The Arboricity of the Random Graph

Paul A. Catlin*
Zhi-Hong Chen*

INTRODUCTION

The arboricity $a(G)$ of a graph G is the minimum number of forests in G whose union contains G. Nash-Williams [6] proved

$$a(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)| - 1} \right\rceil, \tag{1}$$

where the maximum runs over all nontrivial subgraphs H of G We shall show that if G is the random graph, then the expression $|E(H)|/(|V(H)| - 1)$ attains its maximum in (1) if and only if $H = G$. This result also gives the maximum number of edge-disjoint spanning trees in the random graph.

Let p be a fixed real number between 0 and 1. Write $\mathcal{G}(n,p)$ for the probability space of simple graphs of order n, where the probability that any two distinct vertices are adjacent is p, and where these probabilities are independent. Except in a concluding remark, when we write of "the random graph" G or "almost every graph" G, we are in the space $\mathcal{G}(n,p)$ and G has order n. This is Model A of Palmer [7].

We shall follow the notation of Bondy and Murty [2], and we use Landau's notation $O(f(n))$ for a term which, after division by $f(n)$, remains bounded as $n \longrightarrow \infty$; and $o(f(n))$ is a term which, after division by $f(n)$, approaches 0 as $n \longrightarrow \infty$.

*Department of Mathematics, Wayne State University, Detroit, MI 48202.

SOME KNOWN RESULTS

For any connected graph G, define

$$\gamma(G) = \max_{H \subseteq G} \frac{|E(H)|}{|V(H)| - 1}, \tag{2}$$

where the maximum is taken over all nontrivial subgraphs of G. Also define

$$\eta(G) = \min_{E \subseteq E(G)} \frac{|E|}{\omega(G - E) - 1}, \tag{3}$$

where $\omega(G-E)$ is the number of components of $G-E$. Let $t(G)$ denote the maximum number of edge-disjoint spanning trees in G. Tutte [10] and Nash-Williams [7] proved

$$t(G) = \lfloor \eta(G) \rfloor. \tag{4}$$

By (1) and (2),

$$a(G) = \lceil \gamma(G) \rceil. \tag{5}$$

<u>Lemma 1</u> [3] For any connected graph G of order n, these are equivalent:
(a) $|E(G)| = \gamma(G)(n-1)$;
(b) $|E(G)| = \eta(G)(n-1)$;
(c) $\eta(G) = \gamma(G)$. □

Also,

$$\eta(G) \le \frac{|E(G)|}{n-1} \le \gamma(G) \tag{6}$$

if G is connected of order n. Although $\gamma(G)$ and $\eta(G)$ may not be integers, they are often easier to use than $a(G)$ and $t(G)$.

<u>Lemma 2</u> For almost every graph G, the minimum degree is

$$\delta(G) = pn + O((n \log n)^{1/2}). \quad \square$$

Stronger versions of Lemma 2 appear in [1].

<u>Lemma 3</u> (Bollobás [1, Lemma 18]) Let $\epsilon > 0$. For almost every graph G, if $r > n^\epsilon$ then every induced subgraph H of order r has

$$|E(H)| = p\binom{r}{2} + o(r^2). \quad \square \tag{7}$$

THE MAIN RESULTS

Let G be a connected graph. (Almost all graphs are connected [7, p. 14].) Define $\mathcal{F}(G)$ to be the family of nontrivial subgraphs H of G such that

$$\gamma(G) = \frac{|E(H)|}{|V(H)| - 1}. \tag{8}$$

Thus, $H \in \mathcal{F}(G)$ implies $\gamma(H) = \gamma(G)$. Payan [8] introduced the invariant $\gamma(G)$ and he called G decomposible if $G \in \mathcal{F}(G)$. Ruciński and Vince [9] called G strongly balanced if $G \in \mathcal{F}(G)$, and they proved that there is a strongly balanced graph with order n and with m edges if and only if

$$1 \leq n - 1 \leq m \leq \binom{n}{2}.$$

Also, they remarked [9, p. 255] that for such values of m and n, either $n - 1 = m$ or there is a simple graph G of order n and size m with $\mathcal{F}(G) = \{G\}$. Condition (a) of Lemma 1 holds if and only if $G \in \mathcal{F}(G)$.

Theorem 4 For the random graph G, $\mathcal{F}(G) = \{G\}$.

Proof: Let G be a random graph of order $n > 1$. We may assume that G is connected. Let $H \in \mathcal{F}(G)$ and denote $|V(H)|$ by r. We shall prove $H = G$. Clearly $r > 1$ since $G \neq K_1$.

Since $H \in \mathcal{F}(G)$, H is an induced subgraph of G and

$$\gamma(H) = \gamma(G) = \frac{|E(H)|}{r - 1}. \tag{9}$$

Since H is simple of order r, (9) gives

$$r = \frac{2}{r-1}\binom{r}{2} \geq \frac{2}{r-1}|E(H)| = 2\gamma(H). \tag{10}$$

By Lemma 3, with G in place of H,

$$|E(G)| = p\binom{n}{2} + o(n^2). \tag{11}$$

By (10), (9), (6), and (11),

$$r \geq 2\gamma(H) = 2\gamma(G) \geq \frac{2|E(G)|}{n-1} = pn + o(n),$$

and so r is large enough so that Lemma 3 applies to the induced subgraph H. Thus,

(12) $$|E(H)| = p\binom{r}{2}(1 + o(1)).$$

By (9) and (12),

(13) $$\gamma(H) = \frac{|E(H)|}{r-1} = \frac{pr}{2}(1 + o(1)).$$

By (6) and (11),

(14) $$\gamma(G) \geq \frac{|E(G)|}{n-1} = \frac{pn}{2} + o(n),$$

and so by (13), (9), and (14),

(15) $$\frac{pr}{2}(1 + o(1)) = \gamma(H) = \gamma(G) \geq \frac{pn}{2} + o(n).$$

This gives

(16) $$|V(G) - V(H)| = n - r = o(n).$$

By way of contradiction, suppose that there is a vertex $v \in V(G) - V(H)$. Define

$$H_v = G[V(H) \cup \{v\}].$$

Then $|V(H_v)| = r + 1$. By (6) (with H_v in place of G),

(17) $$|E(H_v)| \leq \gamma(H_v)r.$$

Since $H \in \mathcal{F}(G)$,

(18) $$\gamma(H_v) \leq \gamma(H).$$

By (17), (18), and (9),

(19) $$|E(H_v)| \leq \gamma(H_v)r \leq \gamma(H)r = |E(H)| + \gamma(H).$$

Notice that (19) implies

(20) $$|N(v) \cap V(H)| \leq \gamma(H).$$

By (20), (16), (13), and $r \leq n$, a bound on the degree of v is

$$\begin{aligned} d(v) &< |N(v) \cap V(H)| + |V(G) - V(H)| \\ &\leq \gamma(H) + o(n) \\ &= \frac{pr}{2}(1 + o(1)) + o(n) \\ &< \frac{pn}{2}(1 + o(1)). \end{aligned}$$

contrary to Lemma 2. Hence, v does not exist, and so H must equal G. This proves Theorem 4. □

Corollary 5 Almost every graph G satisfies

$$a(G) = \left\lceil \frac{|E(G)|}{n-1} \right\rceil$$

and

$$t(G) = \left\lfloor \frac{|E(G)|}{n-1} \right\rfloor .$$

Proof: Combine Theorem 4 and (5) to get $a(G)$. By Theorem 4, G satisfies (a) of Lemma 1. Use Lemma 1 and (4) to get $t(G)$. □

Corollary 6 For almost any graph G, $a(G) - t(G) = 1$.

Proof: By Corollary 5, $0 \leq a(G) - t(G) \leq 1$, and by (4), (5), and (6),

$$t(G) \leq \frac{|E(G)|}{n-1} \leq a(G).$$

Since $t(G)$ and $a(G)$ are integers, we see that to prove Corollary 6 it suffices to show that $|E(G)|/(n-1)$ is almost never an integer. This is routine and hence omitted. □

REMARKS

Frieze and Luczak [4] determined $t(G)$ for the graph G, when G is the random graph underlying the digraph chosen randomly according to Palmer's Model C. For positive integers r and n with $1 \leq r \leq n-1$, the sample space in Model C consists of all labelled digraphs of order n in which each vertex has outdegree r. For each vertex v, there are $\binom{n-1}{r}$ choices for the neighborhood of v in the digraph. The underlying graph thus has rn edges and hence cannot have $r+1$ edge-disjoint spanning trees. Frieze and Luczak [4] showed that the underlying graph almost always has r edge-disjoint spanning trees.

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