A note on uniformly dense matroids

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April 5, 1991

Abstract

This note gives a characterization of uniformly dense matroids in terms of contractions of its restricted submatroids.

INTRODUCTION

The matroids considered in this note are loopless matroids on finite nonempty sets. For convenience, we use M to denote the matroid as well as its underlying set. The notation of Welsh [8] will be used. Let M be a matroid and let $N \subseteq M$ be a subset of M. Then when no confusion arises, we use N to denote the restricted submatroid $M|N$. For convenience, we call N a submatroid of M. The closure of N in M is denoted by $\sigma(N)$. If $M = \sigma(N)$, then we say that N spans M. If N does not span M, then we define the (loopless) contraction by

$$M/N = M.(M - \sigma(N)),$$

where $M.X$ is the contraction notation defined in [8]. Thus if ρ, ρ_c denote the rank functions of M and M/N, respectively, then by a formula in [8] (page 62), for any subset $X \subseteq M/N$,

$$\rho_c(X) = \rho(X \cup N) - \rho(N). \tag{1}$$

Let M be a loopless matroid with rank function ρ and let

$$g(M) = \frac{|M|}{\rho M}.$$

Define the fractional arboricity of M to be

$$\gamma(M) = \max_{\emptyset \neq N \subseteq M} g(N),$$

Utilitas Mathematica **40** (1991), pp. 251-256

and the strength of M to be

$$\eta(M) = \min_{N \subset M} \frac{|M - N|}{\rho(M) - \rho(N)},$$

where the minimun is taken over all subsets $N \subset M$ such that $\rho(M) > \rho(N)$. Call a loopless matroid M unformly dense if $\gamma(M) \geq \gamma(N)$, for any $\emptyset \neq N \subseteq M$.

Uniformly dense matroids and their graph counterparts have been studied by many. See [1] - [7], among others. In [1], the following are shown:

Theorem A ([1], Theorem 3) Let M be a matroid with rank function ρ. The following are equivalent:
(a) M is uniformly dense;
(b) $\eta(M)\rho(M) = |M|$;
(c) $\gamma(M)\rho(M) = |M|$;
(d) $\eta(M) = \gamma(M)$;
(e) ([2]) there are integers $t, s > 0$ and a family of bases $\{B_1, B_2, ..., B_s\}$ of M such that each element $x \in M$ is in exactly t of the B_i's, and such that $\gamma(M) = s/t$.

Theorem B ([1], Theorem 4) Let M be a non-uniformly dense matroid. Then there is a unique non-spanning closed submatroid $M_+ \subset M$ with $\eta(M_+) > \eta(M)$ such that

$$\gamma(M/M_+) = \eta(M/M_+) = \eta(M).$$

In this note, we shall give another characterization of uniformly dense matroids in terms of contractions of its submatroids.

Theorem Let M be a matroid. The following are equivalent:
(i) $\eta(M) = \gamma(M)$.
(ii) For any nonspanning closed submatroid N of M, the following equivalence holds:

$$\eta(M) = \eta(M/N) = \gamma(M/N) \Longleftrightarrow \eta(M) = \eta(N). \tag{2}$$

PROOF OF THEOREM

Let $t \geq 2$ be an integer and let M be a matroid. For each $x \in M$, we replace x by a set of parallel elements $E(x) = \{x(1), \ldots, x(t)\}$ such that $E(x) \cap E(x') = \emptyset$, whenever $x \neq x'$ and $x, x' \in M$; and denote by M_t the resulting t-parallel extension of M (A subset $\{x_1(i_1), x_2(i_2), \ldots, x_b(i_b)\}$ is a base of M_t if and only if $\{x_1, x_2, \ldots, x_b\}$ is a base of M). Thus if $W \subseteq M_t$ and if $X \subseteq M$ is minimal with respect to

$$W \subseteq \bigcup_{x \in X} E(x), \tag{3}$$

then $\rho_t(W) = \rho(X)$, where ρ_t and ρ are the rank functions of M_t and M, respectively. For contractions, we have, by definition,

$$(M/N)_t \cong (M_t/N_t), \text{ if } \rho(N) < \rho(M). \tag{4}$$

<u>Lemma 1</u> Let $t \leq 2$ be an integer and let M be a matroid with rank function ρ. Then each of the following holds.

(a) $t\eta(M) = \eta(M_t)$;

(b) $t\gamma(M) = \gamma(M_t)$.

<u>Proof</u>: By definition of η, we can find a submatroid N of M such that

$$\eta(M) = \frac{|M-N|}{\rho(M)-\rho(N)}.$$

Hence

$$t\eta(M) = t\frac{|M-N|}{\rho(M)-\rho(N)} = \frac{|M_t - N_t|}{\rho_t(M_t)-\rho_t(N_t)} \geq \eta(M_t).$$

On the other hand, let $W \subseteq M_t$ be such that

$$\eta(M_t) = \frac{|M_t - W|}{\rho_t(M_t)-\rho_t(W)}.$$

Since $\eta(M_t)$ is the minimun of such ratios, the minimum is achieved only when equality holds in (3) and so $W = X_t$ for some $X \subseteq M$. Thus

$$\eta(M_t) = \frac{|M_t - W|}{\rho_t(M_t)-\rho_t(W)} = \frac{t|M-X|}{\rho(M)-\rho(X)} \geq t\eta(M).$$

This proves (a). The proof for (b) is similar. □

<u>Lemma 2</u> Let M be a uniformly dense matroid. For any nonspanning closed submatroid N of M, (2) holds.

<u>Proof</u>: Since M is uniformly dense, by (a) $\Longrightarrow$ (d) of Theorem A, we have

$$\eta(M) = \gamma(M). \tag{5}$$

Assume first that

$$\eta(M) = \eta(N) = r, \tag{6}$$

where $r \geq 1$ is a rational number. We shall show

$$\eta(M) = \eta(M/N) = \gamma(M/N). \tag{7}$$

Suppose further that r is an integer. By (5) and by the definitions of $\eta(M)$ and $\gamma(M)$, we have

$$\eta(N) \leq g(M) \leq \gamma(N) \leq \gamma(M) = \eta(M), \tag{8}$$

and so by (6) and (8), we have $\eta(N) = \gamma(N) = r$. By (d) $\Longrightarrow$ (e) of Theorem A, N is the union of r pairwise disjoint bases $X_1, X_2, ..., X_r$ of N. Thus N itself is an independent set in $M^{(r)}$, the union matroid of r copies of M (see [8], page 121).

Since M is uniformly dense, by (d) $\Longrightarrow$ (e) of Theorem A, we can find r pairwise disjoint bases $B_1, B_2, ..., B_r$ such that M is the union of these bases. Hence M is a base of $M^{(r)}$. Applying the Augmentation Theorem ([8], page 13) to $M^{(r)}$, we can regard $X_i = B_i \cap N$ $(1 \leq i \leq r)$. Since X_i is a base of N, B_i/X_i is a base in $M/X_i = M/N$, $(1 \leq i \leq r)$. It follows that M/N is the union of the r pairwise disjoint bases $B_1/X_1, ..., B_r/X_r$. By (e) $\Longrightarrow$ (d) of Theorem A, $\eta(M/N) = \gamma(M/N) = r$ and so by (6), (7) follows.

Suppose that $r = \frac{s}{t}$ with $t > 1$. We consider M_t, the t-parallel extension of M. Then $\eta(M_t) = \gamma(M_t) = s$ and so we have,

$$\eta(N_t) = \gamma(N_t) = \eta(M_t) = s.$$

By Lemma 1, we have

$$\eta(N) = \gamma(N) = \eta(M) = \frac{s}{t}.$$

Thus (7) follows in general.

Conversely, we assume that

$$\eta(M) = \eta(M/N) = \gamma(M/N) = r, \tag{9}$$

and we want to show that (6) holds also.

Assume that r is an integer. By (9) and by (d) $\Longrightarrow$ (e) of Theorem A, M/N is the pairwise disjoint union of r bases $Y_1, Y_2, ..., Y_r$. Note that each of the Y_i's is also independent in M. By (5) and (9), $\eta(M) = \gamma(M) = r$ and so we can apply the Augmentation Theorem in $M^{(r)}$ to augment the Y_i's to r pairwise disjoint bases $B_1, B_2, ..., B_r$ of M such that

$$M = \bigcup_{i=1}^{r} B_i,$$

and such that

$$Y_i \subseteq B_i, \ (1 \leq i \leq r).$$

Let $X_i = B_i - Y_i$, $(1 \leq i \leq r)$. Since N is closed, we have the following set equation:

$$N = M - (M/N) = \bigcup_{i=1}^{r} X_i.$$

Since the B_i's are bases of M and the Y_i's are the bases of M/N, all the X_i's have the same cardinality. In fact, letting ρ, ρ_c denote the rank functions of M and M/N, respectively, we have, by (1) and for each i, $(1 \leq i \leq r)$,

$$\rho(N) = \rho(M) - \rho_c(M/N) = |B_i| - |Y_i| = |X_i|.$$

It follows that N is the pairwise disjoint union of r bases and so by (e) $\Longrightarrow$ (d) of Theorem A, $\eta(N) = \gamma(N) = r = \eta(M)$, and so (6) follows.

Now assume that $r = \frac{s}{t}$. We turn to M_t again to get

$$\eta(M_t) = \eta(M_t/N_t) = \gamma(M_t/N_t) = s.$$

Thus by Lemma 1, we have

$$\eta(M) = \eta(M/N) = \gamma(M/N) = \frac{s}{t},$$

and so (6) follows in general. □

<u>Proof of Theorem</u>: By Lemma 2, (i) $\Longrightarrow$ (ii). Suppose that (ii) holds and by contradiction, we assume that

$$\eta(M) < \gamma(M).$$

Let M_+ be defined as in Theorem B. Then by Theorem B,

$$\eta(M_+) > \eta(M), \tag{10}$$

and M_+ is closed. Moreover, it follows by Theorem B that

$$\eta(M) = \eta(M/M_+) = \gamma(M/M_+),$$

and so by (ii), $\eta(M_+) = \eta(M)$, contrary to (10). □

REFERENCES

1. P. A. Catlin, J. W. Grossman, A. M. Hobbs and H.-J. Lai, Fractional arboricity, strength, and principal partition in graphs and matroids. Discrete Appl. Math., to appear.

2. W. H. Cunningham, Optimal attack and reinforcement of a network. J. Assoc. Comp. Mach. 32 (1985) 549 - 561.

3. A. M. Hobbs, Computing edge-toughness and fractional arboricity. Contemporary Mathematics, 89 (1989) 89 - 106.

4. H. Narayanan and N. Vartak, An elementary approach to the principal partition of a matroid. Trans. IECE Japan E64 (1981) 227-234.

5. C. Payan, Graphes équillibrés et arboricité rationnelle. Europ. J. Combinatorics 7(1986) 263-270.

6. A. Ruciński and A. Vince, Strongly balanced graphs and random graphs. J.Graph Theory 10(1986) 251-264.

7. N. Tomizawa, Strongly irreducible matroids and principal partition of a matroid into strongly irreducible minors. Elect. and Comm. in Japan 59A, No. 2(1976) 1-10.

8. D. J. A. Welsh, "Matroid Theory". Academic Press, London, New York (1976).