Double cycle covers and the Petersen graph, III

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Abstract

Any graph with no cut edge and with at most 13 edge cuts of size 3 either has a cycle double cover formed by three subgraphs with eulerian components, or it is contractible to the Petersen graph. We correct an error in an earlier proof of this result.

We allow loops but not multiple edges. Define $O(G)$ to be the set of odd-degree vertices of G. A graph is even if $O(G) = \emptyset$. Let $\mathcal{S}_3$ denote the family of graphs for which there is a partition $E(G) = E_1 \cup E_2 \cup E_3$ such that $O(G[E_i]) = O(G)$ $(1 \le i \le 3)$. For 3-regular graphs, $\mathcal{S}_3$ is the family of graphs having a 1-factorization. The Petersen graph, denoted P, is the smallest 2-edge-connected graph not in $\mathcal{S}_3$.

Since we are simply correcting an erroneous proof of a lemma of a prior paper (Lemma 13 of [1]), we shall confine this note to a proof of that lemma, plus background for the proof. For more details, see [1].

Jaeger [2] proved the following result for graphs with no edge cut of size 1 or 3.

Theorem 1 Let G be a graph with no cut edge. If G has at most 13 edge cuts of size 3, then exactly one of these holds:
(a) $G \in \mathcal{S}_3$;
(b) G is contractible to P.

For any graph G with nonparallel edges xy and yz incident with a vertex y, where $d(y) \ge 3$, the graph G_0 (say) obtained from $G - \{xy, yz\}$ by adding a new edge xz is said to be obtained from G by lifting $\{xy, yz\}$; and that pair of edges is said to be lifted to form G_0.

Theorem 2 (Mader [3]) Suppose that $y \in V(G)$ is not a cutvertex of G. If $d(y) \ge 4$, then some pair of edges incident with y can be lifted, so that in the resulting graph G_0, any pair of distinct vertices $v, w \in V(G) - y$ satisfy

$$\kappa'_{G_0}(v,w) = \kappa'_G(v,w). \quad \Box$$

<u>Theorem 3</u> For G and G_0 of Theorem 2, $G_0 \in \mathcal{S}_3 \implies G \in \mathcal{S}_3$.

Theorem 3 is an easy consequence of the definition of $\mathcal{S}_3$.

As in [1], we suppose that G is a counterexample to Theorem 1 having the fewest number of edges. Then (see Lemmas 11 and 12 of [1]),

(P1) The girth of G is at least 5;

(P2) G is 3-edge-connected.

Lemma 13 of [1] is the following result, which is what we want to prove:

<u>Lemma 4</u> All vertices of G have odd degree.

<u>Proof</u>: Suppose instead that G has a vertex y of even degree. By (P2), $d(y) \geq 4$, and so G and y satisfy the hypothesis of Theorem 2. Let G_0 be the graph obtained in Theorem 2 by lifting a pair of edges at y, so that edge-connectivity is preserved. We shall find contradictions.

If G_0 is not contractible to P, then $G_0 \in \mathcal{S}_3$, since G is a smallest counterexample to Theorem 1. By Theorem 3, $G \in \mathcal{S}_3$, and so G is not a counterexample to Theorem 1, a contradiction.

Suppose that G_0 is contractible to P. Let $v_1, v_2, ..., v_{10}$ be the vertices of P, and let $\theta : V(G_0) \longrightarrow V(P)$ be a contraction of G_0 onto P. Since G is a smallest counterexample, G is not contractible to P. Then θ induces a contraction of G onto a graph H that is one of two graphs: H can be transformed into P either by lifting edges at a single vertex of degree 4 (if H has order 11) and dissolving the resulting degree 2 vertex; or by lifting edges at a vertex of degree 5 (if H has order 10). In either case, H does not have girth at least 5, contrary to (P1). $\Box$

References

[1] P. A. Catlin, Double cycle covers and the Petersen graph, II. Congressus Numerantium **76** (1990) 173-181.

[2] F. Jaeger, Flows and generalized coloring theorems in graphs. J. Combinatorial Theory (B) **26** (1979) 205-216.

[3] W. Mader, Edge-connectivity preserving reductions, in "Advances in Graph Theory" (ed. by B. Bollobás) North-Holland (1978).