Group Connectivity of Graphs

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- $D(G)$: = an orientation of G.
- $D = (d_{ij})_{n \times m} :=$ vertex-edge incidence matrix, where

$$d_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is oriented away from } v_i \\ -1 & \text{if } e_j \text{ is oriented into } v_i \\ 0 & \text{otherwise} \end{cases}$$

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- A function $b : V \mapsto A$ can be viewed as an n-dimensional vector

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Nowhere-zero A-flows (or A-NZFs)

- **Assumption:** For any graph G, we assume that a fixed orientation $D(G)$ of G is given.

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- For any $f \in F(G, A)$, the boundary of f is $\partial f := Df$. That is, $\forall v_i \in V, \partial f(v_i) = Df(v_i)$, which is the v_ith component of the vector Df.

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- A function $f \in F^*(G, A)$ is a nowhere-zero A-flow (or just an A-NZF) if $Df = \mathbf{0}$ (the all zero vector).

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- Tutte: If G has a k-NZF, then G has a $(k+1)$-NZF.
- Tutte: A graph G has an A-NZF if and only if G has an $|A|$-NZF.

Some Properties

- If some orientation $D(G)$ has an A-NZF or a k-NZF, then for any orientation of G also has the same property, and so having an A-MZF or a k-NZF is independent of the choice of the orientation.

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- If for an abelian group A, a connected graph G has an A-NZF, then G must be 2-edge-connected. (That is, G does not have a cut edge).
- We shall only consider 2-edge-connected graphs G and define

$$\Lambda(G) = \min\{k : \ G \text{ has a } k - NZF\}.$$

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- (3-flow) Every 4-edge-connected graph has a 3-NZF.
- (Jaeger's weak 3-flow conjecture) There exists an integer $k > 0$ such that every k-edge-connected graph has a 3-NZF.

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- These conjectures are theorems when restricted to planar graphs (need 4 Color Theorem for the 4-flow conjecture).

What do we know?

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- The 5-flow conjecture and 3-flow conjecture have also been verified for projective planes and some other surfaces.

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- Z. H. Chen, H. Y. Lai and HJL (2002, DM): Tutte's flow conjectures are valid if and only if they are valid within line graphs.

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- Any $b : V \mapsto A$ with $\sum_{v \in V(G)} b(v) = 0$ is an A-zero-sum function. The set of all A-zero-sum functions is $Z(G, A)$.

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- $\Lambda(G) \leq \Lambda_g(G)$.

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- Jeager et al (1992) and HJL (1998): For the n-cycle C_n, $\Lambda_g(C_n) = n + 1$.

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- Kral, Pangrac and Voss, (2006, JGT): There exists a family of 4-edge-connected planar graphs G with $\Lambda_g(G) = 4$.

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Complete Bipartite Graphs

J. Chen, E. Eschen and HJL (2008, Ars Comb): Let $m \geq n \geq 2$ be integers. Then

$$\Lambda_g(K_{m,n}) = \begin{cases} 5 & \text{if } n = 2 \\ 4 & \text{if } n = 3 \\ 3 & \text{if } n \geq 4 \end{cases}.$$

Let G be a graph with $u'v' \in E(G)$ and H be a graph with $uv \in E(H)$. $G \oplus H$ denotes the graph obtained from the disjoint union of $G - \{u'v'\}$ and H by identifying u' and u and identifying v' and v.

Chordal Graphs:

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- As examples, all complete graphs or order at least 3 are chordal.
- **Problem:** Determine the group connectivity of chordal graphs.

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- Let G be a 3-connected chordal graph. Then $\Lambda_g(G) = 3$ if and only if $G \ncong K_4$.
- Let G be 2-connected (but not 3-connected) chordal graph. Then $\Lambda_g(G) = 4$ if and only if $G \in \{K_3, K_4\}$ or G has two subgraphs G_1 and G_2 such that both $\Lambda_g(G_1)$ and $\Lambda_g(G_2)$ are 4, and such that $G = G_1 \oplus G_2$.

Graphs with Diameter at most 2

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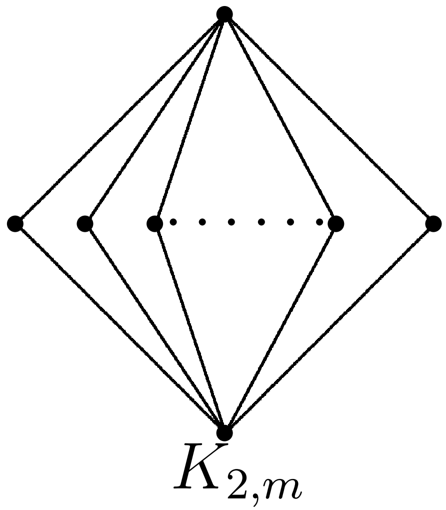
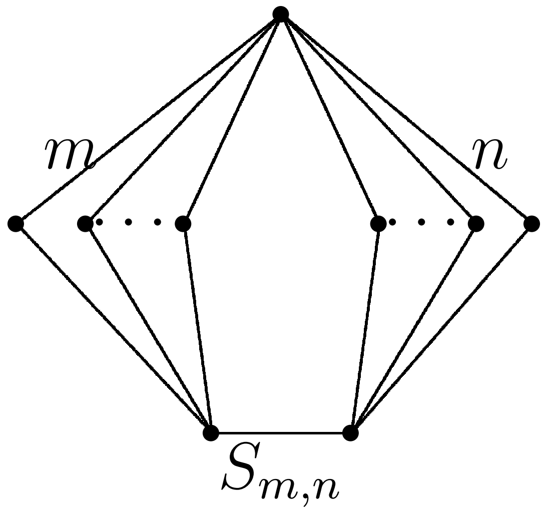
- X. Yao and HJL (2006, EJC): If G is a 2-edge-connected graph with diameter at most 2, then
  (i) $\Lambda(G) \leq 6$, and $\Lambda_g(G) = 6$ if and only if $G = C_5$.
  (ii) If $G \neq C_5$, then $\Lambda_g(G) \leq 5$, where equality holds if and only if $G = P_{10}$, the Petersen graph, or $G \in \{S_{m,n}, K_{2,n}\}$.

Graphs with Diameter at most 2





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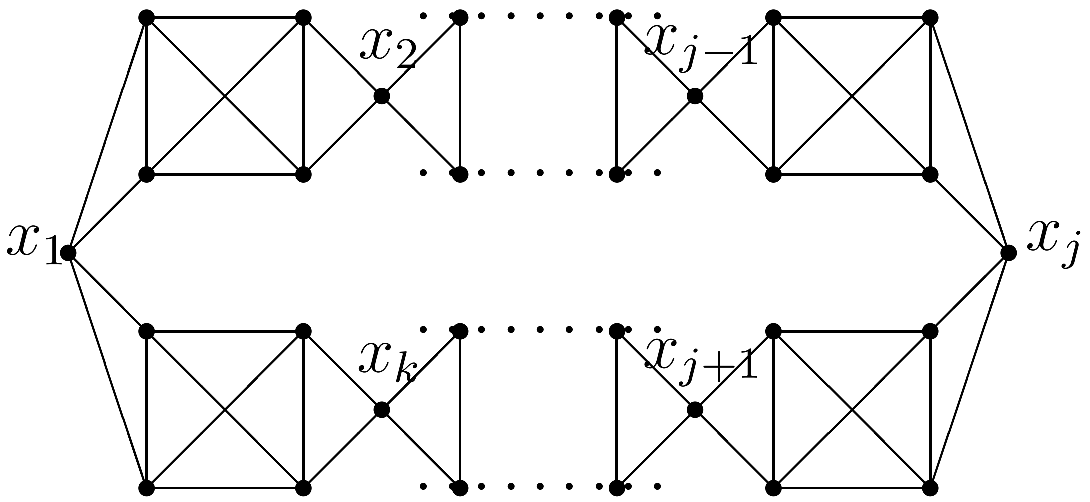
- **Conjecture (Davos, 2003)** If G is a 4-edge-connected triangulated graph, then $\Lambda_g(G) \leq 3$.

Triangulated Graphs

Xu, Zhou and HJL (2008 GC) found an infinite family of 4-edge-connected triangulated graphs G with $\Lambda_g(G) = 4$.



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- **Corollary** If G is a 3-edge-connected, triangularly-connected graph, then $\Lambda_g(G) \leq 3$.

Barat-Thomassen's Approach

- A graph G with $|E(G)| \equiv 0$ (mod 3) has a **claw-decomposition** if $E(G)$ is a disjoint union $E(G) = X_1 \cup X_2 \cup \cdots \cup X_k$ such that for each i with $1 \le i \le k$, $G[X_i]$ is a generalized claw.

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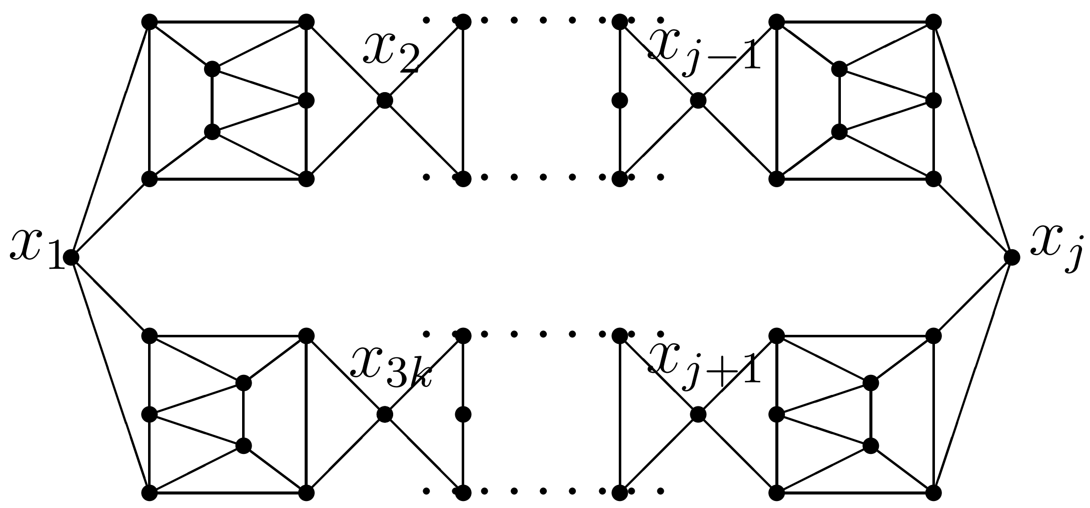
- **Theorem (Barat and Thomassen, 2006, JGT)** There exists a function $f(k)$ such that If every k-edge-connected graph G with $|E(G)| \equiv 0 \pmod 3$ has a claw-decomposition, then every $f(k)$-edge-connected graph G has a 3-NZF.

Barat-Thomassen's Approach

- **Conjecture (Barat and Thomassen, 2006, JGT)** Every 4-edge-connected simple planar graph G with $|E(G)| \equiv 0 \pmod 3$ has a claw-decomposition.

Counterexample

- (HJL, SIAM J of DM, 2007) An infinite family of 4-edge-connected simple planar graph G with $|E(G)| \equiv 0 \pmod 3$ is constructed which does not have a $K_{1,3}$-decomposition.



Justification

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- Let H_i ($i = 1, 2, ..., 3k$ denote a "building block".

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- $16k = 2|W| = \sum_{i=1}^{3k} |V(H_i \cup H_{i+1} - \{y_{i+1}\}) \cap W| \le 5 \times 3k = 15k$, a contradiction.

Thank You!