

Uniformly dense generalized prisms over graphs

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Abstract

The *density* of a nontrivial graph G is $g(G) = |E(G)|/(V(G) - \omega(G))$, where $\omega(G)$ denotes the number of components of G . A graph is *uniformly dense* if for any subgraph H of G , $g(H) \leq g(G)$. The problem of building large uniformly dense graphs with a given uniformly dense as an induced subgraph is one which can arise in various applications. Given vertex-disjoint graphs G_1 and G_2 of order n , and given a k -regular bipartite graph B having the two sets of vertices for its bipartition, we call $G_1 \cup G_2 \cup B$ a *generalized prism* $A_k(G_1, G_2)$ over G_1 and G_2 . In the paper of which this note is a summary, we show that if G_1 and G_2 are uniformly dense graphs with the same density, then for k depending only on G_1 and G_2 and for any k -regular bipartite graph B , $A_k(G_1, G_2)$ is also a uniformly dense graph. Further, we prove that every simple vertex-transitive graph is uniformly dense, and we show that the Cartesian product of two simple connected graphs, each being vertex-transitive or edge-transitive, is uniformly dense.

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1. Introduction.

We follow the notation of Bondy and Murty [BM] for graphs and Welsh [W] for matroid theory unless otherwise noted. Graphs and matroids considered in this note are loopless, but multiple edges are allowed. The *rank* ρG of a graph G is the number $|V(G)| - \omega(G)$, where $\omega(G)$ is the number of components of G . If X is a subset of $E(G)$, then the *rank* ρX of X is the rank of $G[X]$. For a nontrivial graph G with $X \subseteq E(G)$, the *contraction* G/X is the graph obtained from G by identifying the ends of each edge in X and by deleting the resulting loops. If M is a matroid on set S and if $X \subseteq S$, the *contraction* M/X is the matroid $M.(S \setminus X)$. The *density* of a nontrivial graph G is $g(G) = |E(G)|/\rho G$. Similarly, the *density* of a matroid M on set S with rank function ρ is $g(M) = |S|/\rho M$, and for any subset $X \subseteq S$ of S , the *density* of X is $g(X) = |X|/\rho X$. A graph or matroid M on (edge) set S is *uniformly dense* if for every subset $X \subseteq S$ of S , $g(X) \leq g(M)$. Uniformly dense graphs and matroids have been studied by many workers; see [CGH], [LL], [N1], [N2], and [T], among others.

There are two invariants associated with uniformly dense graphs and matroids. The *strength* of graph or matroid M on (edge) set S is defined as

$$\eta(M) = \min_{X \subseteq S} g(M/X),$$

where the minimum is taken over the subsets X with $\rho X \neq \rho M$. The *fractional aboricity* is defined as

$$\gamma(M) = \max_{\emptyset \neq X \subseteq S} g(X).$$

Note that

$$\eta(M) \leq g(M) \leq \gamma(M).$$

for any matroid or graph M , and that M is uniformly dense if and only if $g(M) = \gamma(M)$.

It has been indicated by several authors ([Cu], [G] and [H], among others) that these parameters can be used as a measure of invulnerability of networks. Other applications of these parameters in the electrical network analysis can be found in [K], [O] and [T], among others.

In this note we summarize the results in a recent completed paper [HLLW]. The proofs of these results use the following theorem from [CGH]:

Theorem A ([CGHL], Theorem 3) Let M be a loopless matroid. Then the following are equivalent:

- (i) $\gamma(M) = g(M)$.
- (ii) $\eta(M) = g(M)$.
- (iii) $\eta(M) = \gamma(M)$. \square

2. Prisms.

Given vertex-disjoint graphs G_1 and G_2 of order n , and given a k -regular bipartite graph B having the two sets $V(G_1)$ and $V(G_2)$ for its bipartition, we define $A = A_k(G_1, G_2) = G_1 \cup G_2 \cup B$ as a generalized prism over G_1 and G_2 . This extends the generalized prisms defined in [PR], among others.

We start with some lemmas. The most interesting of these is the following:

Lemma 3 Let G be a loopless graph. Then $\delta(G) \geq \eta(G)$. \square

Our first theorem gives us a very wide class of uniformly dense graphs.

Theorem 1 Let G_1 and G_2 be connected uniformly dense graphs on disjoint sets of $n \geq 2$ vertices. Suppose $g(G_1) = g(G_2)$. Let k be a positive integer such that $(k-1)n(n-1) < |E(G_1)| \leq kn(n-1)$. Let B be a k -regular bipartite graph whose two sides are the vertex sets of G_1 and G_2 . Then $A = A_k(G_1, G_2) = G_1 \cup G_2 \cup B$ is uniformly dense, $g(A) \geq g(G_1) = g(G_2)$, and $g(A) > k$. \square

The example below shows that even if $A_k(G, H)$ is uniformly dense for some G and H , it does not necessarily follow that either G or H is uniformly dense.

Example. Form a graph G from a path of length three by adding one edge in parallel to each of the end edges of the path. Let H be vertex-disjoint from G but isomorphic to G . Number the vertices of G along the path as 1, 2, 3, 4 and number the vertices of H along the corresponding path as 2, 1, 3, 4. To $G \cup H$ add the four edges joining like-numbered vertices of G and H . The result is one of the graphs $A = A_1(G, H)$. It is easy to see that A is the union of 2 edge-disjoint spanning trees; consequently A is

uniformly dense. However, one can easily find that $\eta(G) = 1$ and $\gamma(G) = 2$, (for example, by using the η -reduction method in [CGHL]), and so G and H are not uniformly dense, by Theorem A.

3. Transitivity.

A matroid M on a finite set S is *transitive* if for every non-empty proper subset E of S , there is an automorphism α_E of M such that $\alpha_E(E) \neq E$.

This definition of transitivity is not quite standard, but its equivalence to a standard definition (M is transitive if, for each $x, y \in S$, there is an automorphism α of M such that $\alpha x = y$) is easily seen by showing, for each $x \in S$, that the set

$$\{y \in S : \text{there is an automorphism } \alpha \text{ of } M \text{ such that } \alpha x = y\}$$

is all of S .

Narayanan stated in [N1] and [N2] the following theorem (included here for completeness). We present a short proof here.

Theorem 2 If a loopless matroid M on a finite set S is transitive, then it is uniformly dense.

Proof: Let T be a maximal subset of S such that $g(T) = \gamma(M)$; then $T \neq \emptyset$ and T is closed ($\rho\sigma(T) = \rho(T)$). Suppose $T \neq S$. Then there is an automorphism α of M such that $\alpha(T) \neq T$. Since $|\alpha(T)| = |T|$ and $\rho(\alpha(T)) = \rho(T)$, we have $g(\alpha(T)) = g(T) = \gamma(M)$. By Theorem B, $g(T \cup \alpha(T)) = \gamma(M)$, a contradiction of the choice of T as maximal. Hence $T = S$ and M is uniformly dense. \square

A graph G is *edge-transitive* if for any pair of edges $e, f \in E(G)$, there is a automorphism ϕ of G such that $\phi(e) = f$. Similarly, a graph G is *vertex-transitive* if for any pair of vertices $u, v \in V(G)$, there is a automorphism ϕ of G such that $\phi(u) = v$.

In the case of graphs, matroid transitivity becomes edge-transitivity. What about vertex-transitivity? Let H be a graph consisting of two vertices joined by three parallel edges. If $R = A_1(H, H) = H \times K_2$, then $g(R) = 8/3$ while $g(H) = 3 > 8/3$. Since R is vertex-transitive, this shows that vertex-transitivity need not entail uniform density. However, we can get

uniform density by restricting ourselves to simple graphs.

Theorem 3 If simple graph G with n vertices is vertex transitive, then G is uniformly dense. \square

4. Cartesian Products of Graphs.

We use the definition of Chartrand et al [CL] for Cartesian products of graphs. Let G_1 and G_2 be two graphs. The *Cartesian product* (or *product*, for short), denoted by $G_1 \times G_2$, has $V(G_1 \times G_2) = V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$, or $u_2 = v_2$ and $v_1u_1 \in E(G_1)$.

In this section of the paper, we are interested in finding uniformly dense Cartesian products of graphs. Edge-transitivity is not preserved when taking Cartesian products. For example, $K_3 \times K_2 = A_1(K_3, K_3)$ is not edge-transitive, even though both K_3 and K_2 are. Further, the Cartesian product of two arbitrary graphs need not be uniformly dense. For example, letting H be a graph with two vertices joined by three parallel edges, H is both edge-transitive and vertex-transitive. But the graph R formed in the previous section using H is not uniformly dense. However, if we restrict ourselves to simple graphs, we can get uniform density.

To get at this, we begin by showing that the Cartesian product of two simple vertex-transitive graphs is vertex-transitive.

Theorem 4 Let G_1 and G_2 be two simple graphs, and let $v_i, u_i \in V(G_i)$, ($1 \leq i \leq 2$). If there are automorphisms ψ_i of G_i such that $\psi_i(u_i) = v_i$, then $G_1 \times G_2$ has an automorphism ϕ sending (u_1, u_2) to (v_1, v_2) . \square

Corollary 1 If G_1 and G_2 are simple vertex-transitive graphs, then $G_1 \times G_2$ is also vertex-transitive. Thus $G_1 \times G_2$ is uniformly dense. \square

Corollary 2 For any positive integer n and m with $n + m > 2$, $K_n \times K_m$, $K_n \times C_m$ and $C_n \times C_m$ are uniformly dense. \square

Next we take care of edge-transitive graphs.

Theorem 5 If G_1 and G_2 are edge-transitive connected simple graphs, then $G_1 \times G_2$ is uniformly dense. \square

Our last theorem combines an edge-transitive graph with a vertex-transitive graph.

Theorem 6 Suppose G_1 and G_2 are connected simple graphs. Suppose G_1 is edge-transitive and G_2 is vertex-transitive. Then $G_1 \times G_2$ is uniformly dense. \square

As an example of a consequence of this theorem, let R' be the Cartesian product of K_3 and K_2 ; this graph is vertex-transitive but not edge-transitive. Let S be any star with at least 3 end vertices; this graph is edge-transitive but not vertex-transitive. In accordance with Theorem 6, $S \times R'$ is uniformly dense.

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