Discrete Mathematics 69 (1988) 43–53
North-Holland

ON THE HAMILTONIAN INDEX

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Received 18 August 1986
Revised 9 March 1987

For simple connected graphs that are neither paths nor cycles, we define $h(G) = \min\{m: L^m(G)$ is Hamiltonian$\}$ and $l(G) = \max\{m: G$ has an arc of length m that is not both of length 2 and in a $K_3\}$, where an *arc* in G is a path in G whose internal vertices have degree two in G. We prove that $h(G) \le l(G) + 1$. As consequences, we obtain theorems of Chartrand and Wall and of Lesniak-Foster and Williamson. We also characterize those graphs that satisfy $l(G) + 1 = h(G)$. This characterization provides counterexamples to a previous result in [5].

We use Bondy and Murty [1] for basic notation and terminology, except for arcs, line graphs and contractions. Consider simple graphs that are not paths. Let G be a graph. A path P of G is called an *arc* in G if all the internal vertices of P are divalent vertices of G. A *D-circuit* of G is a closed trail C of G such that $E(G - V(C)) = \emptyset$. The graphs K_1 and $K_{1,m}$ are considered as having a D-circuit, where $m > 0$ is an integer. An *S-circuit* of G is a spanning closed trail of G. The graph K_1 is considered to have an S-circuit. We will speak of *line graphs* instead of edge graphs; the line graph of G is denoted by $L(G)$ or $L^1(G)$. For a positive integer m, we define $L^m(G) = L(L^{m-1}(G))$ with $L^0(G) = G$. It is known (see [4, Theorem A]) that if G is connected and is not a path, then $L^m(G)$ is defined for every integer $m \ge 0$. The edge connectivity of G is denoted by $\kappa'(G)$ and the number of components of G is denoted by $\omega(G)$. We use $d(v)$ to mean the degree of v in G, $\forall v \in V(G)$. We define

$$\begin{aligned} D_1(G) &= \{v \in V(G);\ d(v) = 1\},\\ h(G) &= \min\{m;\ L^m(G) \text{ is Hamiltonian}\},\\ s(G) &= \min\{m;\ L^m(G) \text{ has an } S\text{-circuit}\},\\ l(G) &= \max\{m;\ G \text{ has an arc of length } m \text{ that is not both of length 2 and in a } K_3\}. \end{aligned}$$

$h(G)$ is called the *Hamiltonian index* of G. In [3] it was shown that if G is a connected graph that is not a path, then $h(G)$ exists and is a finite number.

If G is a cycle, then $L^n(G) \cong G$, for all $n \ge 0$. Thus we are more interested in the case when G is not a cycle. Denote by $\mathcal{G}$ the collection of connected simple graphs that are not paths nor cycles nor isomorphic to $K_{1,3}$.

For $xy \in E(G)$, an *elementary contraction* of G is the graph G/xy obtained from G by deleting $\{x, y\}$ and inserting a new vertex v and edges joining v to each $w \in V(G - \{x, y\})$ with as many edges as $\{x, y\}$ was joined to w by edges in G. A *contraction* of G is a graph G/H obtained by a sequence of elementary contractions of edges of H.

The following concept is posed by Catlin.

A graph is called *collapsible* if for every even set $S \subseteq V(G)$, there is a subgraph H of G such that

(i) $G - E(H)$ is connected, and

(ii) S is the set of vertices of odd degrees in H.

For example, K_3 is a collapsible graph.

Let $E' \subseteq E(G)$ be a minimal edge set such that each component of $G - E'$ is collapsible. Let $X_1, X_2, \ldots, X_c$ denote the components of $G - E'$. Denote by G_1 the graph of order c obtained from G by contracting the subgraphs $X_1, X_2, \ldots, X_c$ to distinct vertices. We refer to G_1 as the *reduction of* G. In the rest of this paper, we use G_1 or $(G)_1$ to mean the reduction of a graph G. Catlin [2, Corollary 2] showed that the set E' is unique, and hence that G_1 is well defined. He also proved

Theorem 1 (Catlin [2, Theorem 2, Theorem 9, Theorem 3 and Theorem 4]).

(i) *If G is collapsible, then $s(G) = 0$.*

(ii) *$s(G) = 0$ if and only if $s(G_1) = 0$.*

(iii) *If H is a collapsible subgraph of G, then G is collapsible if and only if G/H is collapsible.*

(iv) *Let H_1 and H_2 be subgraphs of H such that $H_1 \cup H_2 = H$ and $H_1 \cap H_2 \neq \emptyset$. If H_1 and H_2 are collapsible, then so is H.*

As a consequence of Theorem 1(iv), and since K_3 is collapsible, we get

Lemma 2. *If every edge of G is in a K_3 of G, then G is collapsible.*

The following is shown in [6]:

Theorem 3 (Harary and Nash-Williams [6]). *The line graph $L(G)$ of a connected graph G is Hamiltonian if and only if G has a D-circuit and $G \notin \{K_1, K_2, K_{1,2}\}$.*

Recall that K_1 and $K_{1,m}$ are considered to have a D-circuit, for $m > 0$.

Since a Hamiltonian cycle is an S-circuit and an S-circuit is a D-circuit, we have immediately from Theorem 3,

Proposition 4. *If G is a connected graph that is not a path, then*

$$s(G) \leqslant h(G) \leqslant s(G) + 1. \tag{1}$$

Our first result is the following

Theorem 5. *Let $G \in \mathcal{G}$ and $r = l(G)$. Then $L^r(G)$ is collapsible.*

Proof. We prove Theorem 5 by induction on $l(G)$. Assume that $l(G)=1$. Since $l(G)=1$, for any $x \in V(G)-D_1(G)$, if x is not in a K_3, there are at least three incident edges. Thus every edge in G is either in a K_3 or in a $K_{1,3}$ and so every
4 edge of $L(G)$ is in a K_3. By Lemma 2, $L(G)$ is collapsible.

Now we assume that $l(G)>1$ and Theorem 5 holds for graphs with smaller values of $l(G)$. Let $A=e_0e_1\ldots e_m$ be an arc in $L(G)$. If $m>2$, then $G[\{e_0, e_1, \ldots, e_m\}]$ is an arc of length $m+1$. Thus for $l(G)>2$,

$$l(L(G))=l(G)-1. \tag{2}$$

Suppose $l(G)=2$. Let $A=e_1e_2e_3$ be an arc of $L(G)$. If $e_1e_3 \notin E(L(G))$, then $G[e_1, e_2, e_3]$ is an arc of length 3 in G, contrary to the fact that $l(G)=2$. Hence every arc of length 2 in $L(G)$ is in a K_3 and so (2) holds for $l(G)\geqslant 2$. By induction and (2), we conclude that $L^r(G)$ is collapsible. □

Corollary 6. *Let* $G \in \mathcal{G}$. *Then*

(a) $s(G)\leqslant l(G)$,

(b) $h(G)\leqslant l(G)+1$.

Proof. If $r=l(G)$, then by Theorem 5, $L^r(G)$ is collapsible, and so by (i) of Theorem 1, $L^r(G)$ has an S-circuit. Hence (a) holds. By (a) and (1), we get (b). □

Both bounds in Corollary 6 are best possible in some sense. Let $r>0$ be an integer. Let $P(r)=v_0v_1\ldots v_{r+2}$ be a path. We define $G(r)$ as follows:

$$V(G(r))=V(P(r))\cup\{u_0, u_1\},$$

$$E(G(r))=E(P(r))\cup\{u_0v_1, v_{r+1}u_1\}.$$

It is easy to check that

$$s(G(r))=l(G(r))=r \quad \text{and} \quad h(G(r))=r+1.$$

Our main result (Theorem 20) is a constructive characterization, for $l(G)\geqslant 2$, of graphs satisfying (b) with equality.

Corollary 7 (Chartrand and Wall [4]). *If G is a connected graph and $\delta(G)\geqslant 3$, then $h(G)\leqslant 2$.*

Proof. $\delta(G)\geqslant 3$ implies that $l(G)=1$. □

Let $G \in \mathcal{G}$ and define

$$\mathbb{M}(G)=\{A\colon A \text{ is an arc of } G \text{ of length } l(G)\}.$$

For $A \in \mathbb{M}(G)$, consider the following condition:

$$A \text{ is in a cycle containing no arc of } \mathbb{M}(G) \text{ except } A. \tag{3}$$

Define

$$\mathbb{M}_1(G) = \{A \in \mathbb{M}(G)\colon V(A) \cap D_1(G) \neq \emptyset\};$$
$$\mathbb{M}_2(G) = \{A \in \mathbb{M}(G)\colon A \text{ satisfies } (3)\};$$
$$\mathbb{M}_0(G) = \mathbb{M}(G) - (\mathbb{M}_1(G) \cup \mathbb{M}_2(G)).$$

Let $G \in \mathcal{G}$ with $l(G) \geq 2$ and let

$$Z = \{v \in D_1(G)\colon v \in V(A), \text{ for some } A \in \mathbb{M}_1(G)\}. \tag{4}$$

Note that if $l(G) = l(G - Z)$, then $\mathbb{M}_0(G - Z) = \mathbb{M}_0(G)$, $\mathbb{M}_1(G - Z) = \emptyset$ and $\mathbb{M}_2(G - Z) = \mathbb{M}_2(G)$.

Lemma 8. *Let $G \in \mathcal{G}$. If $r = l(G) \geq 2$ and $\mathbb{M}_0(G) = \emptyset$, then $L^{r-1}(G - Z)$ is collapsible.*

Proof. We proceed by induction on $r = l(G)$, and prove the case when $r = 2$ by contradiction. Suppose that $L(G - Z)$ is not collapsible. By Theorem 1(iv), there is an edge $e \in E(L(G - Z))$ that is in no collapsible subgraph of $L(G - Z)$. Let x_1y_1 and x_1y_2 be two edges in $G - Z$ that correspond to the ends of e.

Case 1. If $y_1x_1y_2$ is not an arc in $G - Z$, then we can find an edge x_1y_3 in $G - Z$. Hence e is in a K_3 of $L(G - Z)$, a contradiction.

Case 2. Suppose that $y_1x_1y_2$ is an arc in $\mathbb{M}_0 \cup \mathbb{M}_1$. Since $l(G - Z) = 2 = l(G)$, $\mathbb{M}_0(G - Z) = \mathbb{M}_0(G) = \emptyset$ and $\mathbb{M}_1(G - Z) = \emptyset$, a contradiction.

Case 3. Suppose that $y_1x_1y_2$ is an arc in $\mathbb{M}_2(G - Z)$. By (3), either $y_1x_1y_2y_1$ is a K_3 or there is a (y_1, y_2)-path P in $G = (Z \cup \{x_1\})$ such that all the vertices of P are of degree at least 3 in $G - Z$. By the choice of x_1y_1 and x_1y_2, $y_1x_1y_2y_1$ cannot be a K_3. Let H_P be the subgraph of $L(G - Z)$ generated by $E(P)$ and those edges in $G - (Z \cup \{x_1\})$ incident with a vertex of P. Since each edge in $H_P - e$ is in a K_3 of it, by Lemma 2, $H_P - e$ is collapsible. Since $H_P - e$ is collapsible to a vertex, H_P is collapsible; but H_P contains e, a contradiction. Thus $L(G - Z)$ must be collapsible when $r = 2$.

Since $\mathbb{M}_0(G) = \emptyset$, it follows from (2) that $\mathbb{M}_0(L^S(G)) = \emptyset$ for $0 \leq s \leq r - 2$, and so by (2) and by induction, we are done. □

Corollary 9. *Let $G \in \mathcal{G}$ and $r = l(G)$. If one of the following holds*
(i) $\mathbb{M}(G) = \mathbb{M}_1(G)$;
(ii) $\mathbb{M}(G) = \mathbb{M}_1(G) \cup \mathbb{M}_2(G)$ *and* $l(G) \geq 2$;
then $h(G) \leq l(G)$.

Proof. If $l(G) \geq 2$, then by Lemma 8 with $r = l(G)$, $L^{r-1}(G - Z)$ is collapsible.

Hence by Theorem 1, $L^{r-1}(G-Z)$ has an S-circuit C. Note that

$$D_1(L^{r-1}(G)) = V(L^{r-1}(G)) - V(L^{r-1}(G-Z)).$$

It follows that C is a D-circuit of $L^{r-1}(G)$, and $h(G) \leq l(G)$ follows by Theorem 3.

If $l(G)=1$ and $\mathbb{M}(G)=\mathbb{M}_1(G)$, then $G=K_{1,s}$ with $s \geq 3$. By Theorem 3, $h(G) \leq l(G)$. □

Corollary 10. *If $G \in \mathcal{G}$ and $\mathbb{M}(G)=\mathbb{M}_1(G)$, then $l(G)=h(G)$.*

Proof. By Corollary 9(i), we get $l(G) \geq h(G)$. Let $r=l(G)$. Since an arc $A \in \mathbb{M}_1(G)$ will mean a vertex in $D_1(L^{r-1}(G))$, $h(G) \geq l(G)$. □

Corollary 11 (Lesniak-Foster and Williamson [7]). *Let G be a connected graph with at least four edges. If every vertex of degree 2 is adjacent to a vertex of degree one, then $h(G) \leq 2$.*

Proof. If $l(G)=1$, then Corollary 6(b) applies; if $l(G) \geq 2$, then the hypothesis of Corollary 11 forces $l(G)=2$ and (i) of Corollary 9 applies. □

We are going to characterize those graphs G with $h(G)=l(G)+1$. When $l(G) \geq 2$, Corollary 9 indicates that this can happen only when $\mathbb{M}_0(G) \neq \emptyset$. For any arc A with length greater than one, we denote

$$A^0 = \{v \in V(A): \quad d(v)=2 \text{ in } G\}.$$

That is, A^0 is the set of all internal vertices of A. Let

$$V_1 = \{v \in A^0: \quad A \in \mathbb{M}_0(G)\}.$$

In other words, V_1 is the union of all sets A^0, where $A \in \mathbb{M}_0(G)$.

For $G \in \mathcal{G}$ with $l(G) \geq 2$, we define $H(G)$ to be the graph obtained from G by contracting each component of $G - V_1$ to a single vertex and then replacing each $A \in \mathbb{M}_0(G)$ by an edge that joins the corresponding vertices of $H(G)$. The graph $H(G)$ may have multiple edges, but since arcs in $\mathbb{M}_2(G)$ are contracted, $H(G)$ has no loops. Note that if the components of $G - V_1$ are $X_1, X_2, \ldots, X_m$, then we have

$$A \in \mathbb{M}_1(G) \cup \mathbb{M}_2(G) \quad \text{iff} \quad A \in \mathbb{M}(G) \quad \text{and} \quad A \subseteq X_i \quad (1 \leq i \leq m) \tag{5}$$

Lemma 12. *$\kappa'(H(G))=1$ if and only if G has a cut edge e such that $e \in E(A)$, for some $A \in \mathbb{M}_0(G)$.*

Proof. If $H(G)$ has a cut edge, then this cut edge corresponds to an arc $A \in \mathbb{M}_0(G)$ such that every edge of A is a cut edge of G. Conversely, if G has an arc $A \in \mathbb{M}_0(G)$ that contains a cut edge of G, then $G - A^0$ will have two components. Hence $H(G)$ has a cut edge. □

A graph G is said to be $K_{1,3}$-*free* if G has no induced $K_{1,3}$ subgraphs.

Lemma 13. *If $\kappa'(G - D_1(G)) \geq 2$ and G is $K_{1,3}$-free and $l(G) = 1$, then G has a D-circuit if and only if $G - D_1(G)$ has an S-circuit.*

Proof. Suppose that G has a D-circuit. Let C be a longest D-circuit of G. If C were not an S-circuit of $G - D_1(G)$, then there would be $xy \in E(G - D_1(G))$ such that $x \in V(C)$ and $y \notin V(C)$. Since $G - D_1(G)$ is 2-edge-connected and since C is a D-circuit, $\exists z \in V(C) - \{x\}$ such that $zy \in E(G)$. Since $l(G) = 1$ and since C is a D-circuit, $\exists w \in V(C) - \{x, z\}$ such that $yw \in E(G)$. Since G is $K_{1,3}$-free, there would be an edge connecting two of $\{x, z, w\}$. It would follow that C could be extended to include y, contrary to the assumption that C is a longest D-circuit. Hence C must be an S-circuit of $G - D_1(G)$.

The other direction of the statement is trivial, since every S-circuit of $G - D_1(G)$ is a D-circuit of G. □

Corollary 14. *If $r = l(G) \geq 2$ and $\kappa'(L^{r-1}(G)) \geq 2$, then $L^{r-1}(G)$ has a D-circuit if and only if $L^{r-1}(G)$ has an S-circuit.*

Proof. By (2), we can see $l(L^{r-1}(G)) = 1$. The hypothesis of the corollary implies that $D_1(L^{r-1}(G)) = \emptyset$. Since a line graph is $K_{1,3}$-free, we can replace G of Lemma 13 by $L^{r-1}(G)$, and Corollary 14 follows. □

Lemma 15. *$H(G) \cong H(L(G))$, if $G \in \mathcal{G}$ and $l(G) \geq 3$.*

Proof. By the definition of line graphs, $E(G) = V(L(G))$. By (2) and $l(G) \geq 3$, this edge-vertex correspondence induces a bijection between $\mathbb{M}_0(G)$ and $\mathbb{M}_0(L(G))$. Let

$$V_1 = \{v \in A^0; \quad A \in \mathbb{M}_0(G)\}, \tag{6}$$

$$V_2 = \{v \in A^0; \quad A \in \mathbb{M}_0(L(G))\}. \tag{7}$$

Let $\mathcal{X}$, $\mathcal{Y}$ denote the collections of components of $G - V_1$ and $L(G) - V_2$, respectively. Let $\Phi : \mathcal{X} \to \mathcal{Y}$ be the mapping induced by the correspondence $E(G) = V(L(G))$. Thus if $X \in \mathcal{X}$ contains exactly one end of $e^1, e^2, \ldots, e^k \in \bigcup_{A \in \mathbb{M}_0(G)} E(A)$, where $k \geq 3$, then $\Phi(X)$ is the component in $\mathcal{Y}$ containing the vertices $e^1, e^2, \ldots, e^k$; and if $X \in \mathcal{X}$ is nontrivial, then $\exists e \in E(X)$ such that e is not an edge of any arc in $\mathbb{M}_0(G)$ and hence $\Phi(X)$ is the component in $\mathcal{Y}$ that contains the vertex e.

Note that the correspondence $E(G) = V(L(G))$ induces a bijection between

$\mathbb{M}_0(G)$ and $\mathbb{M}_0(L(G))$. The following are equivalent:

(a) e_1 and e_2 are in or incident with the same component X of $G - V_1$;

(b) e_1 and e_2 cannot be separated by arcs in $\mathbb{M}_0(G)$;

(c) in $L(G)$, the vertices e_1 and e_2 cannot be separated by arcs in $\mathbb{M}_0(L(G))$;

(d) in $L(G)$, the vertices e_1 and e_2 are in the same component Y of $L(G) - V_2$.

Thus Φ is well defined and one-to-one.

For any $Y \in \mathscr{Y}$, we can find some $e \in E(G)$ such that $e \in Y$. It follows that $\Phi(X) = Y$, where X is the component of $G - V_1$ that contains e. Hence Φ is surjective.

If X_1X_2 is an edge of $H(G)$, then there is some $A \in \mathbb{M}_0(G)$ such that A connects X_1 and X_2. Let A' denote the subgraph generated by the vertex set $E(A)$ in $L(G)$. Then $A' \in \mathbb{M}_0(L(G))$ is an arc connecting $\Phi(X_1)$ and $\Phi(X_2)$ in $H(L(G))$. Thus Φ preserves adjacency.

Conversely, if $Y_1Y_2 \in E(H(L(G)))$, since Φ is surjective, $\exists X_1, X_2 \in \mathscr{X}$ such that $\Phi(X_1) = Y_1$ and $\Phi(X_2) = Y_2$. That Y_1Y_2 is an edge implies that there is an arc in $\mathbb{M}_0(L(G))$ which connects Y_1 and Y_2. It follows that there is an arc in $\mathbb{M}_0(G)$ which connects X_1 and X_2. Hence Φ carries X_1X_2 to Y_1Y_2.

Summing up, we can see that Φ is an isomorphism between $H(G)$ and $H(L(G))$. $\square$

Lemma 16. *If $G \in \mathscr{G}$, $l(G) = 2$, and $\mathbb{M}_1(G) = \emptyset$, then*

$$(H(G))_1 \cong (L(G))_1.$$

Proof. Let V_1 be defined as in (6). Let $\mathscr{X} = \{X_1, X_2, \ldots, X_m\}$ be the components of $G - V_1$. Let X_i^* denote the subgraph of G generated by the edge subset E_i, where E_i consists of those edges at least one of whose ends is incident with some vertex in $V(X_i)$, $1 \leq i \leq m$.

Since $l(G) = 2$ and $\mathbb{M}_1(G) = \emptyset$, by the definition of X_i^*, we have $l(X_i^*) = 1$. By Theorem 5, $L(X_i^*)$ is collapsible, $1 \leq i \leq m$. Let

$$L = L(X_1^*) \cup L(X_2^*) \cup \cdots \cup L(X_m^*).$$

Note that $l(G) = 2$ implies that the $L(X_i^*)$'s are vertex-disjoint subgraphs of $L(G)$. Recall that $L(G)/L$ is a contraction of $L(G)$. Set $V(L(G)/L) = \{v_1, v_2, \ldots, v_m\}$, where $v_i = L(X_i^*)/L$ is the vertex corresponding to $L(X_i^*)$, $1 \leq i \leq m$. Define $\theta: V(H(G)) \to V(L(G)/L)$ by $\theta(X_i) = v_i$ for $1 \leq i \leq m$. We are going to show that θ is an isomorphism.

Note that

$$V(H(G)) = \{X_1, X_2, \ldots, X_m\},$$
$$V(L(G)/L) = \{v_1, v_2, \ldots, v_m\}.$$

It follows that θ is surjective and hence one to one. If X_iX_j is an edge of $H(G)$, then X_i and X_j are joined by an arc $A \in \mathbb{M}_0(G)$. Since $l(G) = 2$, by the definition of $L(X_i^*)$ and $L(X_j^*)$, this arc A becomes an edge in $L(G)$ connecting $L(X_i^*)$ and $L(X_j^*)$. This implies that θ preserves adjacency. It there is an edge in $L(G)/L$ joining v_i and v_j, then this edge corresponds to an arc in $\mathbb{M}_0(G)$ since otherwise it would be contracted. Hence $v_iv_j \in E(L(G)/L)$ if and only if $X_iX_j \in E(H(G))$. Thus θ is an isomorphism between $H(G)$ and $L(G)/L$. Since each $L(X_i^*)$ is collapsible, by Theorem 1(iii) we have $(H(G))_1 \cong (L(G)/L)_1 \cong (L(G))_1$. □

Corollary 17. *If $G \in \mathcal{G}$, $\mathbb{M}_1(G) = \emptyset$, and $r = l(G) \geq 2$, then*

$$(L^{r-1}(G))_1 \cong (H(G))_1.$$

Proof. If $r = 2$, then Lemma 15 gives Corollary 17. If $r \geq 3$, then by Lemma 15,

$$H(G) \cong H(L(G)) \cong \cdots \cong H(L^{r-2}(G)), \tag{8}$$

and by (2), we have $l(L^{r-2}(G)) = 2$, and so by (8) and Lemma 16,

$$(H(G))_1 \cong (H(L^{r-2}(G)))_1 \cong (L^{r-1}(G))_1. \quad \square$$

Corollary 18. *If $G \in \mathcal{G}$, $\mathbb{M}_1(G) = \emptyset$, and $r = l(G) \geq 2$, then $\kappa'(H(G)) \geq 2$ if and only if $\kappa'(L^{r-1}(G)) \geq 2$.*

Proof. By Theorem 1(i), we know that a graph with a cut edge cannot be collapsible. Hence a graph G has cut edge if and only if G_1 has a cut edge. The corollary follows from this result and Corollary 17. □

Theorem 19. *Let $G \in \mathcal{G}$ with $r = l(G) \geq 2$, $\mathbb{M}_1(G) = \emptyset$ and $\kappa'(L^{r-1}(G)) \geq 2$. The following are equivalent*

(i) $s(H(G)) = 0$;
(ii) $h(G) \leq l(G)$.

Proof. Consider the following statements:

(a) $s(H(G)) = 0$.
(b) $s((H(G))_1) = 0$.
(c) $s((L^{r-1}(G))_1) = 0$.
(d) $s(L^{r-1}(G)) = 0$.
(e) $h(G) \leq l(G)$.

By Theorem 1, (a)⇔(b) and (c)⇔(d). By Corollary 17, (b)⇔(c), and hence (a)⇔(d). By Proposition 4, (d) implies that $h(L^{r-1}(G)) \leq 1$. Hence $h(L^r(G)) \leq 0$ and (a)⇒(d)⇒(e). By Theorem 3, (e) implies that $L^{r-1}(G)$ has a D-circuit. Since $\kappa'(L^{r-1}(G)) \geq 2$, by Corollary 14, $s(L^{r-1}(G)) = 0$. Thus (e)⇒(d)⇒(a). □

Theorem 20. *If $G \in \mathcal{G}$, then $h(G) = l(G) + 1$ if and only if one of the following* *holds*:

(i) $l(G)=1$ *and* G *has no D-circuits*;

(ii) $l(G)\geq 2$ *and* $\exists A\in \mathbb{M}_0(G)$ *such that* $\omega(G-A^0)=2$;

(iii) $l(G)\geq 2$, $\mathbb{M}_0(G)\neq\emptyset$, $\kappa'(H(G))\geq 2$, *and* $s(H(G-Z))\geq 1$, *where* Z *is defined by* (4).

Proof. Let $r=l(G)$. Suppose first that $h(G)=l(G)+1$. If $r=1$, then $h(G)=2$. If G had a D-circuit, then by Theorem 3, we would have $h(G)=1$, a contradiction. Thus (i) holds. Assume that $r\geq 2$. We consider $H(G)$. By Lemma 12, either (ii) holds or $\kappa'(H(G))\geq 2$. Suppose that $\kappa'(H(G))\geq 2$. Since the conclusion of Corollary 9 is false, the hypothesis (ii) of Corollary 9 is false, and so $\mathbb{M}_0(G)\neq\emptyset$. Hence $l(G)=l(G-Z)$, where Z is defined by (4), and so $\mathbb{M}_1(G-Z)=\emptyset$. By Corollary 18 and Theorem 19, $s(H(G-Z))\geq 1$.

Conversely, let us assume (i) first. By Theorem 3, (i) implies $h(G)\geq 2$. On the other hand, by Corollary 6, $h(G)\leq r+1=2$. Thus $l(G)+1=2=h(G)$. If (ii) holds, then $L^r(G)$ has a cutvertex. Hence $h(G)>r$. By Corollary 6, $h(G)=l(G)+1$. Now we assume (iii). Since $\mathbb{M}_0(G)\neq\emptyset$, we have $l(G-Z)=l(G)$ and $\mathbb{M}_1(G-Z)=\emptyset$. Theorem 19 and (iii) imply that $h(G-Z)>l(G-Z)$ and so Corollary 6(b) gives

$$h(G-Z)=l(G-Z)+1=l(G)+1. \tag{9}$$

Note that

$$V(L^{r-1}(G-Z))=V(L^{r-1}(G))-D_1(L^{r-1}(G)). \tag{10}$$

By way of contradiction, suppose

$$L^{r-1}(G) \quad \text{has a } D\text{-circuit}. \tag{11}$$

They by (10) and (11), $L^{r-1}(G-Z)$ has a D-circuit. Since $\kappa'(H(G))\geq 2$, we must have $\kappa'(L^{r-1}(G-Z))\geq 2$, and so by Corollary 14, $L^{r-1}(G-Z)$ has an S-circuit. Hence, $h(G-Z)\leq l(G-Z)$, contrary to (9), and so (11) is false. Thus by Theorem 3, $L^r(G)$ is not Hamiltonian. It follows that $h(G)>l(G)$. By Corollary 6, $h(G)=l(G)+1$. □

Example. Let $H=K_{2,3}$ and let $r>1$ be an integer. Obtain $G[r]$ by subdividing each edge $e\in E(H)$ into an arc A_e of length r in $G[r]$ and by replacing each divalent vertex of H by a K_4. By Theorem 20, $h(G[r])=l(G[r])+1$. This is another class of extremal graphs for Corollary 6.

In [5], it is claimed that if G is a connected graph of order at least 3 such that no cut edge in G is incident to a vertex of degree 2 and no path in G contains three or more consecutive vertices of degree 2, then $h(G)\leq 2$. Note that $G[2]$ is a counterexample to this result. In fact, when $l(G)=2$ and when G satisfies (iii) of Theorem 20, G is a counterexample to the above result. Furthermore, this characterizes the counterexamples to the above claim in [5].

In [4], Chartrand and Wall called an arc P a *poly-path* if P is an arc such that both ends of P are of degree at least 3, and called an arc Q and *end-path* if Q is an arc such that one end of Q has degree at least 3 and the other end of Q has degree one.

Corollary 21 (Chartrand and Wall [4]). *If G is a tree which is not a path, then*

$$h(G) = \max\{\{l(Q)\}, \{1 + l(P)\}\}, \tag{12}$$

over all end-paths Q and poly-paths P of G.

Proof. If the maximum on right hand side of (12) is attained by a poly-path P of length $l(P)$, then since G is a tree, P satisfies (ii) of Theorem 20. By Theorem 20, (12) holds. Suppose that the maximum of the right hand side of (12) is attained by an end-path Q such that $l(Q) > l(P) + 1$ for all poly-path P of G. Then we have $\mathbb{M}(G) = \mathbb{M}_1(G)$, and by Corollary 10, (12) holds. □

A block B of G is called an *acyclic block* if B consists of a single edge, and otherwise B is called a *cyclic block* (see [4]).

In [4], it was claimed ([4, Theorem 2]) that if G is a graph containing cyclic and acyclic blocks such that each cyclic block is Hamiltonian, then (12) holds. The following graph is a counterexample to this claim. Let $C_m = v_1v_2 \ldots v_mv_1$ denote a cycle of length m, where $m > 3$ is an integer. We define G_m as follows:

$$V(G_m) = V(C_m) \cup \{w_1, w_2\},$$

$$E(G_m) = E(C_m) \cup \{v_1w_1, v_2w_2\}.$$

Then G_m has three blocks: two K_2's and a cycle C_m, which is Hamiltonian. If Q denotes a longest end-path and P denotes a longest poly-path in G_m, then $l(Q) = 1$ and $l(P) = m - 1$. But by Theorem 3, $h(G_m) = 1$, whereas $\max\{\{l(Q)\}, \{1 + l(P)\}\} = m$.

Acknowledgement

The author wishes to thank P.A. Catlin, the author's Ph.D. supervisor, for his many helpful suggestions.

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