Small cycle covers of planar graphs

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We follow the notation of Bondy and Murty [3], unless stated otherwise. Two edges e and e' of G are parallel or are a pair of multiple edges if e and e' have the same ends in G. Graphs in this note are finite and may have multiple edges but loops are not allowed. Let G be a graph and let $X \subseteq E(G)$, the contraction G/X is the graph obtained from G by identifying the ends of each edge in X and deleting the resulting loops. When H is a connected subgraph, we use G/H for $G/E(H)$.

Let G be a 2-edge-connected graph with n vertices. A cycle cover (CC) of G is a collection $\mathcal{C}$ of cycles in G such that every edge of G lies in at least one cycle in $\mathcal{C}$. A cycle double cover (CDC) of G is is cycle cover $\mathcal{C}$ of G such that every edge of G lies in exactly two cycles of $\mathcal{C}$. Let $cc(G)$ denote the minimum number of cycles in a CC of G and $sc(G)$ the minimum number of cycles in a CDC of G. Bondy posed the following conjectures in [1]:

Conjecture SCDC: If G is a 2-edge-connected simple graph with n vertices, then

$$sc(G) \leq n - 1. \tag{1}$$

Conjecture SCC: If G is simple and 2-edge-connected, then

$$cc(G) \leq \frac{2n-1}{3}. \tag{2}$$

These conjectures, as pointed out by Bondy (see [1] and [2]), are closely related to the Cycle Double Cover Conjecture ([11] and [12]) and the Hajós' conjecture (see [1] and [2]) on decomposing even graphs into cycles. Previously, Bondy and Seyffarth proved the following results.

Theroem 1 (Bondy and Seyffarth [2], [10]) Let G be a simple plane triangulation with n vertices. Then

$$sc(G) \leq n - 1.$$

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Theorem 2 (Seyffarth [9], [10]) If G is a simple plane triangulation on n vertices with $\Delta(G) \geq 8$ and $n > (3\Delta(G)/2) + 1$, then there exists a CDC of G with at most $n-2$ cycles.

Theorem 3 (Seyffarth [10]) If G is a simple 4-connected planar graph with n vertices, then

$$cc(G) \leq n - 1.$$

The method used by Seyffarth in the proof of Theorem 2 can be applied to show the following:

Theorem 4 For any integer $m > 0$, there is an integer $N(m)$ such that for any simple plane triangulation G with diameter of G at least $N(m)$,

$$sc(G) \leq n - m.$$

Proof: For the sake of completeness, we repeat some of Seyffarth's argument here. For given $m > 0$, we choose $N(m) \geq 3m + 1$. Let G be a simple plane triangulation with diameter at least $N(m)$. Then G has a path $P = v_1 v_2 \cdots v_k$ where k is the diameter of G such that the distance of v_i and v_j in G is the same as the distance of v_i and v_j in P, for any $i, j \in \{1, 2, \cdots, k\}$. Since G is a plane triangulation, for any $v \in V(G)$, let C_v be the cycle formed by the neighbors of v in G. Then $\mathcal{C} = \{C_v : v \in V(G)\}$ is a CDC of G.

Fix an i, $(1 \leq i \leq m)$. For each of the vertex $v \in N(v_{3i-1})$, we replace the segment $v^- v v^+$ of C_v by $v^- v_{3i-1} v v^+$ and denote the resulting cycle by C'_v. This can be done for all i, $(1 \leq i \leq m)$ since v_{3i-1} and v_{3j-1} have distance at least 3 in G. Thus

$$\mathcal{C}' = \{C_v : v \in \bigcup_{i=1}^{m} (N(v_{3i-1}) \cup \{v_{3i-1}\})\} \bigcup (\bigcup_{i=1}^{m} \{C'_v : v \in N(v_{3i-1})\})$$

is a CDC of G. This proves Theorem 4. □

We consider the SCC conjecture for planar graphs and we start with a multigraph approach. For a graph G, define a relation on $E(G)$ such that e is relatred to e' if and only if either $e = e'$ or e and e' are parallel in G. It is easy to check that this is an eqivalence realtion. Let $[e]$ denote the equivalence class containing e and let $[G]$ denote the collection of all equivalence classes. Define

$$\mu(G) = \sum_{[e] \in [G]} (\|[e]\| - 1).$$

Then G is simple if and only if $\mu(G) = 0$. The multiple version of Conjectures SCDC and SCC can then be stated below.

Conjecture SCDCM (multigraph version): If G is a 2-edge-connected graph with n vertices, then

$$sc(G) \leq n - 1 + \mu(G).$$

Conjecture SCCM (multigraph version) If G is a 2-edge-connected graph with n vertices, then

$$cc(G) \leq \frac{2n-1}{3} + \frac{\mu(G)}{2}.$$

Proposition 5 Each of the following holds:
(i) SCCM implies SCC.
(ii) SCDCM and SCDC are equivalent.

Proof: Only part (ii) needs a proof. Since G is simple if and only if $\mu(G) = 0$, SCDCM imples SCDC. Conversely, we assume that truth of SCDC and consider a 2-edge-connected graph G with n vertices. By contradiction, we assume that G is a counterexample to Conjecture SCDCM with $|V(G)| + \mu(G)$ minimized. By the truth of SCDC, $mu(G) > 0$, and so there is some $e \in E(G)$ with $|[e]| > 1$. Let $e, e' \in [e]$ and let $G' = G - e$. Since $|[e]| > 1$ and since G is 2-edge-connected, G' is also 2-edge-connected. Note that $\mu(G') = \mu(G) - 1$. By the minimality of G, we have $cc(G') \leq n - 1 + \mu(G')$. Let $\mathcal{C}$ be a CDC of G' with $|\mathcal{C}| = cc(G')$ and let $C_1, C_2 \in \mathcal{C}$ be the two cycles that contain e'. Thus by letting $C_2' = C_2 - e' + e$ and $F = G[\{e, e'\}]$, we obtain a CDC $(\mathcal{C} - \{C_2\}) \cup \{C_2', F\}$ of G and so by $\mu(G') = \mu(G) - 1$, we have

$$cc(G) \leq cc(G') + 1 \leq n - 1 + \mu(G),$$

contrary to the assumption that G is a counterexample. □

For a planar multigraph G, G is a triangulation if there is a plane embedding of G in which every face has degree 2 or 3. By an inductive argument, we proved the following:

Theorem 6 ([6]) If G is a planar triangulation with $n \geq 6$ vertices, then

$$cc(G) \leq \frac{2n-3}{3} + \frac{\mu(G)}{2}.$$

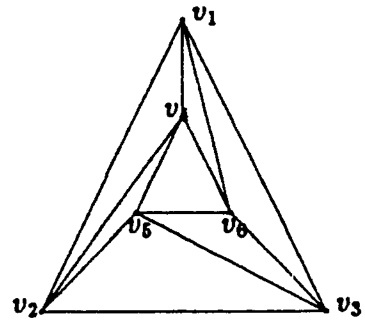
Theorem 7 ([5]) If G is a 2-edge-connected planar graph with $n \geq 6$ vertices, then

$$cc(G) \leq \frac{2n-2}{3} + \frac{\mu(G)}{2}.$$

Some reduction techniques for planar graphs are developed in the investigation of conjecture SCCM. Lemma 8 below is a typical one of such reduction lemmas.

Lemma 8 ([6]) Let G be a graph and $H = \Gamma_1$ (see Figure 1) be a subgraph of G such that the vertices of attachment of H in G are lying in $\{v_1, v_2, v_3\}$. Let $e \notin E(G)$ be an edge parallel to v_2v_3. Then let $G' = (G - V_H) + e_2$ and we have

$$cc(G) \leq cc(G') + 1. \tag{3}$$



Γ_1

Figure 1 : The graph Γ_1

There are other similar reduction lemmas using deletion and/or contraction. These lemmas are used to reduce the order of a minimum counterexample, and so we basically need to consider graphs with small orders in the proofs of Theorems 4 and 5.

It is natural to approach these problems by considering the extremal graphs. We restricted ourselves to graphs without subdivisions of K_4 and were able to characterize all extremal graphs within this family.

To describe these results, we introduce some terms. An arc of a graph G is an (x,y)-path P of G with $x, y \in V(G)$ possibly $x = y$), such that all internal vertices of P have degree 2 in G. A maximal arc is one that cannot be extended in G. The length of an arc P is $|E(P)|$. We regard K_2 as an arc of length 1 and K_1 as an arc of length 0 (with identical ends).

Let $\mathcal{A}(G)$ denote the collection of all maximal arcs A with $|E(A)| \geq 2$. For any $A \in \mathcal{A}(G)$, A is a cycle arc if $G[E(A)]$ is a cycle of G; A is a cyclic arc if $G[E(A)]$ is not a cycle but there is an arc A' in G such that $G[E(A) \cup E(A')]$ is a cycle; A is an acyclic arc if G is either a cycle arc nor a cyclic arc.

For each $A \in \mathcal{A}(G)$, define $b_G(A)$ as follows:

$$b_G(A) = \begin{cases} |E(A)| - 3 & \text{if } A \text{ is a cycle arc} \\ |E(A)| - 2 & \text{if } A \text{ is a cyclic arc} \\ |E(A)| - 1 & \text{if } A \text{ is an acyclic arc} \end{cases}$$

and define

$$b(G) = \sum_{A \in \mathcal{A}(G)} b_G(A).$$

Let $t \geq 3$ and $s_t \geq \ldots \geq s_2 \geq s_1 \geq 1$ be integers. Let the t arcs of length 2 of $K_{2,t}$ be labeled by $A_1, A_2, \ldots, A_t$. Define $K_{2,t}(s_1, \ldots, s_t)$ to be the graph obtained from $K_{2,t}$ by replacing A_i by a path of length s_i, $(1 \leq i \leq t)$. For convenience, we regard a cycle of length $s_1 + s_2$ as a $K_{2,2}(s_1, s_2)$.

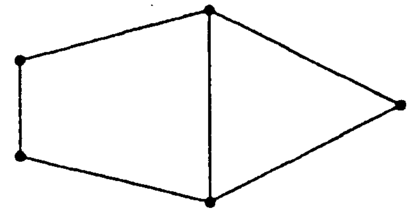


Figure 2: $K_{2,3}(1,2,3)$.

Let $\mathcal{K}$ denote the collection of graphs such that $G \in \mathcal{K}$ if and only if each block of G is a $K_{2,3}(1,s_2,s_3)$, for some $s_3 \geq s_2 > 1$. Let $\mathcal{K}'$ denote the subcollection of $\mathcal{K}$ such that $G \in \mathcal{K}'$ if and only if each block of G is a $K_{2,3}(1,2,2)$. Note that by definition, every graph in $\mathcal{K}$ is simple.

Theorem 9 ([7]) Let G be a 2-edge-connected simple graph with n vertices. If G has no subdivision of K_4, then

$$cc(G) \leq \frac{2(n-1-b(G))}{3}, \tag{4}$$

where equality holds if and only if $G \in \mathcal{K}$. Moreover, if $b(G) = 0$, then equality holds in (2) if and only if $G \in \mathcal{K}'$.

Let k be a nonnegative integer. Given graphs G_1 and G_2, if for $i \in \{1,2\}$, G_i has an arc P_i with $|E(P_i)| = k$ and with the ends of P_i being $x_i, y_i \in V(G_i)$, then one can define the k-arc-sum of G_1 and G_2 to be the graph obtained from the vertex disjoint union of G_1 and G_2 by deleting all the internal vertices of P_2 and identifying x_1 with x_2 and y_1 with y_2. Thus the k-arc-sum of G_1 and G_2 contains G_1 and G_2 as subgraphs. If G is a k-arc-sum of G_1 and G_2 with

$$|E(G_i)| < |E(G)|, (1 \leq i \leq 2), \tag{5}$$

then G is called a proper k-arc-sum of G_1 and G_2.

If G is a proper 1-arc-sum of G_1 and G_2, then the edge e shared commonly by G_1 and G_2 is called a sum-edge of G. For each integer $i \geq 3$, define $\mathcal{K}(i)$ to be the family of simple graphs satisfying each of the following:

(i) all k-cycles, $3 \leq k \leq i$, are in $\mathcal{K}(i)$;

(ii) $G \in \mathcal{K}(i)$ if and only if either G is a cycle of length at most i, or G is a 0-arc-sum or a 1-arc-sum of G_1 and G_2 for some $G_1, G_2 \in \mathcal{K}(i)$, such that every k-cycle of G, $3 \leq k \leq i$, has at most two sum-edges of G, and such that if a k-cycle C has exactly two sum-edges in G, $3 \leq k \leq i$, then these two sum-edges are adjacent in C.

Define $\mathcal{K} = \cup_{i \geq 3} \mathcal{K}(i)$.

Theorem 10 ([9]) Let G be a 2-edge-connected simple graph with n vertices. If G has no subdivision of K_4, then

$$sc(G) \leq n - 1 - b(G), \tag{6}$$

where equality holds if and only if $G \in \mathcal{K}$. Moreover, if $b(G) = 0$, then equality holds in (2) if and only if $G \in \mathcal{K}(3)$.

The proofs of both Theorems 8 and 9 depends on the following proposition, which can be derived from Dirac's theorem ([4]) that if G is a nontrivial simple graph without subdivision of K_4, then $\delta(G) \leq 2$.

Proposition 11 ([9]) Let G be a nontrivial 2-edge-connected graph. If G contains no subdivision of K_4, then either G is a cycle or G is a proper k-arc-sum of some graphs G_1 and G_2, for some $k \leq 0$, with $\kappa'(G_i) \geq 2$, $(1 \leq i \leq 2)$. Moreover, if G is simple and not a cycle, then both G_1 and G_2 can be chosen as simple graphs.

In view of Theorem 4, we conclude this note with the following conjecture:

Conjecture: For any integer $m > 0$, there exists an integer $N(m)$ such that for any simple plane triangulation G with $|V(G)| \geq N(m)$,

$$cc(G) \leq \frac{2n}{3} - m.$$

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