Reduced Graphs of Diameter Two

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ABSTRACT

A graph H is *collapsible* if for every subset $X \subseteq V(H)$, H has a spanning connected subgraph whose set of odd-degree vertices is X. In any graph G there is a unique collection of maximal collapsible subgraphs, and when all of them are contracted, the resulting contraction of G is a *reduced* graph. Interest in reduced graphs arises from the fact [4] that a graph G has a spanning closed trail if and only if its corresponding reduced graph has a spanning closed trail. The concept can also be applied to study hamiltonian line graphs [11] or double cycle covers [8]. In this article, we characterize the reduced graphs of diameter two. As applications, we obtain prior results in [12] and [14], and show that every 2-edge-connected graph with diameter at most two either admits a double cycle cover with three even subgraphs or is isomorphic to the Petersen graph.

INTRODUCTION

We shall use the notation of Bondy and Murty [3], except for contractions, and we allow graphs to have multiple edges but not loops. We shall use $d_G(u, v)$ to denote the distance between a vertex u and a vertex v in G. When no confusion arises, we use $d(u, v)$ instead of $d_G(u, v)$. The *girth* of a graph G, denoted by $g(G)$, is the length of a shortest cycle of G. For positive integers $r \geq 2$ and $g \geq 3$, a smallest regular graph of degree r and girth g is called an (r, g)-*cage*. We shall use C_5 to denote the 5-cycle. As in [3], $\delta(G)$ denotes the minimum degree of G and $\kappa(G)$ denotes the connectivity of G.

The *diameter* of G, denoted by diam(G), is defined as follows:

$$\mathrm{diam}(G) = \max_{u, v \in V(G)} d(u, v) .$$

For a set $X \subseteq E(G)$, we define the *contraction* G/X to be the graph obtained from G by contracting the edges of X and deleting all resulting loops. For

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$e \in E(G)$, we denote $G/\{e\}$ by G/e, and for a connected subgraph H of G, G/H denotes the contraction $G/E(H)$.

An *even graph* is a graph whose vertices are of even degrees. An *eulerian* graph is a connected even graph. The trivial graph K_1 is regarded as an eulerian graph. A graph with a spanning eulerian subgraph is called *supereulerian*. It is clear that every eulerian graph is supereulerian. A *dominating eulerian subgraph H* of G is a eulerian subgraph of G such that every edge of G is incident with at least one vertex of H in G. By definition, every supereulerian graph has a dominating eulerian subgraph.

Let G be a graph with at least three edges. The *line graph* of G, denoted by $L(G)$, has vertex set $E(G)$, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G

In [4], Catlin defines the collapsible graphs. A graph G is *collapsible* if for every subset $X \subseteq V(G)$ with $|X|$ even, G has a spanning connected subgraph whose set of odd-degree vertices is X. It is routine to show that G is collapsible if and only if for any subset $R \subseteq V(G)$ with $|R|$ even, G has a subgraph Γ such that $G - E(\Gamma)$ is connected and such that the set of odd-degree vertices of Γ is R. The subgraph Γ is called an *R-subgraph* of G. Catlin shows that in [4] every vertex of a graph G is in a unique maximal collapsible subgraph of G. The *reduction* of G is the graph G' obtained from G by contracting all nontrivial collapsible subgraphs of G. A vertex $v' \in V(G')$ is called *trivial* if the preimage of v' under the contraction mapping from G to G' consists of a single vertex. A graph is called *reduced* if it is the reduction of some graph.

Catlin proves the following in [4]:

Theorem A (Catlin [4]). Let G be a graph.

(i) G is reduced if and only if G has no nontrivial collapsible subgraphs.
(ii) If G is reduced, then G is simple, K_3-free, $\delta(G) \leq 3$, and G can be covered by at most two edge-disjoint forests.
(iii) If each edge of a spanning tree of G is in a collapsible subgraph of G, then G is collapsible.
(iv) If H is a collapsible subgraph of G, then G is collapsible if and only if G/H is collapsible.
(v) If H is a collapsible subgraph of G, then G is supereulerian if and only if G/H is supereulerian.
(vi) $L(G)$ is hamiltonian if and only if G', the reduction of G, has a dominating eulerian subgraph that contains all nontrivial vertices of G'. ∎

Thus it is easy to see that the only reduced graph of diameter one is K_2. In this note, we shall characterize all reduced graphs of diameter two.

We shall use the following theorems:

Theorem B (Singleton [13]). Every graph with diameter d and girth $2d + 1$ is regular. ∎

Theorem C. The Petersen graph is the only $(3, 5)$-cage. ∎

A proof of Theorem C can be found in [1, page 63].

Theorem D. Any $(r, 5)$-cage has at least $r^2 + 1$ vertices. ∎

A proof of Theorem D can be found in [2].

The following tool will also be employed:

Let H be a graph and let π be a partition of $V(H)$ into two nonempty sets V_1, V_2. We shall denote this by $\pi = \langle V_1, V_2 \rangle$. Then H is called *π-collapsible* if for every subset $R \subseteq V(H)$ of even cardinality, the following hold:

(i) if $|R \cap V_1|$ is odd, then H has an R-subgraph Γ;
(ii) if $|R \cap V_1|$ is even, then $H + e$ has an R-subgraph Γ, for any newly added edge $e = v_1 v_2$ with $v_1 \in V_2$ and $v_2 \in V_2$.

As examples, complete graphs of order at least 3 are collapsible; collapsible graphs are π-collapsible, for any partition π of $V(H)$. The 4-cycle is π-collapsible (but not collapsible), when π is the bipartition of the 4-cycle.

Suppose that H is a π-collapsible subgraph of G with $\pi = \langle V_1, V_2 \rangle$. Denote by G/π the graph obtained from G by identifying all vertices of V_1 to form a single vertex v_1, by identifying all vertices of V_2 to form a single vertex v_2, and by joining v_1 and v_2 with exactly one edge.

Theorem E (Catlin [6]). Let H be a π-collapsible subgraph of G. If G/π is collapsible, then G is collapsible. ∎

MAIN RESULT

Lemma 1. If G is a connected triangle-free graph with $\mathrm{diam}(G) = 2$ and $\delta(G) = 1$, then $G \cong K_{1,t}$ for some $t > 1$.

Proof. Obvious. ∎

Lemma 2. If G is a connected reduced graph with $\mathrm{diam}(G) = 2$ and $\delta(G) \geq 2$, then $\kappa(G) \geq 2$ and every $K_{1,2}$ is in a 4-cycle or a 5-cycle.

Proof. Suppose that G has a cut-vertex x. Then since $\delta(G) \geq 2$, we can find $y, z \in V(G)$ such that $xy, xz \in E(G)$ and y and z are in distinct components of $G - x$. Since G is reduced, by (ii) of Theorem A, G is simple. By the fact that G is simple and by $\delta(G) \geq 2$, we can find $y_1, z_1 \in V(G) - \{x, y, z\}$ such that $yy_1, zz_1 \in E(G)$. Since G is reduced, G contains no K_3's and so the distance from z_1 to y_1 is at least three, contrary to the assumption of $\mathrm{diam}(G) = 2$. Hence $\kappa(G) \geq 2$.

Since $\kappa(G) \geq 2$, by Menger's Theorem (see [3], page 42) every $K_{1,2}$-subgraph of G is in a cycle of G. Since $\text{diam}(G) = 2$, the shortest cycle containing each $K_{1,2}$-subgraph must have length less than 6. ∎

Let m, n be two positive integers. Let $H_1 \cong K_{2,m}$ and $H_2 \cong K_{2,n}$ be two complete bipartite graphs. Let v_1, u_1 be two nonadjacent vertices of degree m in H_1, and v_2, u_2 be two nonadjacent vertices of degree n in H_2. Let $S_{n,m}$ denote the graph obtained from H_1 and H_2 by identifying v_1 and v_2, and by connecting u_1 and u_2 with a new edge u_1u_2. Note that $S_{1,1}$ is the same as C_5, the 5-cycle.

Theorem 3. Let G be a reduced graph. If $\text{diam}(G) = 2$, then exactly one of the following holds:

(a) $G \cong K_{1,t}$, $t \geq 2$;
(b) $G \cong K_{2,t}$, $t \geq 2$;
(c) $G \cong S_{n,m}$, $n, m \geq 1$;
(d) G is P, the Petersen graph.

Proof. It is easy to see that the conclusions are mutually exclusive.

We argue by contradiction. Let G be a counterexample with as few vertices as possible. Clearly $G \neq K_1$. By (i) of Theorem A, we shall use the following equivalence:

$$G \text{ is reduced} \Longleftrightarrow G \text{ has no nontrivial collapsible subgraphs}. \tag{1}$$

By Lemma 1 and (ii) of Theorem A, either G satisfies (a) of Theorem 3, contrary to the assumption that G is a counterexample, or

$$2 \leq \delta(G) \leq 3. \tag{2}$$

Lemma 4. G does not have a 4-cycle C that contains a vertex of degree 2 in G.

Proof of Lemma 4. By way of contradiction, we assume that $v \in V(G)$ is a vertex of degree 2 in G and v is in a 4-cycle of G. Then

$$\text{diam}(G - v) = \text{diam}(G) = 2.$$

Since G is reduced, it follows by (1) that $G - v$ is also reduced. Since $G - v$ has smaller order than G, the theorem holds for $G - v$.

If $G - v \cong K_{1,t}$, then since $\delta(G) = 2$, it follows that $t = 2$ and G is a 4-cycle, namely, $K_{2,2}$.

If $G - v \cong K_{2,t}$, then $G \cong K_{2,t+1}$, since G contains no 3-cycles and G has diameter at most 2.

If $G - v \cong S_{m,n}$, then $G \cong S_{m,n+1}$ or $G \cong S_{m+1,n}$, for otherwise G would have a 3-cycle or $\text{diam}(G)$ would exceed 2.

If $G - v \cong P$, then $\mathrm{diam}(G) = 3$, a contradiction.

Since in any case, a contradiction arises, Lemma 4 obtains. ∎

Lemma 5. *G does not have any 4-cycle.*

Proof of Lemma 5. By way of contradiction, we assume that G has a 4-cycle $C = u_1u_2u_3u_4u_1$. Define $V_1 = \{u_1, u_3\}$ and $V_2 = \{u_2, u_4\}$. Then $\pi = \langle V_1, V_2 \rangle$ is a partition of $V(C)$. Let G/π denote the graph obtained from $G - E(C)$ by identifying u_1, and u_3 to form a new vertex v_1, and by identifying u_2 and u_4 to form a new vertex v_2, and by joining v_1 and v_2 by a new edge e_π. Let G' denote the reduction of G/π. If $G' \cong K_1$, then by the definition of reduction and by (iii) of Theorem A, G/π is collapsible. Since 4-cycles are π-collapsible [6], by Theorem E, G is collapsible, contrary to the assumption that G is reduced. Thus $G' \neq K_1$.

If $e_\pi \notin E(G')$, then G/π has a collapsible subgraph containing e_π and so it follows by Theorem E that G has a nontrivial collapsible subgraph containing C, contrary to the assumption that G is reduced, by (1). Hence we assume

$$e_\pi \in E(G')\,. \tag{3}$$

By definition

$$G' \text{ is reduced}\,. \tag{4}$$

Since contracting the edges does not increase the diameter, and since the operation to obtain G/π from G does not increase the diameter either, the diameter of G' is not bigger than that of G; in other words, $\mathrm{diam}(G') \leq 2$.

Case 1. $\mathrm{diam}(G') = 1$.

Then $G' = K_2$, since K_2 is the only reduced graph of diameter 1. By (3), $E(G') = \{e_\pi\}$. Thus $E(C)$ is an edge-cut of G. Let G_1 and G_2 be the two subgraphs of $G - E(C)$ such that

$$V_i \subseteq V(G_i), \quad 1 \leq i \leq 2, \quad \text{and} \quad G - E(C) = G_1 \cup G_2\,.$$

Claim. Either G_1 or G_2 is edgeless.

If not, then suppose that $x_1x_2 \in E(G_1)$ and $y_1y_2 \in E(G_2)$. Since G is reduced, G has no 3-cycles, by (ii) of Theorem A. Hence we may assume that $x_1y_1 \notin V(C)$. Since $E(C)$ is an edge-cut of G, it follows that $d(x_1, y_1) \geq 3$ in G, contrary to the assumption that the diameter of G is 2. Hence the claim. ∎

By the claim, we may assume that $V(G_1) = V_1$. Hence u_1 has degree 2 in G and so G has a 4-cycle C that contains a vertex of degree 2 in G, contrary to Lemma 4.

Case 2. $\mathrm{diam}(G') = 2$.

Note that G' has smaller order than G. By the minimality of G, G' satisfies a conclusion of Theorem 3.

By (3), $e_\pi \in E(G')$. By the conclusions of Theorem 3, we can choose an edge $e = xy \in E(G') - \{e_\pi\}$ such that $x \notin \{v_1, v_2\}$ and $y \in \{v_1, v_2\}$. Without loss of generality, we assume $y = v_1$. Note that e is an edge of G. For convenience we regard y as an end vertex of e in G, where $y \in \{u_1, u_3\}$. Without loss of generality, we assume $y = u_1$. If e and the edges u_1u_2, u_2u_3 were in a 5-cycle of G, then e would be in a 3-cycle of G/π and so by the fact that K_3 is collapsible, $e \notin E(G')$, contrary to the choice of e. Hence u_1u_2, u_2u_3 and the edge e are not in any 5-cycles of G. It follows $d_G(x, u_3) \geq 3$, contrary to the assumption of $\mathrm{diam}(G) = 2$.

This completes the proof of Lemma 5. ∎

Proof of Theorem 3, continued. By Lemmas 2 and 5, the girth of G is 5. Note that the diameter of G is 2. By (2) and by Theorem B with $d = 2$, G is either 2-regular or 3-regular. We consider two cases.

Case 1. G is 2-regular.

By $\mathrm{diam}(G) = 2$, by $g(G) = 5$ and since G is 2-regular, G is a 5-cycle, and so (c) of Theorem 3 holds for $m = n = 1$, contrary to the assumption that G is a counterexample.

Case 2. G is 3-regular.

Let v be a vertex of G and for positive integer i, set

$$S_i = \{u \in V(G) : d(u, v) = i\}. \tag{5}$$

Since G is 3-regular and since the girth of G is 5, we have

$$|S_1| = 3 \quad \text{and} \quad |S_2| = 6. \tag{6}$$

It follows by $\mathrm{diam}(G) = 2$ that for $i \geq 3$,

$$|S_i| = 0. \tag{7}$$

Combining (5) and (6), we obtain

$$|V(G)| = |\{v\} \cup S_1 \cup S_2| = 1 + 3 + 6 = 10. \tag{8}$$

Thus G is a regular graph of girth 5 and order 10. By Theorem D with $r = 3$, G is a $(3,5)$-cage, and so it follows from Theorem C that G is the Petersen graph. Thus (d) of Theorem 3 holds, contrary to the assumption that G is a counterexample.

The contradictions establish the theorem. ∎

APPLICATIONS

A *double cycle cover* of a graph G is a collection of even subgraphs H_1, $H_2, \ldots, H_m$ of G, such that each edge of G occurs in exactly two of the H_i's. If $m = 3$, then we say that G admits a double cycle cover with three even subgraphs. In [10], Jaeger gives a survey on the double cycle cover conjecture that any 2-edge-connected graph admits a double cycle cover. The following conjecture, due to Tutte, is well known:

Conjecture (Tutte). Every 2-edge-connected graph with no subgraph contractible to the Petersen graph admits a double cycle cover with three even subgraphs.

In [8], Catlin proves the following:

Theorem F (Catlin [8]). If H is a 4-cycle or a collapsible subgraph of G, then G admits a double cycle cover with three even subgraphs if and only if G/H admits a double cycle cover with three even subgraphs. ∎

As an application of Theorem 3, we have

Corollary 6 Let G be a 2-edge-connected graph with diameter at most 2. Then exactly one of the following holds:

(a) G admits a double cycle cover with three even subgraphs.
(b) $G = P$, the Petersen graph.

Proof. It is known that the Petersen graph cannot have a double cycle cover with three even subgraphs. Thus (a) and (b) are mutually exclusive.

Let G' denote the reduction of G. If $G'' = K_1$, then G' admits a double cycle cover with three even subgraphs trivially. Hence by Theorem F, G also admits a double cycle cover with three even subgraphs, and so (a) of Corollary 6 holds. Thus we assume that $G' \neq K_1$. Since contraction will not decrease the edge-connectivity, G' is 2-edge-connected. Since contraction will not increase the diameter, $\mathrm{diam}(G') \leq 2$. By these facts and by Theorem 3, either (b) or (c) or (d) of Theorem 3 holds for G'.

Case 1. Either $G' \cong K_{2,t}$ for some $t \geq 2$, or $G' \cong S_{n,m}$ for some $n, m \geq 1$.

Then it is routine to check that the $K_{2,t}$'s ($t \geq 2$) or the $S_{n,m}$'s ($n, m \geq 1$) admit double cycle covers with three even subgraphs. Thus (a) of Corollary 6 follows from Theorem F. (We can also apply Theorem F by contracting the 4-cycles in the $K_{2,t}$'s or in the $S_{n,m}$'s to get the same conclusions, when G' is not the 5-cycle; and when G' is the 5-cycle, it is clear that G' has a double cycle cover with three even subgraphs.)

Case 2. G' is the Petersen graph.

What we have to show is $G = G'$. If not, then some vertex v' of the Petersen graph G' is the contraction image of a nontrivial collapsible subgraph H of G. Let u' be a vertex of G' that has distance two from v', and let K be the preimage of u' under the contraction. Note that since G' is the Petersen graph,

$$\text{there is a unique } (u', v')\text{-path } P' \text{ of length 2 in } G'. \tag{9}$$

Since H is nontrivial, H contains more than one vertex. We may assume that there is a vertex $v \in V(H)$ that is not incident with edges in $E(P')$. Let $u \in V(K)$ be such that u is incident with an edge of $E(P')$. Then by (9), we must have $d_G(u, v) \geq 3$, contrary to the assumption of $\text{diam}(G) \leq 2$. Thus $G = G'$ and so (b) of Corollary 6 holds. ∎

In the way to prove Case 2 of Corollary 6, we get the following by-product:

Corollary 7. If G has diameter two and the reduction of G is the Petersen graph, then G itself is the Petersen graph. ∎

Corollary 8. (Veldman [14]). Let G be a connected graph with $\text{diam}(G) \leq 2$ and with at least three edges. Then $L(G)$ is hamiltonian.

Proof. Let G satisfy the hypotheses of Corollary 8, and let G' denote the reduction of G. Clearly $\text{diam}(G') \leq 2$ follows from $\text{diam}(G) \leq 2$. By Theorem 3, either $G' \in \{K_1, K_2\}$ or one of the conclusions of Theorem 3 holds for G'. The hypotheses of $\text{diam}(G) \leq 2$ and $|E(G)| \geq 3$ force that when $G = K_2$, exactly one of the vertices of G' is trivial; when $G' = K_{1,t}$ ($t \geq 2$), all vertices of degree 1 of G' are trivial; and when $G' \in \{K_{2,t}, S_{n,m}\}$($t \geq 2, n, m \geq 1$), one vertex of degree 2 of G' is a trivial vertex. When G' is the Petersen graph, then Corollary 7 says that $G = G'$ is the Petersen graph. Thus in any case, G' has a dominating eulerian subgraph that contains all nontrivial vertices and so by (vi) of Theorem A, $L(G)$ is hamiltonian. ∎

Corollary 9. Let G be a simple graph with $|V(G)| \geq 5$ and $\delta(G) \geq 2$. If for every pair of nonadjacent vertices $u, v \in V(G)$,

$$\deg_G(u) + \deg_G(v) \geq |V(G)| - 1, \tag{10}$$

then exactly one holds:

(a) $G \in \{K_{2,3}, C_5\}$,
(b) for any vertices $x, y \in V(G)$ (possibly $x = y$), G has a spanning (x, y)-trail.

Proof. Since G is simple, (10) implies that $\text{diam}(G) \leq 2$. Let G' denote the reduction of G. Since $\delta(G) \geq 2$, G' is either $K_1, K_{2,t}, S_{n,m}$ or the Petersen graph, by Theorem 3.

If $|V(G)| = 5$, then either G is collapsible or $G = G' \in \{K_{2,3}, C_5\}$. When $|V(G)| \geq 6$, (10) forces that $G' = K_1$ and so either (a) of Corollary 9 holds or G is collapsible.

Suppose that G is collapsible. If $x = y$, then Corollary 9 follows from (v) of Theorem A. Suppose $x \neq y$. Let R denote the symmetric difference of $\{x, y\}$ and the set of all odd-degree vertices of G. Since G is collapsible, G has an R-subgraph Γ. Since x, y are the only odd-degree vertices of $G - E(\Gamma)$ and since $G - E(\Gamma)$ is connected, $G - E(\Gamma)$ is the desired trial. ∎

Corollary 9 strengthens both of the following:

Corollary 10 (Lesniak-Foster and Williamson [13]). Let G be a simple graph with $|V(G)| \geq 6$ and $\delta(G) \geq 2$. If (10) holds for every pair of nonadjacent vertices of G, then G has a spanning closed trail. ∎

Corollary 11 (Lesniak-Foster and Williamson [13]). Let G be a simple graph with $|V(G)| \geq 5$. If for every pair of nonadjacent vertices $u, v \in V(G)$,

$$\deg_G(u) + \deg_G(v) \geq |V(G)|, \tag{11}$$

then for any vertices $x, y \in V(G)$ (possibly $x = y$), G has a spanning (x, y)-trail.

Proof. (11) implies that $\delta(G) \geq 2$ and $G \notin \{K_{2,3}, C_5\}$. Thus Corollary 11 follows from Corollary 9. ∎

OPEN PROBLEMS

For a graph G, let $F(G)$ denote the minimum number of extra edges that must be added to G to create a spanning supergraph of G having two edge disjoint spanning trees.

We have the following conjecture:

Conjecture 1. If G is a nontrivial connected reduced graph with $F(G) = 2$, then $\text{diam}(G) = 2$.

Catlin has conjectured the following:

Conjecture 2. If G is a nontrivial connected reduced graph with $F(G) = 2$, then $G \in \{K_{2,t}: t \geq 1\}$.

As an application of Theorem 3, we have

Proposition 12. Conjecture 1 and Conjecture 2 are equivalent.

Proof. Clearly Conjecture 2 implies Conjecture 1, since $K_{2,t}$ is of diameter 2.

Conversely, among the reduced graphs of diameter 2 listed by Theorem 3, only $G = K_{2,t}$ satisfies $F(G) = 2$ and so Conjecture 1 implies Conjecture 2. ∎

The following result, conjectured by Catlin in [7], is a weaker form of Conjecture 2:

Theorem 13 (Catlin and Lai [9]). Let G be a connected graph. If $F(G) \leq 2$, then exactly one of the following holds:

(a) G is supereulerian;
(b) G has a cut-edge;
(c) the reduction of G is $K_{2,t}$, for some odd integer $t \geq 3$. ∎

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