$ and by Lemma 1 (when $|N(X)| \geq 3$) or by Lemma 2 (when $|N(X)| \leq 2$), G^c is collapsible.

Case 2. $|D_1(G)| = 2$. Let $X = \{v_1, v_2\} = D_1(G)$. Then

$$G^c[X \cup Y'] \text{ is spanned by a } K_{2,s}\,, \tag{8}$$

where $s = |Y'| \geq n - 4 \geq 57$. Let $L = G[Y']$. If L is not a complete subgraph of G, or if $v_1v_2 \notin E(G)$, then by (8), $G^c[X \cup Y']$ is spanned by a $K_{2,s} + e$, and so by Corollary 1, $G^c[X \cup Y']$ is collapsible. It then follows by $\kappa'(G^c) \geq 2$ and by (i) of Lemma 2 that G^c is collapsible.

Thus we may assume that L is a complete subgraph of G and $v_1v_2 \in E(G)$. Since $v_1, v_2 \in D_1(G)$, we must have $N(X) = \varnothing$ and so $V(G) = X \cup Y$. Therefore $G^c = G^c[X \cup Y] = K_{2,n-2}$, by (8), and so (ii) of Lemma 7 holds.

Case 3. $|D_1(G)| = 1$. Let $X = \{v_1\} = D_1(G)$. Then

$$G^c[X \cup Y'] \text{ is spanned by a } K_{1,s} \text{ with } d_{G^c}(v_1) = s\,, \tag{9}$$

By (7), we denote $N(X) = \{w\}$ and let $N(w)$ denote the vertices that are adjacent to w in $G - v_1$. Since $\kappa'(G^c) \geq 2$,

$$|Y' - N(w)| = d_{G^c}(w) \geq 2\,. \tag{10}$$

By (9), every vertex in $Y' - N(w)$ is adjacent in G^c to both v_1 and w, and so $F = G^c[\{v_1, w\} \cup (Y' - N(w))]$ is spanned by a $K_{2,p}$, where $p = |Y' - N(w)|$. Note that by (10), $p \geq 2$.

Claim. one of the conclusions of Lemma 7 must hold if there is an edge in G^c joining two vertices in $Y' - N(w)$.

For in this case, F is collapsible by Corollary 1, and so G^c has a maximal collapsible subgraph F' containing $V(G) - N(w)$. If $N(w) \subseteq V(F')$, then $G^c = F'$ is collapsible and (i) of Lemma 7 holds. Hence we may assume that there is some $w' \in N(w) - V(F')$. By $\kappa'(G^c) \geq 2$, w' must be adjacent in G^c to a vertex $u \in V(G) - \{w', w, v_1\} \subseteq Y'$. If $u \in N'(w)$, then since $C = w'uv_1w'$ is a collapsible subgraph of G^c (Theorem C), and since $V(C) \cap V(F') = \{v_1\} \neq \varnothing$, $F' \cup C$ is a collapsible subgraph in G^c, by (vi) of Theorem D, contrary to the maximality of F'. If $u \in Y' - N(w)$, then w' is adjacent in G^c to two vertices (v_1 and u) in F', and so by Lemma 1, $G^c[V(F') \cup \{w'\}]$ is collapsible, contrary to the maximality of F' again. ∎

By the claim, we may assume that either G^c is collapsible or

$$F = G^c[\{v_1, w\} \cup (Y' - N(w))] \cong K_{2,p}. \tag{11}$$

The argument in the proof of the claim also enables us to assume, when (i) of Lemma 7 fails, that

$$\text{there are no edges in } G^c \text{ joining } N(w) \text{ to } Y' - N(w). \tag{12}$$

By (12) and since $\kappa'(G^c) \geq 2$, every vertex in $N(w)$ must be adjacent in G^c to another vertex in $N(w)$ and so by Theorem C, $G^c[N(w) \cup \{v_1\}]$ is collapsible. This, together with (11) and (12), implies that $G^c[N(w) \cup \{v_1\}]$ is the only nontrivial collapsible subgraph of G^c and so the reduction of G^c is $K_{2,p}$.

This completes the proof of Lemma 7. ∎

Proof of Theorem 2. Let G be a simple graph with $n \geq 61$ vertices. Assume that G is a counterexample to Theorem 2. Then

$$D_1(G) = \emptyset \text{ or } D_1(G^c) = \emptyset. \tag{13}$$

By Lemma 4, we may assume that $\kappa'(G) \leq 1$. Since G is a counterexample, (i) and (ii) of Lemma 6 cannot hold, and so by $\kappa'(G) \leq 1$ and by Lemma 6, $D_1(G) \neq \emptyset$. Thus by (13), $D_1(G^c) = \emptyset$ and so by Lemma 6 with G^c replacing G, we must have $\kappa'(G^c) \geq 2$. By Lemma 7, one of the conclusions of Theorem 2 must hold, a contradiction. ∎

4. AN APPLICATION

A graph H is *even* if every vertex of H has even degree. A collection of even subgraphs $H_1, H_2, \ldots, H_m$ of G is called a *cycle double cover* of G if every edge of G lies in exactly two of the H_i's. (See [8] for a survey of cycle double covers.) If G has a cycle double cover with 3 even subgraphs, then we say that G has a *3-colorable* cycle double cover. It is well known that a graph has a 3-colorable cycle double cover if and only if G has a 4-flow. (See [9] for a survey of flows.) Tutte has conjectured [8,9] that any 2-edge-connected graph without subgraph contractible to the Petersen graph has a 3-colorable cycle double cover. In [4], Catlin showed

Theorem F (Catlin [4]). Let H be a subgraph of G such that either H is collapsible or H is isomorphic to the 4-cycle. Then G/H has 3-colorable cycle double cover if and only if G has a 3-colorable cycle double cover. ∎

Corollary 3. Let G be a simple graph with $n \geq 61$ vertices such that both G and G^c are connected. Then one of the following holds:

(i) G has a 3-colorable cycle double cover.
(ii) G^c has a 3-colorable cycle double cover.
(iii) Both G and G^c have vertices of degree one.

Proof. Assume that (iii) of Corollary 3 fails. By Theorem 2 and since (iii) and (v) of Theorem 2 do not hold, either G or G^c is collapsible, or the reduction of G or G^c is iosmorphic to a $K_{2,t}$ for some $t \geq 2$. In either case, Theorem F says that G or G^c must have a 3-colorable cycle double cover. ∎

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