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Supereulerian Graphs and the Petersen Graph

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Any 3-edge-connected graph with at most 10 edge cuts of size 3 either has a spanning closed trail or it is contractible to the Petersen graph.

INTRODUCTION AND NOTATION

We shall follow the notation of Bondy and Murty [1], but with minor variations. An *arc* in a graph G is a path in G whose internal vertices have degree 2 in G. Denote

$$O(G) = \{\text{odd-degree vertices of } G\}.$$

A graph G is called *even* if $O(G) = \emptyset$, and G is called *eulerian* if G is even and connected. If G has a spanning eulerian subgraph, then G is called *supereulerian*, and we write $G \in \mathcal{SL}$.

Tutte [19, 20] and Matthews [17] conjectured that if a 2-edge-connected graph G has no subgraph contractible to the Petersen graph, then G has a 3-colorable double cycle cover (i.e., a collection of three even subgraphs such that each edge of G lies in exactly two of them). We showed before (see [5 or 6]) that any supereulerian graph has a 3-colorable double cycle cover. In this context, our present result, that any 3-edge-connected graph with at most 10 edge cuts of size 3 is either supereulerian or is itself contractible to the Petersen graph, is of interest. Jaeger [12] had previously shown that any 4-edge-connected graph supereulerian. Catlin [5] recently showed that

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a graph with no cut edge, and with at most 13 edge cuts of size 3, either has a 3-colorable double cycle cover, or it is contractible to the Petersen graph.

Kelmans and Lomonosov [15] and Ellingham, Holton, and Little [10] had proved this analogous result: a 3-connected cubic graph G has a cycle passing through any 10 given vertices if and only if G is not contractible to the Petersen graph in a way such that the 10 given vertices each map to a distinct vertex of the Petersen graph. This result improved a prior version due to Holton, McKay, Plummer, and Thomassen [11] and to Kelmans and Lomonosov [14], in which 10 was replaced by 9 and the Petersen graph exception did not arise (except as an example to show that 9 was best possible). Lomonosov [16] showed that the number 10 of the former result can be improved to 11, but Kelmans [13] obtained an infinite class of graphs to show that 10 could not be improved to 14.

When S is an even subset of $V(G)$, and S-*subgraph* Γ of G is a subgraph Γ such that both $G - E(\Gamma)$ is connected and $O(\Gamma) = S$. We call a graph G *collapsible* if for every even subset S, G has an S-subgraph. Denote the family of collapsible graphs by $\mathscr{CL}$. Of course, $K_1 \in \mathscr{CL} \subseteq \mathscr{SL}$.

Let G be a graph, and let $X \subseteq E(G)$. Then the contraction G/X is the graph obtained from G by contracting all edges in X and by deleting any resulting loops. Contractions can create multiple edges. If H is a subgraph of G, then G/H denotes $G/E(H)$; if $e \in E(G)$ then G/e denotes $G/\{e\}$.

Catlin [2] showed that for any graph G, there is a unique collection $H_1, H_2, \ldots, H_c$ of maximal collapsible subgraphs of G, and each vertex of G is in some H_i. The reduction G' of G is obtained from G by contracting those H_i's $(1 \leqslant i \leqslant c)$. A graph is reduced if its reduction is itself, and K_1 is the only collapsible reduced graph.

1. Prior Results

The *arboricity* of G, denoted $a(G)$, is the minimum number of forests whose union contains G.

THEOREM 1.1 (Nash-Williams [18]). *For any graph G,*

$$a(G) = \max \lceil E(H)/(|V(H)| - 1) \rceil,$$

where this maximum is taken over all nontrivial subgraphs of G.

For a graph G, let $F(G)$ denote the minimum number of extra edges that must be added to G, to obtain a spanning supergraph having two edge-disjoint spanning trees. If $a(G) \leqslant 2$, then

$$F(G) = 2\,|V(G)| - 2 - |E(G)|. \tag{1}$$

THEOREM 1.2 [2]. *Let G be a graph, and let G' be the reduction of G. Then*

(a) $G \in \mathcal{SL} \Leftrightarrow G' \in \mathcal{SL}$;

(b) $G \in \mathcal{CL} \Leftrightarrow G' = K_1$;

(c) *G is reduced if and only if G has no nontrivial collapsible subgraphs*;

(d) *If G is reduced, then G has no cycle of length at most* 3, $a(G) \leqslant 1$, $\delta(G) \leqslant 3$, *and either* $G \in \{K_1, K_2\}$ *or* $2\,|V(G)| - |E(G)| \geqslant 4$.

THEOREM 1.3 [6]. *Let G be a graph with $F(G) \leqslant 2$. Exactly one of these holds*:

(a) $G \in \mathcal{CL}$;

(b) $F(G) \in \{1, 2\}$ *and the reduction of G is K_2*;

(c) $F(G) = 2$ *and the reduction of G is either $2K_1$ or $K_{2,t}$* $(t \geqslant 1)$.

THEOREM 1.4 [2]. *Let G be a graph. Then $G \in \mathcal{SL}$ if and only if G has a spanning tree T such that each component of $G - E(T)$ has evenly many vertices in $O(G)$.*

THEOREM 1.5 [2]. *Let G be a graph with $F(G) = 1$. If no nontrivial subgraph H has $F(H) = 0$, then for any $u \in V(G)$, $G - u$ has a spanning tree U such that $G - E(U)$ is a spanning tree of G.*

THEOREM 1.6 [4]. *Let G be a graph, let $wxyzw$ be a* 4*-cycle in G, and define the partition $\pi = \{w, y\} \cup \{x, z\}$. Define G/π to be the graph obtained from $G - \{wx, xy, yz, zw\}$ by identifying w and y to form a single vertex u, by identifying x and z to form a single vertex v, and by adding the extra edge uv. Each of the following holds*:

(a) $G/\pi \in \mathcal{CL} \Rightarrow G \in \mathcal{CL}$;

(b) $G/\pi \in \mathcal{SL} \Rightarrow G \in \mathcal{SL}$;

(c) [6] *If G is reduced, then* $F(G/\pi) = F(G) - 1$.

THEOREM 1.7 [8]. *If a* 3*-edge-connected graph G has order at most* 13, *then either $G \in \mathcal{SL}$ or the reduction of G is the Petersen graph.*

2. ASSOCIATED RESULTS

THEOREM 2.1. *Let G be a graph with $\kappa'(G) \geqslant 2$, with $F(G) = 2$ and with no nontrivial* 2*-edge-connected subgraph H satisfying $F(H) \leqslant 1$. Let $yw \in E(G)$.*

Then $G-\{y, w\}$ *has a spanning tree* U *such that* $T=G-EU)$ *is a spanning tree of* G.

Proof. If $G/\{yw\}$ has a nontrivial subgraph H' with $F(H')=0$, then by definition of F, $F(G[E(H' \cup \{yw\}])=1$, contrary to the assumption that no such subgraph exist. Therefore, $a(G/\{yw\}) \leqslant 2$, and so by (1), $F(G/\{yw\})=1$. Let u denote the vertex of $G/\{yw\}$ corresponding to $yw \in E(G)$. By Theorem 1.5, $G/\{yw\}-u$ has a spanning tree U such that $G/\{yw\}-E(U)$ is a spanning tree of $G/\{yw\}$. Thus U is a spanning tree of $G-\{y, w\}$ and $T=G-E(U)$ is a spanning tree of G. ∎

Given a graph G and an edge $yw \in E(G)$ such that Theorem 2.1 is satisfied by trees T and U, call that ordered pair (T, U) of trees a *tree decomposition of* G *with respect to* yw. A tree decomposition with respect to yw is called *Type* 1 if $d(y)$ is odd and $d(w)$ is even. Because of asymmetries arising later, a tree decomposition with respect to yw is not regarded the same as a tree decomposition with respect to wy.

Let (T, U) be a tree decomposition of G with respect to yw. Denote

$$N(y)=\{w, y_1, y_2, ..., y_r\}, \tag{2}$$

and let Y_i denote the component of $T-\{y, w\}$ containing y_i $(1 \leqslant i \leqslant r)$.

DEFINITION 2.2. Let $\mathscr{F}_w$ denote the family of graphs G with a distinguished vertex w, such that these properties hold:

(i) $\kappa'(G) \geqslant 3$;

(ii) $F(G)=2$;

(iii) No nontrivial 2-edge-connected subgraph H of G has $F(H) \leqslant 1$;

(iv) $G-w$ is reduced and contains no 4-cycle.

THEOREM 2.3. *Let* G *be a graph and let* $w \in V(G)$. *Suppose* $G \in \mathscr{F}_w$, *and let* $y \in N(w)$. *There is a tree decomposition of* G *with respect to* yw, *such that* $Y_1=\{y_1\}$ *and* $Y_2=\{y_2\}$, *where* y_1 *and* y_2 *are defined in* (2).

Proof. By Theorem 2.1, G has a tree decomposition (T, U) with respect to yw. In the rest of the proof, we shall omit the phrase "with respect to yw" since yw is understood. It suffices to show that both $Y_1 \cong K_1$ and $Y_2 \cong K_1$. Begin with a tree decomposition (T, U) with

$$|V(Y_1)| \text{ minimized.} \tag{3}$$

Claim 1. $Y_1 \cong K_1$. If $U[V(Y_1)]$ is connected, then $U[V(Y_1)]$ has two edge-disjoint spanning trees, and so by Definition 2.2(iii), $Y_1=K_1$. Hence we assume that $U[V(Y_1)]$ has components $U_0, U_1, ...,$ with $y_1 \in V(U_0)$.

Choose a tree decomposition such that, subject to (3),

$$|V(U_0)| \text{ is maximized.} \tag{4}$$

Choose $e_1 = u_0 u_1 \in E(Y_1)$ nearest to y_1 in Y_1 with $u_0 \in V(U_0)$ and $u_1 \in V(U_1)$ (say). Then $Y_1 - e_1$ has two components Y_1' and Y_1'', where $y_1, u_0 \in V(Y_1')$ and $u_1 \in V(Y_1'')$. Note that U has a unique path (v_0, v_k)-path $P_1 = v_0 v_1 \cdots v_k$ such that $v_0 \in V(U_0)$ and $v_k \in V(U_1)$ with $(V(P_1) - \{v_0, v_k\}) \cap V(U_0 \cup U_1) = \emptyset$. Note that $v_{k-1} \notin V(Y_1)$.

If $v_k \in V(Y_1'')$, then let

$$U' = U - v_{k-1} v_k + u_0 u_1, \qquad T' = T - u_0 u_1 + v_{k-1} v_k.$$

Since $v_{k-1} v_k$ is in the unique cycle of $U + u_0 u_1$, U' is a tree with $V(U') = V(U)$. Since $v_{k-1} v_k$ connects Y_1'' to $T - V(Y_1)$, T' is a tree with $V(T') = V(T)$. Thus (T', U') is a tree decomposition violating (3).

Hence we must have $v_k \in V(Y_1')$. Let P_2 be the unique (u_1, v_k)-path in U_1. Let $e_2 = u_1'' u_1' \in E(P_2)$ be such that $u_1' \in V(Y_1')$ and $u_1'' \in V(Y_1'')$, and such that u_1'' is nearest to u_1 in U_1. Let

$$U'' = U - u_1' u_1'' + u_0 u_1, \qquad T'' = T - u_0 u_1 + u_1' u_1''.$$

Since $u_0 u_1$ connects U_0 and the component of $U_1 - u_1' u_1''$ that does not contain v_k, U'' is a tree with $V(U'') = V(U)$. Since $u_1' u_1''$ connects Y_1' and Y_1'', T'' is a tree with $V(T'') = V(T)$. Thus (T'', U'') is a tree decomposition satisfying (3) but violating (4). This proves Claim 1.

Now choose a tree decomposition (T, U) such that, subject to $Y_1 \cong K_1$,

$$|V(Y_2)| \text{ is minimized.} \tag{5}$$

Claim 2. $Y_2 \cong K_1$. As in Claim 1, we may assume that $U[V(Y_2)]$ has components $H_0, H_1, ..., H_c$ with $y_2 \in V(H_0)$ for some $c > 0$. In Y_2, there is an edge e_i (say) nearest to y_2 such that exactly one end of e_i is in $V(H_i)$. Denote by Y_2' and Y_2'' the two components of $Y_2 - e_i$, where $y_2 \in V(Y_2')$. Then $V(H_i) \subseteq V(Y_2'')$, by the choice of e_i. Also by the choice of e_i, the unique cycle C of $U + e_i$ has an edge e_i'' with exactly one end in $V(H_i)$. If the other end of e_i'' is not y_1, then one can imitate the argument in Claim 1 to obtain a contradiction. Hence that other end of e_i'' is y_1.

Let $H = G[V(Y_2) \cup \{y_1\}]$. By Definition 2.2(iv), H is reduced. Since $E(H)$ is the disjoint union of $E(Y_2) \cup E(U[V(Y_2)]) \cup \{e_1'', ..., e_c''\}$ and since $U[V(Y_2)]$ has c components, we have by (1) that $F(H) = 2$. By Theorem 1.3 and by $y_1, y_2 \in V(H)$, either $H \cong K_{2,t}$ for some t, or $H \in \{K_2, 2K_1\}$. If $H \in \{K_2, K_{2,t}\}$, then $G[V(H) \cup \{y\}]$ must have a 3-cycle or a 4-cycle contrary to Definition 2.2(iv). Hence $H = 2K_1$ and so $Y_2 \cong K_1$. ∎

3. The Main Results

When $F(G) \leqslant 2$, the reduction of G is characterized by Theorem 1.3. Here is a useful partial result for 3-edge-connected graphs with $F(G)=3$.

Theorem 3.1. *Let G be a 3-edge-connected graph with $F(G)=3$. If G is reduced, then either $G \in \mathcal{SL}$ or each of the following holds:*

(a) *G has no edge joining two vertices of even degree;*

(b) *G has girth at least 5;*

(c) *G has no 2-edge-connected subgraph H with $F(H)=2$.*

Proof. Suppose that

$$G \text{ is a counterexample.} \tag{6}$$

Thus, G is 3-edge-connected,

$$F(G)=3, \tag{7}$$

$$G \text{ is reduced,} \tag{8}$$

$$G \notin \mathcal{SL}, \tag{9}$$

and at least one of $\{$(a), (b), (c)$\}$ fails.

Lemma 3.2. *G has girth at least 5.*

Proof. By (8), G is reduced, and so by Theorem 1.2(d), G has girth at least 4. By way of contradiction, suppose that $wxyzw$ is a 4-cycle in G. Define the partition π of $\{w, x, y, z\}$ to be $\{w, y\} \cup \{x, z\}$, and define u and v and G/π as in Theorem 1.6. Since $\kappa'(G) \geqslant 3$, either G/π is 2-edge-connected, or uv is a cut edge of G/π.

Case 1. Suppose that G/π is 2-edge-connected. By Theorem 1.6(c) and by (7), $F(G/\pi)=2$. By Theorem 1.3, this implies that either $G/\pi \in \mathcal{SL}$ or G/π is contractible to $K_{2,t}$ for some odd $t \geqslant 3$ (since $\kappa'(G/\pi) \geqslant 2$, since $\mathcal{CL} \subseteq \mathcal{SL}$, and by Theorem 1.2(a)). If $G/\pi \in \mathcal{SL}$, then Theorem 1.6(b) gives $G \in \mathcal{SL}$, contrary to (9). If G/π is contractible to $K_{2,t}$ $(t \geqslant 3)$, then G cannot be 3-edge-connected, a contradiction. This precludes Case 1.

Case 2. Suppose that uv is a cut edge of G/π. Denote by G_w and G_x the two components of $G - \{wx, xy, yz, zw\}$, where $\{w, y\}$ is the set of vertices of attachment in G_w and where $\{x, z\}$ is the set of vertices of attachment in G_x. We lose no generality assuming

$$F(G_w) \leqslant F(G_x). \tag{10}$$

Applying (1) to G_w, G_x and G and using (7), we get

$$\begin{aligned} F(G_w)+F(G_x) &= 2\,|V(G_w)| - |E(G_w)| + 2\,|V(G_x)| - |E(G_x)| - 4 \\ &= 2\,|V(G)| - |E(G) - \{wx, xy, yz, zw\}| - 4 \\ &= F(G)+2 = 5. \end{aligned} \tag{11}$$

By (10) and (11), $F(G_w) \leqslant 2$. By (8) and by Theorem 1.2(c), G_w is reduced. Therefore if $F(G_w) \leqslant 1$ then $G_w \in \{K_1, K_2\}$, by Theorem 1.3. But $w, y \in V(G_w)$, and so $F(G_w) \leqslant 1$ would force $wy \in E(G_w)$. This would imply that the reduced graph G contains the triangle $wxyw$, contrary to Theorem 1.2(d). Hence, $F(G_w) = 2$. By Theorem 1.3(c), $G_w \cong K_{2,t}$ for some $t \geqslant 1$. However, G_w has only two vertices of attachment (w and y), whereas G_w has at least three vertices of degree less than 3 in G_w. Hence, some vertex of G_w has degree less than 3 in G, contrary to $\kappa'(G) \geqslant 3$. Thus, Case 2 is also impossible, and so G has no 4-cycle. This proves Lemma 3.2. ∎

LEMMA 3.3. *G has no 2-edge-connected subgraph H with $F(H) = 2$.*

Proof. By way of contradiction, suppose that the subgraph H has $F(H) = 2$. By (8), G is reduced, and so by Theorem 1.2(c), H is also reduced. Now, Theorem 1.3 implies that either $H \in \mathscr{CL}$ (in which case Theorem 1.2(b) implies $H = K_1$, since H is reduced, and hence we have the contradiction given by $2 = F(H) = F(K_1) = 0$), or the reduction of H is $K_{2,t}$ for some $t \geqslant 2$. In the latter case, since H is already reduced, $H \cong K_{2,t}$. But then H contains a 4-cycle of G, contrary to Lemma 3.2. ∎

By Lemmas 3.2 and 3.3, neither (b) nor (c) of Theorem 3.1 fails. Therefore, by the remark preceding Lemma 3.2, (a) must fail. Hence G has an edge xz such that

$$d(x) \quad \text{and} \quad d(z) \text{ are even.} \tag{12}$$

Let $y \in N(x) - z$, so that yxz is a path in G, where $F(G) = 3$. We define a *tree decomposition of G with respect to yxz* to be an ordered pair (T, U) of trees in G such that U is a tree spanning $G - \{y, x, z\}$ and $T = G - E(U)$ is a spanning tree of G. A tree decomposition with respect to yxz is not regarded as the same as a tree decomposition with respect to zxy.

Let w be the vertex of G/xz corresponding to xz.

LEMMA 3.4. *Both*

$$F(G/xz) = 2, \tag{13}$$

and G has a tree decomposition (T, U) with respect to yxz, such that $(T/xz, U)$ is the corresponding tree decomposition of G/xz with respect to yw. Furthermore, G/xz has no nontrivial 2-edge-connected subgraph H with $F(H) \leqslant 1$.

Proof. By (8), G is reduced, and so it follows from Theorem 1.2(d) that G has no nontrivial 2-edge-connected subgraph H with $F(H) \leqslant 1$. Hence, $a(G/xz) \leqslant 2$, by Theorem 1.1 (for if $a(G/xz) > 2$ then Theorem 1.1 asserts that G/xz has a 2-edge-connected subgraph H_1 with $|E(H_1)| > 2\,|V(H_1)| - 2$; and so $E(H_1)$ or $E(H_1) \cup \{xz\}$ induces in G a nontrivial 2-edge-connected subgraph H violating $F(H) \leqslant 1$, a contradiction). Thus by (1) and (7), we have (13), and G/xz has no 2-edge-connected subgraph H_0 with $F(H_0) = 0$. By (1) and Lemma 3.3 and by an imitation of that prior argument in parentheses in this proof, G/xz has no nontrivial 2-edge-connected subgraph H satisfying $F(H) \leqslant 1$ (the last part of the lemma). Hence, G/xz satisfies the hypothesis of Theorem 2.1. Recall that w is the vertex of G/xz corresponding to xz. Then $yw \in E(G/xz)$, and so by Theorem 2.1, G/xz has a tree decomposition with respect to yw. This induces a tree decomposition (T, U) of G with respect to yxz, where $(T/xz, U)$ is the corresponding tree decomposition of G/xz with respect to yw. ▮

LEMMA 3.5. $G/xz \in \mathscr{F}_w$.

Proof. We know that $\kappa'(G) = 3$, so $\kappa'(G/xz) \geqslant 3$; i.e., G/xz satisfies (i) of the definition of $\mathscr{F}_w$. By (13) of Lemma 3.4, G/xz satisfies (ii) of the definition of $\mathscr{F}_w$, and by the last part of Lemma 3.4, G/xz satisfies (iii). By Lemma 3.2 and the definition of w, $G/xz - w$ has no 4-cycle, and by (8) and the definition of w, $G/xz - w$ is reduced. Hence (iv) holds, and so Lemma 3.5 is proved. ▮

LEMMA 3.6. *$d(y)$ is odd.*

Proof. By way of contradiction, suppose $d(y)$ is even. By Lemma 3.4, G has a tree decomposition (T, U) with respect to yxz. By (12) and since $d(y)$ is even, $O(G) \subseteq V(U)$. Hence by Theorem 1.4, $G \in \mathscr{S}\mathscr{L}$, contrary to (9). ▮

Define $y_1, ..., y_r$ by

$$N(y) = \{x, y_1, y_2, ..., y_r\}, \tag{14}$$

where $\kappa'(G) \geqslant 3$ implies $r \geqslant 2$. Recall that w is the vertex of G/xz corresponding to xz. By Lemma 3.5 and Theorem 2.3, G/xz has a tree decomposition with respect to yw such that

$$Y_1 = \{y_1\}, \qquad Y_2 = \{y_2\}, \tag{15}$$

where Y_i is the component of $T-\{y, w\}$ containing y_i. Notice that this tree decomposition of G/xz with respect to yw induces a corresponding tree decomposition (T, U) of G with respect to yxz, where the correspondence is given in Lemma 3.4. Furthermore, we can define Y_i $(i=1, 2)$ to be the component of $T-\{y, x, z\}$ containing y_i, and both (14) and (15) hold for this tree decomposition of G with respect to yxz, just as (2) and (15) hold for the corresponding tree decomposition of G/xz with respect to yw. Denote

$$N_U(y_1)=\{u_1, u_2, ..., u_t\}, \qquad N_U(y_2)=\{v_1, v_2, ..., v_k\}. \tag{16}$$

Also, denote by U_i $(1 \leqslant i \leqslant t)$ the component of $U-y_1$ containing u_i, and denote by V_i $(1 \leqslant i \leqslant k)$ the component of $U-y_2$ containing v_i. Without loss of generality, suppose

$$y_2 \in V(U_1), \qquad y_1 \in V(V_1). \tag{17}$$

For each u_i $(1 \leqslant i \leqslant t)$ there is a unique vertex $u'_i \in N(y) \cup N(w) - \{y, w\}$ lying in the same component of $T-\{y, w\}$ as u_i. Likewise, for each v_i $(1 \leqslant i \leqslant k)$, there is a unique vertex $v'_i \in N(y) \cup N(w) - \{y, w\}$ in the same component of $T-\{y, w\}$ as v_i.

A *theta graph* Θ consists of exactly three paths that connect two vertices of degree 3. The proof for Lemma 3.7 is routine.

LEMMA 3.7. *Let T be a connected spanning subgraph of G and let $U=G-E(T)$. Each of the following holds*:

(i) *Suppose that $e \in E(U)$. If $e' \in E(T)$ lies in a cycle of $T+e$, the $T'=T-e'+e$ is also a spanning connected subgraph of G.*

(ii) *Let $e, f \in E(U)$ be such that $T+e+f$ has a theta graph Θ. If $e', f' \in (T) \cap E(\Theta)$ and $\Theta-\{e', f'\}$ is connected, then $T'=T+e+f-e'-f'$ is a spanning connected subgraph of G.*

LEMMA 3.8. *Each U_i $(1 \leqslant i \leqslant t)$ and each V_i $(1 \leqslant i \leqslant k)$ has an odd number of vertices in $O(G)$.*

Proof. We only present the proof for the U_i's, since the proof for the V_i's is similar. By contradiction, we assume that for some i $(1 \leqslant i \leqslant t)$, $|V(U_i) \cap O(G)|$ is even. Note that yy_1 is in the unique cycle of $T+y_1u_i$. By Lemma 3.7(i), $T'=T+y_1u_i-yy_1$ is a spanning tree of G and $G-E(T')$ has four components $\{x\}$, $\{z\}$, U_i, and $U-V(U_i)+yy_1$, each of which has evenly many vertices in $O(G)$. By Theorem 1.4, $G \in \mathcal{SL}$, contrary to (6). ∎

Let P_j denote the unique (u_j, u'_j)-path in $T-\{y, w\}$ and let Q_j denote the unique (v_j, v'_j)-path in $T-\{y, w\}$.

LEMMA 3.9. *Both* $V(P_i) \subseteq V(U_i)$ $(1 \leqslant i \leqslant t)$ *and* $V(Q_i) \subseteq V(V_i)$ $(1 \leqslant i \leqslant k)$.

Proof. We only present the proof for the P_i's, since the proof for the Q_i's is similar. By contradiction, assume that $V(P_j) \nsubseteq V(U_j)$ for some j $(1 \leqslant j \leqslant t)$. Let T_0 be the smallest subtree of T that contains every edge in $\bigcup_{i=1}^{t} E(P_i)$ whose ends are in distinct U_i's. Then $E(T_0) \neq \emptyset$. Let $e \in E(T_0)$ be an edge incident with an endvertex (vertex of degree 1) of T_0, and choose j so that $e \in E(P_j)$. Note that e has exactly one end in U_j (the end closer on P_j to u_j), and that e separates u_i from u_j in T for any $i \neq j$. Suppose, without loss of generality, that the end of e outside $V(U_j)$ is in $V(U_i)$ for some $i \neq j$. Thus u_i and u_j are in distinct components of $T-e$. Also, $T+y_1u_i+y_1u_j$ has a theta graph Θ in which y_1 has degree 3 and $\Theta-e-yy_1$ is connected. By Lemma 3.7(ii), $T'=T+y_1u_i+y_1u_j-e-yy_1$ is a spanning tree of G, and $G-E(T')$ has four components $\{x\}$, $\{z\}$, $(U_i \cup U_j)+e$ and $U-V(U_i \cup U_j)+yy_1$. By Lemma 3.8, each of these four components has evenly many vertices in $O(G)$. By Theorem 1.4, $G \in \mathcal{SL}$, contrary to (6). ∎

LEMMA 3.10. *Each* u'_i $(1 \leqslant i \leqslant t)$ *and each* v'_i $(1 \leqslant i \leqslant k)$ *are adjacent in* G/xz *to* w.

Proof. Suppose, by contradiction, that for some j $(1 \leqslant i \leqslant t)$, $u'_j \in N(y)$. By Lemma 3.9, $u'_j \in V(U_j)$. Note that the cycle in $T+y_1u_j$ contains yu'_j. By Lemma 3.7(i), $T'=T+y_1u_j-yu'_j$ is a spanning tree of G, such that $G-E(T')$ has four components $\{x\}$, $\{z\}$, $U_j+yu'_j$, and $U-V(U_j)$. By Lemma 3.6, each of these four components has an even number of vertices in $O(G)$, and so by Theorem 1.4, $G \in \mathcal{SL}$, contrary to (9). ∎

Hence in G we can define u'_i $(1 \leqslant i \leqslant t)$ to be the sole vertex of $T-\{x, y, z\}$ adjacent in T to one of $\{x, y, z\}$, such that u_i and u'_i are in the same component of $T-\{x, y, z\}$; and since u'_i in G/xz is adjacent to w, we have in G that

$$u'_i \in N(x) \cup N(z). \tag{18}$$

Likewise, define v'_i to be the sole vertex of $T-\{x, y, z\}$ adjacent in T to one of $\{x, y, z\}$, such that v_i and v'_i lie in the same component of $T-\{x, y, z\}$. By Lemma 3.10,

$$v'_i \in N(x) \cup N(z). \tag{19}$$

LEMMA 3.11. $|V(U_1 \cap V_1) \cap O(G)|$ *is odd. Also,* $d(y_1) \not\equiv t \pmod 2$, *and* $d(y_2) \not\equiv k \pmod 2$.

Proof. By (15) and the definitions of U_i and V_i,

$$V(U_1) = V(U_1 \cap V_1) \cup \{y_2\} \cup \bigcup_{i=2}^{k} V(V_i). \tag{20}$$

By Lemma 3.8, $|V(V_i) \cap O(G)|$ is odd ($2 \leqslant i \leqslant k$). By (16) and

$$k+1 = d_U(y_2) + 1 = d(y_2), \tag{21}$$

$y_2 \in O(G)$ if and only if k is even. It follows that $\{y_2\} \cup \bigcup_{i=2}^{k} V(V_i)$ must have evenly many vertices in $O(G)$. This fact, the fact that $|V(U_1) \cap O(G)|$ is odd, and (20) together imply the first part of Lemma 3.11. From (21) comes $d(y_2) \not\equiv k \pmod 2$, and a similar argument gives $d(y_1) \not\equiv t \pmod 2$. ∎

Case 1. Suppose that $u_2', v_2' \in N(z)$. Define

$$T' = T + \{y_1u_2, y_2v_2, y_1u_1\} - \{u_2'z, v_2'z, yy_1\}.$$

We claim that T' is a spanning tree of G and that each component of $G - E(T')$ has evenly many vertices in $O(G)$.

To see the first claim, consider the graph

$$H = T + \{y_1u_2, y_2v_2, y_1u_1\}.$$

Let H_0 be the maximal 2-edge-connected subgraph in H. Either $u_1'x$ or $u_1'z$ is an edge of T, but not both.

Suppose $u_1'x \in E(T)$. Then by Lemma 3.9 and the hypothesis of Case 1, H_0 is a subdivision of K_4 whose degree 3 vertices are $\{y_1, y, x, z\}$ and whose six arcs are yy_1, yx, xz, $P_2 \cup \{u_2'z, y_1u_2\}$, $Q_2 \cup \{v_2'z, y_2v_2, yy_2\}$, and $P_1 \cup \{y_1u_1, u_1'x\}$.

If $u_1' \in E(T)$ then by Lemma 3.9 and the hypothesis of Case 1, H_0 has z of degree 4 and both y and y_1 of degree 3, and these three vertices are joined by the five arcs yy_1, yxz, $P_2 \cup \{u_2'z, y_1u_2\}$, $P_1 \cup \{y_1u_1, u_1'z\}$, and $Q_2 \cup \{v_2'z, y_2v_2, yy_2\}$.

In either case the edges $u_2'z$, $v_2'z$, and yy_1 lie on separate arcs of H_0 and when they are removed from H, the resulting graph T' is a tree spanning G. The first claim thus holds.

In either case the four components of $G - E(T')$ are $\{x\}$, $U_2 \cup V_2 \cup \{u_2'z, v_2'z\}$, $U_1 - V(V_2)$, and $(U \cup \{yy_1) - (V(U_1) \cup V(U_2))$. By Lemmas 3.8 and 3.11 and by (12) and (15), $O(G)$ has evenly many vertices in each component of $G - E(T')$. Hence by Theorem 1.4, $G \in \mathscr{SL}$, contrary to (9). This concludes Case 1.

Case 2. Suppose that $u_2', v_2' \in N(x)$. Imitate Case 1, interchanging z and x. (In addition to the interchange of z and x, there are minor alterations in the list of the six arcs of H_0.)

Case 3. Suppose that $u_2' \in N(x)$ and $v_2' \in N(z)$. (This is essentially the same as the case where $u_2' \in N(z)$ and $v_2' \in N(x)$, and so we need not consider that case.) Define

$$T' = T \cup \{y_1 u_2, y_2 v_1\} - \{u_2' x, xy\}.$$

First we show that T' is a spanning tree of G. Consider the graph

$$H = T \cup \{y_1 u_2, y_2 v_1\},$$

and note that H contains a theta graph whose degree 3 vertices are x and y. The edges xu_2' and xy lie on separate arcs of this theta graph, and when they are removed from H, we get the graph T' that is a tree spanning G.

The four components of $G - E(T')$ are $\{z\}$, $U_2 \cup \{u_2' x, xy\}$, $V_1 - V(U_2)$, and $U - V(V_1)$. By Lemmas 3.8 and 3.11, by (12), and by (15), each of these components has evenly many vertices in $O(G)$. Hence, by Theorem 1.4, $G \in \mathcal{SL}$, contrary to (9). This proves Case 3.

By (18) and (19), these cases exhaust all possibilities. Hence, Theorem 3.1 is proved. ∎

Jaeger [12] proved that a 4-edge-connected graph is supereulerian. A consequence of (c) of Theorem 1.3 is that a 3-edge-connected graph with at most nine edge cuts of size 3 is both supereulerian and collapsible [6]. Here we use Theorem 3.1 to improve that result.

THEOREM 3.12. *Let G be a* 3-*edge-connected graph. If G has at most* 10 *edge cuts of size* 3, *then exactly one of these holds*:

(a) $G \in \mathcal{SL}$;

(b) *The reduction of G is the Petersen graph.*

Proof. Suppose that G is a 3-edge-connected graph with at most 10 edge cuts of size 3. By way of contradiction, suppose that G is a smallest counterexample to Theorem 3.12. This implies that

$$G \notin \mathcal{SL} \tag{22}$$

and that

$$G \text{ is reduced.} \tag{23}$$

The justification for (23) is this: if G' is the reduction of G, then $\kappa'(G') \geqslant \kappa'(G) \geqslant 3$; G' has no more edge cuts of size 3 than does G;

$$G \in \mathcal{SL} \Leftrightarrow G' \in \mathcal{SL},$$

by Theorem 1.2(a); the reduction of G is the Petersen graph if and only if the reduction of G' is the Petersen graph; and G is assumed to be the smallest counterexample to Theorem 3.12.

Since G is reduced, Theorem 1.2(d) implies $a(G) \leqslant 2$, and so (1) holds. Let n_3 be the number of vertices of degree 3 in G. By the hypothesis of Theorem 3.12, $n_3 \leqslant 10$. By estimating in two ways the number of edge-vertex incidences, and by recalling $\delta(G) \geqslant 3$, we get

$$2\,|E(G)| \geqslant 4\,|V(G)| - n_3 \geqslant 4\,|V(G)| - 10, \tag{24}$$

and some inequality in (24) is strict if either $\Delta(G) > 4$ or $n_3 < 10$. By (1) and (24),

$$F(G) = 2\,|V(G)| - 2 - |E(G)| \leqslant 3. \tag{25}$$

If $F(G) \leqslant 2$, then a conclusion of Theorem 1.3 holds: it must be conclusion (a) since $\kappa'(G) \geqslant 3$. This implies $G \in \mathscr{CL} \subseteq \mathscr{SL}$, contrary to the assumption that G is a counterexample to Theorem 3.12.

Hence, $F(G) = 3$ and so Theorem 3.1 applies to G, equality holds in (25), and hence equality holds throughout (24). By the remark following (24), $\Delta(G) \leqslant 4$, and $n_3 = 10$. Let S_i be the set of vertices of degree i in G for $i \in \{3, 4\}$. We have shown

$$|S_3| = 10 \tag{26}$$

and

$$S_3 \cup S_4 = V(G). \tag{27}$$

By (22), both (b) and (a) of Theorem 3.1 hold:

$$G \text{ has girth at least } 5 \tag{28}$$

and

$$S_4 \text{ is an independent set in } G. \tag{29}$$

LEMMA 3.13. *If* $|S_4| \geqslant 4$, *then* $G[S_3]$ *has no isolated vertex.*

Proof. By way of contradiction, suppose that w is an isolated vertex in $G[S_3]$, where $N(w) = \{x, y, z\}$. Hence,

$$d(x) = d(y) = d(z) = 4. \tag{30}$$

Let H_w be the subgraph of G induced by vertices at distance at most 2 from w. By (28), H is acyclic. Hence $N(x)-w$, $N(y)-w$, and $N(z)-w$ are disjoint sets, each of order 3. By (26), (29), (30), and $w \in S_3$, we must have

$$S_3 = N(x) \cup N(y) \cup N(z), \tag{31}$$

and so any vertex of G not in H_w has degree 4. By $|S_4| \geqslant 4$ and since $|S_4 \cap V(H_w)| = \{x, y, z\}$, G has a vertex $v \in S_4 - V(H_w)$. By (29), $N(v) \subseteq N(x) \cup N(y) \cup N(z)$. Since $d(v) = 4$, this implies that v has two neighbors in some member of $\{N(x), N(y), N(z)\}$, say in $N(z)$. These two neighbors and $\{v, z\}$ together induce a 4-cycle in G, contrary to (28). ∎

To obtain a bound on $|S_4|$, we estimate in two ways the number of edges with one end in S_3 and one end in S_4. By Lemma 3.13 and (26), $G[S_3]$, has at least five edges, thus accounting for at least 10 of the $3|S_3| = 30$ edge-vertex incidences at S_3. That leaves at most 20 incidences at S_3 with edges whose other end is in S_4. By (29), each vertex of S_4 is incident with exactly four edges joining S_3 and S_4, and so $4|S_4| \leqslant 20$. Hence

$$|S_4| \leqslant 5,$$

and if equality holds, then $|E(G[S_3])| = 5$.

Case 1. Suppose that $|S_4| = 5$, and define $H = G - E(G[S_3])$. As remarked, $|S_4| = 5$, implies $|E(G[S_3])| = 5$, and so by Lemma 3.13, $G[S_3] \cong 5K_2$. Therefore, each vertex of the subgraph H has degree 2 or 4; in particular, each edge of H has one end of degree 2 and the other of degree 4, and so we can prove $G \in \mathcal{SL}$ merely by showing that H is connected since H is a spanning subgraphs of G with eulerian components.

By way of contradiction, suppose that H is not connected, and let H_6 be the smaller component of H. Since H is bipartite (with bipartition $S_3 \cup S_4$), (28) implies that H has girth at least 6. Since G has order $|S_3| + |S_4| = 15$, H_6 has order at most 7. The only eulerian bipartite graph of order at most 7 with girth at least 6 is the 6-cycle, and so $H_6 \cong C_6$. But each edge of H must have one end of degree 2 and the other of degree 4, and so we have a contradiction that precludes Case 1.

Case 2. Suppose that $|S_4| = 4$. Then by (27) and (29), of the 30 edge-vertex incidences at S_3, exactly 16 are with edges whose other end is in S_4. Hence,

$$|E(G[S_3])| = (30-16)/2 = 7. \tag{32}$$

We claim that

$$G[S_3] \text{ is acyclic.} \tag{33}$$

If not, then by (28), $G[S_3]$ has a cycle C of length s for some $s \geqslant 5$. By (32), $s \leqslant 7$, and there are only $7-s$ edges of $G[S_3]-E(C)$, not enough to satisfy the requirement of Lemma 3.13. This proves (33).

By (32), (33), and (26), $G[S_3]$ has three components, each a tree. We claim

$$\text{each component of } G[S_3] \text{ is a path.} \tag{34}$$

If not, then the three components of $G[S_3]$ collectively have at least seven endvertices. Since each endvertex of $G[S_3]$ is adjacent to two members of S_4, and since $|S_4|=4$, some two of any seven endvertices of $G[S_3]$ are adjacent to the same two vertices in S_4, contrary to (28). This proves (34).

By way of contradiction, suppose that some component of $G[S_3]$ has order 3. By (32), that component is a $K_{1,2}$, and we denote its center vertex by w. Denote $N(w)=\{x, y, z\}$, where $G[\{w, y, z\}]$ is that $K_{1,2}$ of $G[S_3]$. Hence, $d(x)=4$, and both $N(y)-w$ and $N(z)-w$ consist of two vertices of degree 4. These five degree-4 vertices are distinct, by (28). This violates $|S_4|=4$ of Case 2.

Hence, no component of $G[S_3]$ has order 3, and by Lemma 3.13, no component has order 1, either. The only partitions of the integer $|S_3|$ into three integers, none of which is 1 or 3, are $10=2+2+6$ and $10=2+4+4$. This fact and (34) imply that the three components of $G[S_3]$ are each paths of even order. Thus, there is a perfect matching $M \subseteq G[S_3]$, and $|M|=5$. Then $G-M$ is a graph in which each vertex has degree 2 or 4, and, except for two nonadjacent edges of $G[S_3]-M$ whose ends both have degree 2 in $G-M$, each edge of $G-M$ has one end of degree 2 and one end of degree 4 in $G-M$. To prove $G \in \mathcal{SL}$, it suffices to prove that $G-M$ is connected.

By way of contradiction, suppose that $G-M$ is disconnected, and let G_6 be the smallest component of $G-M$. Since $G-M$ has order 14, G_6 is an eulerian graph of order at most 7. The only eulerian graphs of order at most 7 satisfying (28) are cycles of order 5, 6, and 7. But if G_6 is such a cycle, then there are more than two edges in $G-M$ with both ends of degree 2 in $G-M$, a contradiction. Hence, $G-M$ is connected and so $G \in \mathcal{SL}$. This concludes Case 2.

Case 3. Suppose $|S_4| \leqslant 3$. Then by (26) and (27), G has order at most 13. By Theorem 1.7, either $G \in \mathcal{SL}$ or the reduction of G is the Petersen graph.

This proves Theorem 3.12. ∎

THEOREM 3.14. *Let G be a* 3-*edge-connected graph. If G has at most* 11 *edge cuts of size* 3, *then exactly one of these holds*:

(a) $G \in \mathcal{SL}$;

(b) *The reduction of G is the Petersen graph*;

(c) *The reduction of G is nonsupereulerian graph of order between* 17 *and* 19, *with girth at least* 5, *with exactly* 11 *vertices of degree* 3, *and with the remaining vertices independent and of degree* 4;

Outline of Proof of Theorem 3.14. By Theorem 3.12, we can assume that G has exactly 11 edge cuts of size 3. Imitate the proof of Theorem 3.12, to show that G can again be presumed to be reduced, that the girth of G is at least 5, and that G now has 11 vertices of degree 3, 1 vertex of degree 5, and some indeterminate number n_4 of vertices of degree 4. Also, $F(G)=3$, and so Theorem 3.1 can be applied.

By Theorem 3.1(a), the vertices of degree 4 from an independent set in G, and so $4n_4$ edges join them to the odd degree vertices of G. There are $3(11)+5(1)=38$ edge-vertex incidences at the odd degree vertices of G, and so $4n_4 \leqslant 38$. Hence, $n_4 \leqslant 9$, and G has order at most $11+9+1=21$. With further arguments, one can improve this to $|V(G)| \leqslant 19$.

Let w have degree 5 in G, and let H_w be the subgraph induced by the vertices within distance 2 of w. Use Theorem 3.1(b) to show that at least 10 vertices lie at distance 2 from w, and hence that $|V(G)| \geqslant 1+5+10=16$. This bound can be improved to $|V(G)| \geqslant 17$. ∎

Conjecture [5]. Let G be a 3-edge-connected graph. If G has at most 17 edge cuts of size 3, then exactly one of these holds:

(a) $G \in \mathcal{SL}$;

(b) G is contractible to the Petersen graph.

Snarks of order 18 [9] show that "17" of this conjecture is best possible. Notice that conclusion (b) is weaker than (b) of Theorems 3.12 and 3.14. Chen [7] has given examples of 3-regular 3-edge-connected reduced graphs G of order 14 and 16 that are not supereulerian but that are reduced. They are contractible to the Petersen graph, though. The one of order 14 has an induced subgraph H isomorphic to $K_{2,3}$ such that G/H is the Petersen graph. The one of order 16 is similarly constructed, where H is the 3-cube minus a vertex.

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