Hamilton cycles and closed trails in iterated line graphs

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Abstract

Let G be an undirected connected graph that is not a path. We define $h(G)$ (respectively, $s(G)$) to be the least integer m such that the iterated line graph $L^m(G)$ has a Hamiltonian cycle (respectively, a spanning closed trail). To obtain upper bounds on $h(G)$ and $s(G)$, we characterize the least integer m such that $L^m(G)$ has a connected subgraph H, in which each edge of H is in a 3-cycle and $V(H)$ contains all vertices of degree not 2 in $L^m(G)$. We characterize the graphs G such that $h(G) - 1$ (respectively, $s(G)$) is greater than the radius of G.

NOTATION

We use the notation of Bondy and Murty [4], except where noted otherwise. Graphs are undirected and can have multiple edges, but loops are forbidden. The multigraph of order 2 with two edges will be called a 2-cycle and denoted C_2. The graph K_1 will be considered 2-edge-connected.

For the graph G, the line graph $L(G)$, called an edge graph in [4], is a simple graph obtained from G by regarding $E(G)$ as its vertex set, where e and e' are adjacent edges in G if and only if they are adjacent vertices in $L(G)$. We shall denote vertices of $L(G)$ in this paper by using the letter e, i.e., e, e', e_i, etc., because they are also edges of G. The iterated line graph $L^m(G)$ is defined recursively by $L^0(G) = G$ and

$$L^{k+1}(G) = L(L^k(G)) \qquad (k \in \mathbf{N}).$$

If a connected graph G is neither a path nor a 2-cycle, then $L(G)$ has at least as many edges as G and $L(G)$ is not a path, and so $L^k(G)$ is nonempty, for all $k \in \mathbf{N}$. If G is a path or a 2-cycle, then for some $k \in \mathbf{N}$, $L^k(G) \cong K_1$ and $L^m(G)$ is the empty graph if $m > k$.

Let G be a connected graph. For any two vertices $v_1, v_2 \in V(G)$, define the distance $d(v_1, v_2)$ between v_1 and v_2 to be the length of the shortest (v_1, v_2)-path in G. For any vertex $v \in V(G)$, define the eccentricity of v to be

$$\mathrm{ecc}(v) = \max\{d(v, w) \mid w \in V(G)\}.$$

Define the diameter of G to be

$$\mathrm{diam}(G) = \max\{\mathrm{ecc}(v) \mid v \in V(G)\},$$

and define the radius of G to be

$$\mathrm{rad}(G) = \min\{\mathrm{ecc}(v) \mid v \in V(G)\}.$$

SOME PRIOR RESULTS

Chartrand [6] calls a graph G of size m sequential if $E(G)$ has an ordering $e_1, e_2, ..., e_m = e_1$ such that consecutive edges in this sequence are adjacent. A closed trail T in a graph G is called dominating if each edge in $E(G)$ has at least one end in $V(T)$. Harary and Nash-Williams [8] proved the "(a) $\Leftrightarrow$ (c)" part of the following result:

Theorem 1 Let G be a graph with at least three edges. These are equivalent:

(a) $L(G)$ is hamiltonian;

(b) G is sequential;

(c) G has a dominating closed trail, or G is a star.

Corollary 1A Let G be a graph that is not C_2. For any $m \geq 0$ such that $L^m(G)$ is hamiltonian, $L^n(G)$ is hamiltonian for all $n \geq m$.

Both Corollary 1A and Corollary 1B below follow easily from the "(c) $\Rightarrow$ (a)" part of Theorem 1. A graph G is called supereulerian if G has a spanning eulerian subgraph (equivalently, if G has a spanning closed trail). We regard K_1 as supereulerian.

Corollary 1B Let G be a graph with at least 3 edges. If G is supereulerian, then $L(G)$ is hamiltonian.

Our main interest lies in finding upper bounds on the least integer m such that $L^m(G)$ is hamiltonian (respectively, supereulerian). We define

$$h(G) = \min\{m \mid L^m(G) \text{ is hamiltonian}\};$$

$$s(G) = \min\{m \mid L^m(G) \text{ is supereulerian}\}.$$

Catlin [5] developed a reduction method for determining whether a graph G is supereulerian. For a connected subgraph H of G, let G/H denote the graph obtained from G by contracting H to a single vertex, say v_H, in G/H, and deleting any resulting loops. Thus, for any $v \in V(G) - V(H)$, v_H is joined to v in G/H by as many edges as join $V(H)$ to v in G. Define the graph H to be collapsible if for every even subset $S \subseteq V(H)$ there is a subgraph Γ of H such that $H - E(\Gamma)$ is connected and the set of odd degree vertices of Γ is S. For example, C_2 and C_3 are collapsible, but longer cycles are not. Any collapsible subgraph is 2-edge-connected and supereulerian, and K_1 is regarded as collapsible.

Theorem 2 [5] Let H be a collapsible subgraph of G. Then G is supereulerian if and only if G/H is supereulerian. $\square$

Theorem 2 is obtained by setting $S = \{\text{odd-degree vertices of } G\}$ in Theorem 3 of [5]. We shall only use Theorem 2 in conjunction with the following result, which follows from Corollary 1 (page 33) of [5]:

Theorem 3 [5] If each edge of a connected graph H is in a cycle of length 2 or 3, then H is collapsible. $\square$

Since a collapsible graph is supereulerian, Theorem 3 implies immediately a result of Balakrishnan and Paulraja [3], that if every edge of a connected graph G lies in a 3-cycle, then G has a spanning closed trail. By Theorem 1, this implies Oberly and Sumner's result [13], that such a graph has a hamiltonian line graph.

Other papers on Hamiltonian cycles in iterated line graphs, or in subgraphs of iterated line graphs, include [1], [2], [7], [9], [10], [11], and [12].

THE MAIN RESULTS

Let G be a graph and define

$$W = \{v \in V(G) \mid d_G(v) \neq 2\}. \tag{1}$$

A branch in G is a nontrivial path whose ends are in W and whose internal vertices, if any, have degree 2 in G (and thus are not in W). If the branch has length 1, then it has no internal vertex. Any graph has a unique decomposition into an edge-disjoint union of branches.

We define a type A subgraph of G to be a maximal connected subgraph H such that

(i) Every edge of H is in a cycle in H of length at most 3; or

(ii) If $V(H) = \{v\}$ for some $v \in V(G)$, then $v \in W$.

Thus, there are three kinds of type A subgraphs: single vertices of degree 1 in G; single vertices of degree at least 3 in G that lie in no cycle of G of length at most 3; and nontrivial connected subgraphs whose edges all lie in cycles of length at most 3, and that are maximal with this property. Since a line graph is simple, (i) is equivalent to "Each edge of H is in a 3-cycle," if G is a line graph. By Theorem 3, any type A subgraph is collapsible.

Define

$$f(G) = \min \{m \mid L^m(G) \text{ has only one type A subgraph}\}.$$

When G is a cycle of length at least 4, $f(G)$ is undefined, and $s(G) = h(G) = 0$. If $G \cong C_2$ or $G \cong C_3$, then $f(G) = 0$. When G has at least two components with vertices in W, $f(G)$ does not exist. In all other cases, it is easy to check that $f(G)$ is bounded above by the maximum distance between vertices of W.

Theorem 4 Let G be a connected graph that is neither a path nor a cycle. Then

$$s(G) \leq f(G) \quad \text{and} \quad h(G) \leq f(G) + 1.$$

Proof: Let G be a connected graph that is neither a path nor a cycle. Then $f(G)$ exists, and we denote $m = f(G)$. Then $L^m(G)$ has a single type A connected subgraph, say L. Either $L^m(G) = L$ or by the definition of a type A subgraph, $L^m(G)$ consists of L and branches whose internal vertices are not in L, but whose ends are in L. All edges of $E(L^m(G)) - E(L)$, if any, lie within such branches. Therefore, $L^m(G)/L$ is either K_1 or it has a single vertex v_L common to a collection of cycles that are otherwise disjoint. (Each cycle of $L^m(G)/L$ is induced by a branch of $L^m(G)$ with internal vertices lying outside of L.) In either case, $L^m(G)/L$ is supereulerian. By the definition of a type A subgraph and by Theorem 3, L is collapsible. Hence, by Theorem 2, $L^m(G)$ is supereulerian, and so $s(G) \leq m = f(G)$. By Corollary 1B, $L^{m+1}(G)$ is hamiltonian, and so $h(G) \leq m + 1 = f(G) + 1$. This proves Theorem 4. $\square$

In Theorem 6, we shall characterize $f(G)$. By Theorem 4, upper bounds on $s(G)$ and $h(G)$ are thus obtained.

For any $v_1, v_2 \in W$, and for any (v_1, v_2)-path P in G, there is a unique decomposition of P into branches in G. For a given (v_1, v_2)-path P, let $B_1, B_2, ..., B_k$ denote the branches of G such that

$$E(B_1) \cup E(B_2) \cup .... \cup E(B_k) = E(P).$$

Obviously, the $E(B_i)$'s are disjoint sets. If B_i is in a cycle of G of length at most 3 for all $i \in \{1, 2, ...k\}$, or if P has length 0, then define $Z(P) = 0$. Otherwise, denote

$$Z(P) = \max_i |E(B_i)|, \tag{2}$$

where the maximum in (2) is taken only over those branches B_i $(1 \leq i \leq t)$ such that B_i is not in a cycle in G of length at most 3.

For a connected graph G and for $v_1, v_2 \in W$, where W satisfies (1), define

$$Z(v_1, v_2) = \min_P Z(P), \tag{3}$$

where this minimum is taken over all (v_1, v_2)-paths P in G.

If G is a cycle, then define $\zeta(G) = 0$. If G is a connected graph that is not a cycle, then W of (1) is nonempty, and we define

$$\zeta(G) = \max_{v_1, v_2 \in W} Z(v_1, v_2). \tag{4}$$

Thus, $\zeta(G)$ is rather like diam(G) in its definition, except that the distance function $Z(v_1, v_2)$ of (3) is much more complicated than $d(v_1, v_2)$. As an illustration, suppose that G is $K_{3,3}$ minus an edge. Then W is the set of four vertices of degree 3 in G. Since $G[W]$ is connected, $Z(v_1, v_2) = 1$ for any pair of distinct vertices $v_1, v_2 \in W$. Hence, $\zeta(G) = 1$.

Theorem 5 If G is a connected graph and if $\zeta(G) > 0$, then

$$\zeta(L(G)) = \zeta(G) - 1. \tag{5}$$

Furthermore, G has disjoint connected subgraphs, say G_1 and G_2, such that each type A subgraph of G is contained either in G_1 or G_2; such that both G_1 and G_2 contain at least one type A subgraph; such that every branch in G connecting G_1 and G_2 has length at least $\zeta(G)$; and such that at least one branch connecting G_1 and G_2 has length exactly $\zeta(G)$.

Before proving Theorem 5, we present some terminology and lemmas.

Define

$$W_L = \{e \in V(L(G)) \mid d_{L(G)}(e) \neq 2\}. \tag{6}$$

Any branch of G not contained in a type A subgraph of G will be called a type B subgraph of G. A type B subgraph cannot have length 0. If $\{A_1, A_2, ..., A_c\}$ denotes the collection of type A subgraphs and if $\{B_1, B_2, ..., B_t\}$ is the collection of type B subgraphs of G, then

$$G = \left(\bigcup_{i=1}^{c} A_i \right) \cup \left(\bigcup_{i=1}^{t} B_i \right). \tag{7}$$

This decomposition of G into type A and type B subgraphs is unique.

Let B be a type B subgraph of G. Then B is a branch with consecutive edges $e_1, e_2, ..., e_k$, where $k \geq 1$. Denote by B_L the path $e_1 e_2 .... e_k$ in $L(G)$. We call B_L the subgraph of $L(G)$ induced by B, and we say that B induces B_L. If $k = 1$, then B_L is only a single vertex in W_L, and in that case B_L is not a type B subgraph of $L(G)$, because it is contained in a type A subgraph. If $k \geq 2$, then B_L is a branch of $L(G)$, and it is a type B subgraph.

Lemma 1 If B is a type B subgraph of G, then B_L is a branch in $L(G)$, where

(8) $$|E(B_L)| = |E(B)| - 1. \quad \square$$

Since Lemma 1 is straightforward, we omit the details of the proof. The equality (8) is obvious. It remains to check that the ends of B_L are in W_L, for otherwise B_L would not be a branch. Internal vertices of B_L, if any, obviously have degree 2 in $L(G)$.

Let A be a type A subgraph of G. Let ∂A denote the set of edges of $E(G) - E(A)$ with at least one end in $V(A)$. Define A_L to be the subgraph of $L(G)$ induced by $E(A) \cup \partial A$ ($E(A) \cup \partial A$ is a vertex set in $L(G)$). We say that A_L is induced by A, and that A induces A_L. This usage of "induced" is not the same as the usage for type B subgraphs, but since a type A subgraph is not a type B subgraph, there will be no confusion.

Lemma 2 If A is a type A subgraph of G, then A_L is contained in a type A subgraph of $L(G)$.

Proof: Suppose that A is a type A subgraph of G. By definition,

(9) $$A \text{ is connected;}$$

(10) $$\text{Each edge of } A \text{ is in a cycle in } A \text{ of length at most } 3;$$

(11) $$\text{If } V(A) = \{v\}, \text{ then } v \in W;$$

and A is a maximal subgraph of G satisfying (9), (10), and (11).

By (9) and the definition of A_L, A_L is connected. By (10), (11), and the definition of A_L, either each edge of A_L is in a 3-cycle of A_L, or A_L is edgeless and $|V(A_L)| = 1$. If $|V(A_L)| = 1$ then the sole vertex of A_L is an edge $e \in E(G)$, where e is incident with a vertex of degree 1 in G. Then either $d_{L(G)}(e) = 1$, in which case $e \in W_L$, or $d_{L(G)}(e) \geq 2$, in which case e is a vertex of a C_3 in $L(G)$. In any case, A_L satisfies all conditions of a type A subgraph of $L(G)$, except maximality, and so A_L is contained in a type A subgraph of $L(G)$. □

Sometimes disjoint type A subgraphs, say A and A', will induce subgraphs A_L and A'_L that are contained in a common type A subgraph of $L(G)$. This happens if A and A' are joined in G by a branch of length 1.

Lemma 3 Let G be a connected graph that is neither a path nor a cycle. Let $A_1, A_2, ..., A_c$ be the type A subgraphs of G. Then either $c = 1$ or there is a collection $\{B_1, B_2, ..., B_{c-1}\}$ of $c - 1$ type B subgraphs (branches) of G

such that

$$\zeta(G) = \max_{1 \leq j \leq c-1} |E(B_j)|, \tag{12}$$

and such that the subgraph H' defined by

$$H' = \left(\bigcup_{i=1}^{c} A_i \right) \cup \left(\bigcup_{j=1}^{c-1} B_j \right) \tag{13}$$

is connected.

Proof: Let G and $\{A_1, A_2, ..., A_c\}$ satisfy the hypothesis of Lemma 3. Let s be a maximum integer such that there is a collection $\{B_1, B_2, ..., B_s\}$ of branches of G satisfying these three conditions:

(14) Each B_j joins distinct A_i's $(1 \leq j \leq s)$;

(15) $|E(B_j)| \leq \zeta(G) \quad (1 \leq j \leq s)$; and

(16) No branches B_j and B_k $(1 \leq j < k \leq s)$ lie in a common cycle in the subgraph H_s defined by $H_s = (\bigcup_{i=1}^{c} A_i) \cup \left(\bigcup_{j=1}^{s} B_j\right)$.

By way of contradiction, suppose that H_s is disconnected. Since we are done if $c = 1$, there is no loss of generality in assuming that A_1 and A_2 are in distinct components of H_s. Let H denote the component of H_s containing A_1. Since G is connected and not a cycle, every type A subgraph of G has a vertex in W. Thus, we can pick $v_i \in V(A_i) \cap W$, for $i \in \{1, 2\}$. By the definition of $\zeta(G)$,

$$Z(v_1, v_2) \leq \zeta(G),$$

and hence G has a (v_1, v_2)-path, say P, with

$$Z(P) = Z(v_1, v_2) \leq \zeta(G).$$

Since P has ends in different components of H_s, and since one end of P is $v_1 \in V(H)$, there is a branch, say B_{s+1}, of G in P with exactly one end in H. The ends of B_{s+1} are in W and

$$W \subseteq \bigcup_{i=1}^{c} A_i.$$

Therefore, B_{s+1} joins distinct A_i's,

$$|E(B_{s+1})| \leq Z(P) \leq \zeta(G),$$

and since B_{s+1} has exactly one end in the component H of H_s, B_{s+1} does not lie in a cycle in $H_s \cup B_{s+1}$. Thus, $\{B_1, B_2, ..., B_s, B_{s+1}\}$ also satisfies (14), (15), and (16), with s replaced by $s+1$. Since this contradicts the maximality of s, H_s must be connected. Since there are c A_i's, this implies that $s \geq c-1$. Hence by (16), $s = c-1$, and so H_s is the graph H' of (13).

If (15) holds strictly for all $j \in \{1, 2, ..., c-1\}$, then for any $v_1, v_2 \in W$, (2) implies that any (v_1, v_2)-path P in H' satisfies $Z(P) < \zeta(G)$. Thus by (3), $Z(v_1, v_2) < \zeta(G)$ for all $v_1, v_2 \in W$, contrary to (4). Therefore, equality holds in (15) for some branch B_j, and thus (12) holds. This proves Lemma 3. □

<u>Lemma 4</u> For a connected graph G that is not a cycle, $\zeta(G) = 0$ if and only if G has exactly one type A subgraph.

Proof: Suppose that H is the only type A subgraph in G. Since G is not a cycle, $W \neq \emptyset$. For any $v_1, v_2 \in W$, there is a (v_1, v_2)-path in H, say P, and since H is a type A subgraph, $Z(P) = 0$. It follows by (3) and (4) that $\zeta(G) = 0$.

Conversely, suppose that $\zeta(G) = 0$. Since G is not a cycle, $W \neq \emptyset$, and so a type A subgraph exists. By (4) and (3), for any $v_1, v_2 \in W$ there is a (v_1, v_2)-path, say $P(v_1, v_2)$, such that $Z(P(v_1, v_2)) = 0$. By the definition of $Z(P(v_1, v_2))$, each branch of G contained in $P(v_1, v_2)$ is in a 2-cycle or a 3-cycle of G. Define the subgraph

$$H = \bigcup_{v_1, v_2 \in W} P(v_1, v_2).$$

Each edge of H is in a cycle of G with length at most 3, and since H is connected and contains W, H is the only type A subgraph of G. This proves Lemma 4. □

Proof of Theorem 5: Let G be a connected graph with $\zeta(G) > 0$. Hence, G is not a cycle. If G is a path, then $\zeta(G) = |E(G)|$, and Theorem 5 follows easily. Thus, we can assume that G is not a path and not a cycle. Let $A_1, A_2, ..., A_c$ be the type A subgraphs of G. Since $\zeta(G) > 0$, Lemma 4 implies $c \geq 2$. Denote the type B subgraphs (the branches not contained in any type A subgraph) by $B_1, B_2, ..., B_t$, where

$$G = \left(\bigcup_{i=1}^{c} A_i \right) \cup \left(\bigcup_{j=1}^{t} B_j \right).$$

By Lemma 3, we can order these t branches so that

$$H' = \left(\bigcup_{i=1}^{c} A_i\right) \cup \left(\bigcup_{j=1}^{c-1} B_j\right) \tag{17}$$

is the subgraph H' of (13), so that (12) holds, and so that the number, say m, of i's with $1 \leq i \leq c-1$ such that $\zeta(G) = |E(B_i)|$ is minimized. Then

$$m \geq 1. \tag{18}$$

Each A_i, $1 \leq i \leq c$, induces a subgraph $(A_i)_L$ of $L(G)$. Let L_i be the type A subgraph of $L(G)$ containing $(A_i)_L$. (The L_i's may not be distinct, since two type A subgraphs of G may be contained in a common type A subgraph of $L(G)$.) Each B_j $(1 \leq j \leq t)$ induces a branch $(B_j)_L$ in $L(G)$. Only the ends of $(B_j)_L$ are in type A subgraphs of $L(G)$, and so the only type A subgraphs of $L(G)$ are the L_i's. Define

$$L' = \left(\bigcup_{i=1}^{c} L_i\right) \cup \left(\bigcup_{j=1}^{c-1} (B_j)_L\right).$$

Since each internal vertex of each $(B_j)_L$ $(1 \leq j \leq t)$ has degree 2,

$$W_L \subseteq \bigcup_{i=1}^{c} V(L_i),$$

where W_L is defined in (6). Since H' is connected, so is L'. Therefore, for any vertices $e_1, e_2 \in W_L$, L' has an (e_1, e_2)-path, say P_L.

Since each nontrivial branch of $L(G)$ within any type A subgraph L_i is in a cycle in L_i of length 3, we have

$$Z(e_1, e_2) \;\leq\; Z(P_L) = \max_{(B_j)_L \subseteq P_L} |E((B_j)_L)| \tag{19}$$

$$\begin{aligned} &\leq \max_{1\leq j\leq c-1} |E((B_j)_L)| \\ &= \max_{1\leq j\leq c-1} |E(B_i)| - 1 \\ &= \zeta(G) - 1, \end{aligned}$$

by (3), (2), Lemma 1, and (12). Since e_1 and e_2 are arbitrary members of W_L, (4) and (19) imply

$$\zeta(L(G)) = \max_{e_1,e_2\in W_L} Z(e_1,e_2) \leq \zeta(G) - 1. \tag{20}$$

We want to prove that (20) holds with equality. To do so, we shall first prove the latter part of Theorem 5. By (12), there is no loss of generality in assuming

$$|E(B_1)| = \zeta(G).$$

Let v_1 and v_2 denote the ends of B_1. Denote by $H' - \text{int}(B_1)$ the graph obtained from H' of (17) by removing the internal vertices (if any) and the edges of B_1. Let G_1 and G_2 be the two components of $H' - \text{int}(B_1)$, where $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. By way of contradiction, suppose that some branch B_k $(2 \leq k \leq t)$ with $|E(B_k)| < \zeta(G)$ connects G_1 and G_2. Since G_1 and G_2 are separate components of $H' - \text{int}(B_1)$, we cannot have $k \leq c-1$. Hence, $k \geq c$, and so $\{B_k, B_2, B_3, ..., B_{c-1}\}$ has fewer length $\zeta(G)$ B_j's than does $\{B_1, B_2, B_3, ..., B_{c-1}\}$. This contradicts the minimality of m (m is defined before (18)). Therefore, no branch B_k of length less than $\zeta(G)$ connecting G_1 and G_2 exists. This proves the latter part of Theorem 5.

For $h \in \{1, 2\}$, define $(G_h)_L$ to be the subgraph of $L(G)$ containing every $(A_i)_L$ for which $A_i \subseteq G_h$ $(1 \leq i \leq c)$ and containing every $(B_j)_L$ for which both ends of B_j are in the same G_h. Any branch connecting $(G_1)_L$ and $(G_2)_L$ in $L(G)$ is a branch $(B_k)_L$ such that B_k connects G_1 and G_2 in G. Since any such B_k was already shown to have length at least $\zeta(G)$, Lemma 1 gives

$$|E((B_k)_L)| = |E(B_k)| - 1 \geq \zeta(G) - 1. \tag{21}$$

Let $e_1 \in W_L \cap V((G_1)_L)$ and let $e_2 \in W_L \cap V((G_2)_L)$. Any (e_1, e_2)-path P_L in $L(G)$ contains some branch $(B_k)_L$ with one end in $(G_1)_L$ and the other end in $(G_2)_L$, i.e., satisfying (21). Thus by (2) and (21),

$$Z(P_L) \geq |E((B_k)_L)| \geq \zeta(G) - 1. \tag{22}$$

Since (22) holds for all such (e_1, e_2)-paths P_L, (4), (3), and (22) imply

$$\zeta(L(G)) \geq Z(e_1, e_2) = \min_{P_L} Z(P_L) \geq \zeta(G) - 1. \tag{23}$$

Then (5) of Theorem 5 follows from (20) and (23). $\square$

<u>Theorem 6</u> Let G be a connected graph that is neither a path nor a cycle. Then

$$f(G) = \zeta(G). \tag{24}$$

<u>Proof</u>: Let $z = \zeta(G)$. With z applications of Theorem 5, we get

$$\zeta(L^z(G)) = 0,$$

and so by Lemma 4, $L^z(G)$ contains exactly one type A subgraph. Hence, $f(G) \leq z$.

With $z-1$ applications of Theorem 5, we get

$$\zeta(L^{z-1}(G)) = 1,$$

and so by Lemma 4, the number of type A subgraphs of $L^{z-1}(G)$ is not equal to 1. Hence, $f(G) \neq z-1$. By way of contradiction, if $f(G) = m < z-1$ for some integer m, then $L^m(G)$ has exactly one type A subgraph, and since $z-1 > m$, repeated applications of Lemma 2 show that $L^{z-1}(G)$ has a type A subgraph. Since $L^m(G)$ has only one type A subgraph and $m < z-1$, $L^{z-1}(G)$ has exactly one type A subgraph. Then by Lemma 4, $\zeta(L^{z-1}(G)) = 0$, which produces a contradiction. Hence, $f(G) = z$. □

Corollary 6A (H.-J. Lai [10]) Let G be a connected simple graph that is not a path, and let ℓ be the length of the longest branch of G not contained in a 3-cycle. Then $L^{\ell}(G)$ is supereulerian and $L^{\ell+1}(G)$ is hamiltonian.

Proof: By Theorem 6, $f(G) = \zeta(G)$, and by the definitions of ζ and ℓ, $\zeta(G) \leq \ell(G)$. These relations and Theorem 4 give Corollary 6A. □

We now introduce another invariant that is similar to $\zeta(G)$, but much easier to handle.

For W defined by (1), for any $v_1, v_2 \in W$, and for any (v_1, v_2)-path P, let $X(P)$ denote the length $|E(B)|$ of the longest branch B of G in the path P. Define

(25) $$X(v_1, v_2) = \min_P X(P),$$

where the minimum in (25) is taken over all (v_1, v_2)-paths P in G. Thus, $X(v_1, v_2)$ is a distance function that is simpler than $Z(v_1, v_2)$ but more complicated than $d(v_1, v_2)$.

If G is a cycle, then define $\xi(G) = 0$. Otherwise, $W \neq \emptyset$, and we define

$$\xi(G) = \max_{v_1, v_2 \in W} X(v_1, v_2). \tag{26}$$

Thus, $|W| \leq 1$ if and only if $\xi(G) = 0$. The definitions of $\zeta(G)$ and $\xi(G)$ differ only because branches of G lying in cycles of length at most 3 are disregarded in (2) but not in the definition of $X(P)$. The function $\xi(G)$ is a sort of diameter that is based on the distance function $X(v_1, v_2)$, instead of the usual $d(v_1, v_2)$.

Theorem 7 For a connected graph G,

$$\zeta(G) \leq \xi(G) \leq \mathrm{diam}(G). \tag{27}$$

Furthermore, $\zeta(G) \neq \xi(G)$ if and only if both

$$\zeta(G) = 0 \quad \text{and} \quad \xi(G) = 1.$$

Proof: The proof of (27) is an immediate consequence of the definitions. It only remains to characterize graphs with $\zeta(G) \neq \xi(G)$. If $\zeta(G) = 0$ and $\xi(G) = 1$, then $\zeta(G) \neq \xi(G)$ is obvious.

Suppose that $\zeta(G) \neq \xi(G)$. Then by (27), $\xi(G) > \zeta(G)$, and so by (26), there are vertices $v_1, v_2 \in W$ such that

$$X(v_1, v_2) = \xi(G) > \zeta(G). \tag{28}$$

Let P be an arbitrary (v_1, v_2)-path in G. By the definition of $X(P)$, P has a longest branch, say $B(P)$, with

$$|E(B(P))| = X(P) \geq X(v_1, v_2) > \zeta(G), \tag{29}$$

by (25) and (28). Since P is arbitrary, we can pick P to be a shortest (v_1, v_2)-path in G. Hence, no edge $e \in E(G) - E(P)$ connects two vertices that are not consecutive in P i.e., P is chordless. By (29) and the definition of $\zeta(G)$, $B(P)$ must lie in a cycle in G of length at most 3. This implies

$$|E(B(P))| \leq 1, \tag{30}$$

since P is a chordless (v_1, v_2)-path. By (30) and (29), we must have $\zeta(G) = 0$ and $X(v_1, v_2) = 1$. By (28), $X(v_1, v_2) = 1$ implies $\xi(G) = 1$. □

Corollary 7A Let G be a connected graph that is neither a path nor C_2. Then

$$s(G) \leq \zeta(G) \leq \xi(G) \leq \text{diam}(G)$$

and

$$h(G) \leq \zeta(G) + 1 \leq \xi(G) + 1 \leq \text{diam}(G) + 1.$$

Proof: Combine (27), (24), and Theorem 4, if G is not a path and not a cycle. If G is a cycle of length at least 3, then

$$h(G) = s(G) = \zeta(G) = \xi(G) = 0. \quad \square$$

A graph G has a tree T as a subgraph such that $V(T) = W$ if and only if $\xi(G) = 1$. This fact and Corollary 7A imply:

Corollary 7B (Chartrand and Wall [7]) If G is a connected graph with $\delta(G) \geq 3$, then $h(G) \leq 2$. □

For any natural number d, let $\mathcal{G}_d$ be the family of connected graphs G consisting of a vertex u, a complete subgraph H not containing u, and a collection of internally disjoint length d paths connecting u to H, where each vertex of H is at the end of at least one of these paths, and where each internal vertex of these paths has degree 2 in G. Furthermore, when we say above that H is complete, we allow the possibility that H has multiple edges.

Theorem 8 Let $d \geq 1$. A connected graph G satisfies $\xi(G) = d = \text{diam}(G)$ if and only if both $G \in \mathcal{G}_d$ and G is not 2-regular.

Proof: If $G \in \mathcal{G}_d$ and if G is not 2-regular, then $\xi(G) = d = \text{diam}(G)$.

Conversely, suppose that $\xi(G) = d = \text{diam}(G)$ and that G is not 2-regular. If $d = 1$ then G is complete and $G \in \mathcal{G}_d$ follows. Thus, we suppose that $d \geq 2$. By (26) and since G is not 2-regular, there are vertices $u, z \in W$ such that $X(u, z) = d = \text{diam}(G)$. Let $\mathcal{B} = \{B_1, B_2, ..., B_k\}$ be a minimum set of type B branches of length at least d in G, such that u and z are in distinct components of $G - \bigcup_{1 \leq i \leq k} E(B_i)$. Since $X(u, z) = d$, $\mathcal{B}$ exists. Let H_u and H_z, respectively, be the components of $G - \bigcup_{1 \leq i \leq k} E(B_i)$ containing u and z, respectively.

For $1 \leq i \leq k$, each branch $B_i \in \mathcal{B}$ has one end in H_u and one end in

H_z. Let u_i denote the end of B_i in $V(H_u)$, and let z_i denote the end of B_i in $V(H_z)$. Also, let v_i denote the vertex in B_i adjacent to u_i, and let y_i be the vertex of B_i adjacent to z_i. The consecutive vertices of B_i are thus $u_i, v_i, ..., y_i, z_i$, where $v_i = y_i$ if and only if B_i has length 2. (Since $d \geq 2$, B_i cannot have length shorter than 2.)

If $k = 1$ then since B_1 has length at least $d = \text{diam}(G)$, it follows that $G = B_1$ and B_1 has length exactly d. Hence, $G \in \mathcal{G}_d$. Thus, we can suppose that $k \geq 2$.

Suppose that $\mathcal{B}$ contains a branch of G whose length is at least $d+1$, and let B_1 be such a branch. Let $j \in \{2, ..., k\}$. Since $\text{diam}(G) = d$ and since B_j has length at least d, a shortest (v_1, y_j)-path must contain $B_j - z_j$ properly and B_j must have length exactly d. Since $d(v_1, z_1) \leq \text{diam}(G)$, B_1 has length $d+1$. Since $\text{diam}(G) = d$, it follows that G is the union of k internally disjoint (u, z)-paths of $\mathcal{B}$, where B_1 has length $d+1$ and where the other B_j's have length d when $j > 1$. Thus, $G \in \mathcal{G}_d$ holds, where H of the definition of $\mathcal{G}_d$ is $G[\{y_1, z_1\}]$. Henceforth, we suppose that every branch in $\mathcal{B}$ has length exactly d.

Suppose that there are branches in $\mathcal{B}$, say B_1 and B_2, such that $u_1 \neq u_2$ and $z_1 \neq z_2$. Then a shortest (v_1, y_2)-path in G has length at least $d+1 >$ $\text{diam}(G)$, a contradiction. Hence, no such pair of branches exists. Without loss of generality, we may therefore assume $u_1 = u_2 = \; .... \; = u_k$, and since

$d = \text{diam}(G)$ and each branch in $\mathcal{B}$ has length d, we can assume

$$V(H_u) = \{u\} = \{u_i\}, \quad (1 \leq i \leq k).$$

To show that $G \in \mathcal{G}_d$, it suffices to show that H_z satisfies the properties of H in the definition of $\mathcal{G}_d$.

Suppose that there are branches B_1 and B_2 in $\mathcal{B}$ such that $z_1 z_2 \notin E(G)$. Then the shortest (v_1, z_2)-path in G has length at least $d + 1 > \text{diam}(G)$, a contradiction. Therefore, the vertices $z_1, z_2, ..., z_k$ (some of which may be equal) induce a complete subgraph in H_z. Since G has diameter d, the shortest distance between u and any vertex of H_z cannot be more than d, and so each vertex of $V(H_z)$ is an end z_i for some i. Hence, $G \in \mathcal{G}_d$. $\square$

Corollary 8A Let G be a connected graph that is neither a path nor C_2. Then $h(G) \leq \text{diam}(G)$.

Proof: By Corollary 7A, G satisfies

$$h(G) \leq \xi(G) + 1 \leq \text{diam}(G) + 1.$$

These inequalities imply that either Corollary 8A holds or $h(G) = \xi(G) + 1 = \text{diam}(G) + 1$. In the latter case, Theorem 8 implies $G \in \mathcal{G}_d$, where $d = \text{diam}(G)$. When $G \in \mathcal{G}_d$ and G is neither a path nor C_2, it is easy to check that $h(G) \leq d$. Hence, $h(G) \neq \text{diam}(G) + 1$. $\square$

The graphs in $\mathcal{G}_d$ with an odd number of length d paths connecting u and H (of the definition of $\mathcal{G}_d$) show that Corollary 8A is best possible. The

Petersen graph also shows that Corollary 8A is best possible for diameter 2.

Recall that $G - \text{int}(B)$ is the graph obtained from G by removing the edges and internal vertices of the branch B.

Theorem 9 Let G be a connected graph. Then

$$\xi(G) > \text{rad}(G) \tag{31}$$

if and only if G has a branch B such that $G - \text{int}(B)$ has two components, say G_1 and G_2, where the ends of B are $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$, and where

$$\text{ecc}_{G_1}(v_1) + \text{ecc}_{G_2}(v_2) \leq |E(B)| - 2. \tag{32}$$

Furthermore, if the second condition holds, then $|E(B)| = \xi(G)$.

Proof: Suppose that (31) holds, and let $A_1, A_2, ..., A_c$ be the type A subgraphs of G. If $c = 0$, then G is a cycle and Theorem 9 holds. If $c = 1$, then Lemma 4 gives $\zeta(G) = 0$, and so by Theorem 7 and (31), $1 = \xi(G) > \text{rad}(G)$. Then $\text{rad}(G) = 0$, and so G is K_1, a graph satisfying Theorem 9. Suppose $c \geq 2$. Then by Lemma 3, there is a collection $B_1, B_2, ..., B_{c-1}$ of type B branches of G satisfying (12) and (13). By Lemma 4, $c \geq 2$ implies $\zeta(G) > 0$, and so

$$\zeta(G) = \xi(G) \tag{33}$$

by Theorem 7. Therefore, we can apply Theorem 5.

Let G_1 and G_2 be the connected subgraphs of G as described in Theorem 5. By way of contradiction, suppose that B and B' are distinct type B branches in G connecting G_1 and G_2, and let C be the shortest cycle of G containing $B \cup B'$ (since G_1 and G_2 are connected, C exists). By Theorem 5, (33), and (31),

$$|E(C)| \geq |E(B)| + |E(B')| \geq 2\xi(G) \geq 2\,\mathrm{rad}(G) + 2.$$

Since the internal vertices of B and B' have degree 2 in G, this implies that any internal vertex v of B has eccentricity at least $\mathrm{rad}(G) + 1$. Since B and B' are arbitrary, no internal vertex v of any such branch has $\mathrm{ecc}(v) = \mathrm{rad}(G)$. By Theorem 5, (33), and (31), a vertex v not on a branch connecting G_1 and G_2 has $\mathrm{ecc}(v) > \mathrm{rad}(G)$, and so we have a contradiction. Therefore, there is only one branch, say B, connecting G_1 and G_2. By Theorem 5 and (33),

$$|E(B)| = \xi(G). \tag{34}$$

Although Theorem 5 does not definitively state that $G \subseteq G_1 \cup G_2 \cup B$ (there could be other branches with both ends in the same G_i $(i \in \{1, 2\})$, there is no loss of generality in now assuming that $G = G_1 \cup G_2 \cup B$.

Let $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ denote the ends of B in G, and let w be a vertex of minimum eccentricity in G. By (31) and (34), $|E(B)| > \mathrm{rad}(G)$, and so w must be an internal vertex of B. For $i \in \{1, 2\}$, choose x_i to be a vertex in $V(G_i)$ that maximizes $d(w, x_i)$. Let P be a shortest (x_1, x_2)-path in G. Then $B \subseteq P$ and

$$\mathrm{ecc}_{G_1}(v_1) + |E(B)| + \mathrm{ecc}_{G_2}(v_2) \quad = \quad |E(P)|$$

$$\begin{aligned} &= d(w,x_1)+d(w,x_2) \\ &\leq 2\,\mathrm{rad}(G) \leq 2\xi(G)-2 \\ &= 2\,|E(B)|-2, \end{aligned}$$

by (31) and (34). This proves (32), and so G satisfies the second condition of Theorem 9. By (34), the last part of Theorem 9 holds.

Conversely, suppose that G satisfies the second condition of Theorem 9. By (32), for either value of $i \in \{1,2\}$,

$$\mathrm{ecc}_{G_i}(v_i) < |E(B)|.$$

This implies that the maximum in (26) occurs when v_1 and v_2 are the ends of B. It follows that

$$\xi(G) = |E(B)|. \tag{35}$$

Let w be a vertex of minimum eccentricity in G. For $i \in \{1,2\}$, choose $x_i \in V(G_i)$ to maximize $d(w,x_i)$, and let P be a shortest (x_1,x_2)-path. Since G satisfies the second part of Theorem 9, $w \in V(B)$, and so $B \subseteq P$. Without loss of generality, suppose

$$d(w,x_1) \leq d(w,x_2). \tag{36}$$

By the definition of $\mathrm{rad}(G)$ and the choice of w, (36) forces $d(w,x_2) = \mathrm{rad}(G)$. Since $w \in V(B)$, $d(w,x_1) = \mathrm{rad}(G) - k$, where $k \in \{0,1\}$. Then

$$\begin{aligned} 2\,\mathrm{rad}(G) - k &= d(w,x_1)+d(w,x_2) \\ &= \mathrm{ecc}_{G_1}(v_1) + |E(B)| + \mathrm{ecc}_{G_2}(v_2) \\ &\leq 2\,|E(B)|-2, \end{aligned} \tag{37}$$

by (32). By (37) and (35),

$$\text{rad}(G) < |E(B)| = \xi(G),$$

and (31) is proved. $\square$

Corollary 9A Let G be a connected graph that is not a path or C_2. If $\xi(G) \leq \text{rad}(G)$ then

$$(38) \qquad s(G) \leq \text{rad}(G) \quad \text{and} \quad h(G) \leq \text{rad}(G) + 1;$$

if $\xi(G) > \text{rad}(G)$ then

$$s(G) = \xi(G);$$

if $\xi(G) > \text{rad}(G)$ and if the longest branch of G has a vertex of degree 1 as an end, then

$$h(G) = \xi(G);$$

and if $\xi(G) > \text{rad}(G)$ and if the longest branch of G has no vertex of degree 1, then

$$h(G) = \xi(G) + 1.$$

Proof: Let G be a graph, neither a path nor C_2. If $\xi(G) \leq \text{rad}(G)$, then (38) follows from Corollary 7A. Suppose that $\xi(G) > \text{rad}(G)$. By Theorem 9, G has a branch B of length $\xi(G)$ such that $G - \text{int}(B)$ has two components. Let $m = \xi(G)$, and apply Theorem 5 $m-1$ times to G to show that $L^{m-1}(G)$ has a single cut-edge e (induced by B) such that both components of $L^{m-1}(G) - e$ are type A subgraphs. (Since $\text{rad}(G) < |E(B)|$, all type B branches of G other than B have length less than $\text{rad}(G) \leq \xi(G) - 1 = m - 1$.

By Lemma 1, such branches shrink to single vertices before $m-1$ iterations. Thus, the only remaining type B branch in $L^{m-1}(G)$ has e as its only edge.) Hence, $s(G) > m-1$ and $h(G) > m-1$. This and Corollary 7A give $s(G) = m = \xi(G)$, and it remains to find $h(G)$. Denote $e = xy$.

Case 1 If x or y has degree 1 in $L^{m-1}(G)$ (say x) then $L^{m-1}(G) - x$, being a type A subgraph, is collapsible by Theorem 3, and hence has a spanning closed trail that is obviously also a dominating closed trail of $L^{m-1}(G)$. By Theorem 1, $L^m(G)$ is hamiltonian, and so $h(G) = \xi(G)$.

Case 2 If neither x nor y has degree 1 in $L^{m-1}(G)$, then e is a cut-vertex of $L^m(G)$, and so $h(G) > m = \xi(G)$. Hence $h(G) = \xi(G) + 1$, by Corollary 7A. □

There are graphs that show that (31) is best possible. Let $k \in \mathbf{N}$ and let G be a graph containing a complete subgraph H and a vertex v not in $V(H)$, such that v is connected to each vertex of H by at least one path of length exactly k. Also suppose that there are at least two paths from v to H, that all paths from v to H have length exactly k, and that their internal vertices have degree 2 in G. (These paths may not necessarily be branches of G, because one of their ends may have degree 2.) Then $\xi(G) = k = \mathrm{rad}(G)$, but G does not satisfy the latter part of Theorem 9.

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